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On the dimension of Chowla–Milnor space

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Abstract. In a recent work, Gun, Murty and Rath defined the Chowla–Milnor space and proved a non-trivial lower bound for these spaces. They also obtained a conditional improvement of this lower bound and noted that an unconditional improvement of their lower bound will lead to irrationality of $\zeta(k)/\pi^k$ for odd positive integers $k > 1$. In this paper, we give an alternate proof of their theorem about the conditional lower bound.

Keywords. Hurwitz zeta function; Chowla–Milnor spaces.

1. Introduction

For any complex number $s \in \mathbb{C}$, with $\Re(s) > 1$, one defines the Riemann zeta function as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

which has an Euler product

$$\zeta(s) = \prod_{p} (1 - p^{-s})^{-1}.$$

The Riemann zeta function defines an analytic function in the region $\Re(s) > 1$ and can be extended meromorphically to the whole complex plane with a simple pole at $s = 1$ having residue 1. Hurwitz generalized the Riemann zeta function by $\zeta(s, x)$ which is defined as

$$\zeta(s, x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s},$$

where $0 < x \leq 1$ and $s \in \mathbb{C}$ with $\Re(s) > 1$. He proved that $\zeta(s, x)$ can be extended meromorphically to the entire complex plane with a pole at $s = 1$. Note that for $x = 1$, $\zeta(s, 1)$ is the classical Riemann zeta function.

DEFINITION

For integers $k > 1$, $q \geq 2$, define the Chowla–Milnor space $V_k(q)$ by

$$V_k(q) := \mathbb{Q} - \text{span of } \{\zeta(k, a/q) : \ 1 \leq a < q, \ (a, q) = 1\}.$$

As described in [1], the conjecture of Chowla and Milnor is the assertion that the dimension of $V_k(q)$ is equal to $\varphi(q)$, where φ is the Euler's phi-function. Gun *et al.* [1] showed that the dimension of the above spaces is at least $\varphi(q)/2$. They also derived the following theorem.

Theorem. *Let $k > 1$ be an odd integer and $q, r > 2$ be two co-prime integers. Then either*

$$\dim_{\mathbb{Q}} V_k(q) \geq \frac{\varphi(q)}{2} + 1 \quad \textit{or} \quad \dim_{\mathbb{Q}} V_k(r) \geq \frac{\varphi(r)}{2} + 1.$$

The proof in [1] uses the expansion of Bernoulli polynomials. In this note, we give an alternate proof of the theorem by an explicit evaluation of co-tangent derivatives.

2. Proof of the theorem

The following Lemma 1 due to Okada [2] about the linear independence of co-tangent values at rational arguments plays a significant role in proving the theorem.

Lemma 1. *Let k and q be positive integers with $k \geq 1$ and $q > 2$. Let T be a set of $\varphi(q)/2$ representations* mod q *such that the union $T \cup (-T)$ constitutes a complete set of co-prime residue classes* mod q. *Then the set of real numbers*

$$\frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}} \cot(\pi z)|_{z=a/q}, \quad a \in T$$

are linearly independent over $\mathbb{Q}$.

We first have the following lemma.

Lemma 2. *For an integer $k \geq 1$,*

$$D^{k-1}(\pi \cot \pi z) = \pi^k \times \mathbb{Z}\, linear\ combination\ of\ (\csc \pi z)^{2l} (\cot \pi z)^{k-2l},$$

for some non-negative integer l. Here $D^{k-1} = \frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}}$.

Proof. We will prove this by induction on k. For $k = 1$, we have $D^{k-1}(\pi \cot(\pi z)) = \pi \cot(\pi z)$. Assume that the statement is true for $k - 1$, i.e.

$$D^{k-2}(\pi \cot(\pi z)) = \pi^{k-1} \sum a_i (\csc \pi z)^{2l_i} (\cot \pi z)^{(k-1)-2l_i},$$

where a_i's are integers.

Differentiating both sides with respect to z, we get

$$\begin{aligned} D^{k-1}(\pi \cot \pi z) &= \pi^k \sum [b_i (\csc \pi z)^{2l_i} (\cot \pi z)^{k-2l_i} \\ &\quad + c_i (\csc \pi z)^{2l_i+2} (\cot \pi z)^{k-(2l_i+2)}], \end{aligned}$$

where b_i, c_i's are integers. This completes the proof of Lemma 2. □

Lemma 3. *For an integer $k \geq 2$,*

$$\zeta(k, a/q) + (-1)^k \zeta(k, 1 - a/q) = \frac{(-1)^{k-1}}{(k-1)!} D^{k-1}(\pi \cot \pi z)|_{z=a/q}.$$

Proof.

$$\begin{aligned}
\text{L.H.S.} &= \zeta(k, a/q) + (-1)^k \zeta(k, 1 - a/q) \\
&= \sum_{n \geq 0}^{\infty} \frac{1}{(n + a/q)^k} + (-1)^k \sum_{n \geq 0}^{\infty} \frac{1}{(n + 1 - a/q)^k} \\
&= \sum_{n \geq 0}^{\infty} \frac{1}{(n + a/q)^k} + (-1)^k \sum_{n=1}^{\infty} \frac{1}{(n - a/q)^k} \\
&= \sum_{n \geq 0}^{\infty} \frac{1}{(n + a/q)^k} + (-1)^{2k} \sum_{n=1}^{\infty} \frac{1}{(-n + a/q)^k} \\
&= \sum_{n \in \mathbb{Z}} \frac{1}{(n + a/q)^k}.
\end{aligned}$$

Again we know that for $z \notin \mathbb{Z}$,

$$\pi \cot \pi z = \sum_{n \in \mathbb{Z}} \frac{1}{z + n}.$$

This implies that

$$D^{k-1}(\pi \cot \pi z) = (-1)^{k-1}(k-1)! \sum_{n \in \mathbb{Z}} \frac{1}{(z + n)^k}.$$

So,

$$\frac{(-1)^{k-1}}{(k-1)!} D^{k-1}(\pi \cot \pi z)|_{z=a/q} = \sum_{n \in \mathbb{Z}} \frac{1}{(n + a/q)^k},$$

which completes the proof of Lemma 3. □

Finally, we have Lemma 4, whose proof is standard.

