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On the extrema of Dirichlet's first eigenvalue of a family of punctured regular polygons in two dimensional space forms

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Abstract. Let $\wp_1, \wp_0$ be two regular polygons of n sides in a space form $M^2(\kappa)$ of constant curvature $\kappa = 0, 1$ or -1 such that $\wp_0 \subset \wp_1$ and having the same center of mass. Suppose $\wp_0$ is circumscribed by a circle C contained in $\wp_1$. We fix $\wp_1$ and vary $\wp_0$ by rotating it in C about its center of mass. Put $\Omega = (\wp_1 \backslash \wp_0)^0$, the interior of $\wp_1 \backslash \wp_0$ in $M^2(\kappa)$. It is shown that the first Dirichlet's eigenvalue $\lambda_1(\Omega)$ attains extremum when the axes of symmetry of $\wp_0$ coincide with those of $\wp_1$.

Keywords. Laplace Beltrami operator; extremum of first eigenvalues; Green's identities for polygonal domains; Hadamard formula for polygonal domains.

1. Introduction and statement of main result

Let $M^2(\kappa) = (M^2(\kappa), g)$ denote the hyperbolic plane $\mathbb{H}^2$, Riemann sphere S^2, Euclidean plane $\mathbb{R}^2$ for $\kappa = -1, 1, 0$ respectively. We denote the inner metric of $M^2(\kappa)$ as d_κ.

Let D denote the Levi-Civita connection of $M^2(\kappa)$. For a smooth vector field X, the divergence $\mathrm{div}(X)$ is defined as $\mathrm{trace}(DX)$. For a smooth function $f : M^2(\kappa) \to \mathbb{R}$, the gradient vector field ∇f is defined by $g(\nabla f(x), v) = Df(x)(v)$ $(x \in M, v \in T_x M)$ and the Laplace-Beltrami operator Δ is defined by $\Delta f = \mathrm{div}(\nabla f)$. Let $\mathcal{D}(\Omega)$ ($\mathcal{D}(\bar{\Omega})$) denote the space of all infinitely differentiable real-valued functions defined on Ω which have compact support in Ω (respectively, of the restrictions to Ω of functions in $\mathcal{D}(\mathbb{R}^d)$).

DEFINITION 1.1

Let $u \in L^2(\Omega)$ and $f \in L^2(\Omega)$. We say that

$$\Delta u = f \quad \text{on } \Omega \quad \text{in the sense of distributions}$$

if $\int_\Omega u \Delta\phi \mathrm{d}V = \int_\Omega f\phi \mathrm{d}V, \forall \phi \in \mathcal{D}(\Omega)$.

We consider the following eigenvalue problem on Ω:

$$\left.\begin{aligned} -\Delta u &= \lambda u \quad \text{on } \Omega \text{ in the sense of distributions,} \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{1.1.1}$$

The eigenvalues of the positive Laplace–Beltrami operator $-\Delta$ are strictly positive and accumulate only at ∞ (cf. Theorem 1.2.2, p. 6 of [9]). Let $\lambda_1(\Omega)$ denote the first eigenvalue of $-\Delta$. Then

$$\lambda_1(\Omega) = \inf\{\|\nabla\phi\|^2_{L^2(\Omega)} \mid \phi \in H^1_0(\Omega),\ \|\phi\|_{L^2(\Omega)} = 1\}. \tag{1.1.2}$$

The first eigenvalue $\lambda_1(\Omega)$ is simple (cf. Theorem 1.2.5, p. 7 of [9]). Let $y_1(\Omega) \in H^1_0(\Omega)$ be the unique solution of problem (1.1.1) corresponding to the first eigenvalue $\lambda_1(\Omega)$ characterized by

$$y_1(\Omega) > 0 \text{ on } \Omega \quad \text{and} \quad \int_\Omega y_1^2(\Omega)\mathrm{d}V = 1. \tag{1.1.3}$$

Let $R = R(\kappa) > 0$ be an arbitrary constant for $\kappa = -1, 0$. And for $\kappa = 1$, $R(\kappa)$ is any arbitrary constant with $0 < R(\kappa) < \pi/2$. We fix $p \in M^2(\kappa)$. Consider the distance circle $C(p, R) = \{q \in M^2(\kappa) \mid d_\kappa(p, q) = R\}$. Fix $n \in \mathbb{N}$ with $n \geq 3$. Let $\wp_1$ be a regular polygon of n sides in $M^2(\kappa)$ (see §7) circumscribed by the circle $C(p, R)$ and let $r = r(\kappa, R) > 0$ be such that the circle $C(p, r)$ is contained in the interior of $\wp_1$. Throughout the paper the objects p, R, r and $\wp_1$ are all fixed. Let $\wp_0$ denote any regular polygon of n sides in $M^2(\kappa)$ circumscribed by the circle $C(p, r)$ and $\mathcal{F}_\kappa = \mathcal{F}(\kappa, R, r)$ denote family of punctured sets $\Omega = (\wp_1 \backslash \wp_0)^0$, the interior of $\wp_1 \backslash \wp_0$ in $M^2(\kappa)$.

If two vertices of $\wp_0$ and $\wp_1$ lie on the same half-axes of symmetry emanating from p, then we say that $\wp_0$ occupies *on position* in $\wp_1$ (see figure 1). If a vertex of $\wp_0$ lie on the half-axes of symmetry emanating from p and passing through the mid-point of a side of $\wp_1$, then we say that $\wp_0$ occupies *off position* in $\wp_1$ (see figure 2). In the general position of $\wp_0$, neither of the two conditions is satisfied. These concepts were first introduced in [2].

Now we state our main result.

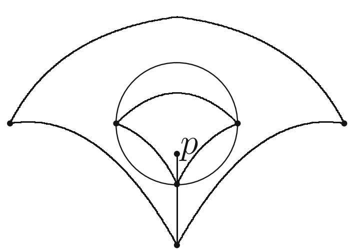

Figure 1. $\wp_0$ in 'on position' ($\kappa = -1, n = 3$).

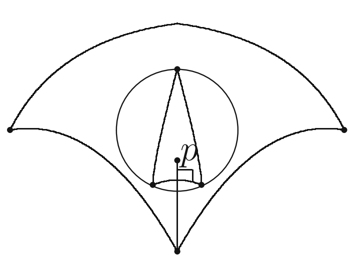

Figure 2. $\wp_0$ in 'off position' ($\kappa = -1, n = 3$).

Theorem 1.2. *For $\Omega \in \mathcal{F}_\kappa$, the first eigenvalue $\lambda_1(\Omega)$ is optimized exactly when the axes of symmetry of $\wp_0$ coincide with those of $\wp