Lemma 4. *Let P be the set of primes. We have*

$$\zeta(k) \prod_{\substack{p \in \mathrm{P}, \\ p \mid q}} (1 - p^{-k}) = q^{-k} \sum_{\substack{a=1 \\ (a,q)=1}}^{q-1} \zeta(k, a/q).$$

Proof of the theorem. First note that the space $V_k(q)$ is also spanned by the following sets of real numbers:

$$\{\zeta(k, a/q) + \zeta(k, 1 - a/q) | (a, q) = 1,\ 1 \leq a < q/2\},$$

$$\{\zeta(k, a/q) - \zeta(k, 1 - a/q) | (a, q) = 1,\ 1 \leq a < q/2\}.$$

Now from Lemma 3, we have the following:

$$\zeta(k, a/q) + (-1)^k \zeta(k, 1 - a/q) = \frac{(-1)^{k-1}}{(k-1)!} D^{k-1}(\pi \cot \pi z)|_{z=a/q}.$$

Applying the above Lemma 1, we see that

$$\dim_{\mathbb{Q}} V_k(q) \geq \frac{\varphi(q)}{2}.$$

Now from Lemmas 2 and 3 for an odd integer k, we have

$$\frac{\zeta(k, a/q) - \zeta(k, 1 - a/q)}{(2\pi i)^k}$$
$$= \frac{i}{2^k} \times \mathbb{Q} \text{ linear combinations of } (\csc \pi a/q)^{2l} (\cot \pi a/q)^{k-2l}.$$

We note that

$$i \cot(\pi a/q) = \frac{1 + \zeta_q^a}{1 - \zeta_q^a}$$

belongs to $\mathbb{Q}(\zeta_q)$ and so do the numbers $\csc(\pi a/q)^{2l}$ and $\cot(\pi a/q)^{2l}$. Since k is odd, we have

$$\frac{\zeta(k, a/q) - \zeta(k, 1 - a/q)}{(2\pi i)^k} \in \mathbb{Q}(\zeta_q) \tag{1}$$

Now we go back to the main part of the proof. Let q and r be two co-prime integers. Suppose that

$$\dim_{\mathbb{Q}} V_k(q) = \frac{\varphi(q)}{2}.$$

Then the numbers

$$\zeta(k, a/q) - \zeta(k, 1 - a/q), \quad \text{where } (a, q) = 1,\ 1 \leq a < q/2$$

generate $V_k(q)$. Now from Lemma 4, we get

$$\zeta(k) \prod_{p|q} (1 - p^{-k}) = q^{-k} \sum_{\substack{a=1, \\ (a,q)=1}}^{q-1} \zeta(k, a/q) \in V_k(q).$$

and hence

$$\zeta(k) = \sum_{\substack{(a,q)=1 \\ 1 \leq a < q/2}} \lambda_a [\zeta(k, a/q) - \zeta(k, 1 - a/q)], \quad \lambda_a \in \mathbb{Q}$$

so that

$$\frac{\zeta(k)}{(2\pi i)^k} = \sum_{\substack{(a,q)=1 \\ 1 \leq a < q/2}} \frac{\lambda_a [\zeta(k, a/q) - \zeta(k, 1 - a/q)]}{(2\pi i)^k}.$$

Thus by (1),

$$\frac{\zeta(k)}{i\pi^k} \in \mathbb{Q}(\zeta_q).$$

Similarly, if

$$\dim_{\mathbb{Q}} V_k(r) = \frac{\varphi(r)}{2},$$

then

$$\frac{\zeta(k)}{i\pi^k} \in \mathbb{Q}(\zeta_r)$$

and hence

$$\frac{\zeta(k)}{i\pi^k} \in \mathbb{Q}(\zeta_q) \cap \mathbb{Q}(\zeta_r).$$

Since any non-trivial finite extension of $\mathbb{Q}$ is ramified, if $\mathbb{Q}(\zeta_q) \cap \mathbb{Q}(\zeta_r) \neq \mathbb{Q}$, there exists a prime which is ramified in $\mathbb{Q}(\zeta_q) \cap \mathbb{Q}(\zeta_r)$, and also in $\mathbb{Q}(\zeta_q)$ and $\mathbb{Q}(\zeta_r)$. Note that a prime which ramifies in this intersection must necessarily divide both q and r. This is impossible because $(q, r) = 1$. So $\mathbb{Q}(\zeta_q) \cap \mathbb{Q}(\zeta_r) = \mathbb{Q}$. Hence we arrive at a contradiction as $\frac{\zeta(k)}{\pi^k}$ is a real number. Thus

$$\dim_{\mathbb{Q}} V_k(q) \geq \frac{\varphi(q)}{2} + 1 \quad \text{or} \quad \dim_{\mathbb{Q}} V_k(r) \geq \frac{\varphi(r)}{2} + 1.$$

This completes the proof of the theorem. □

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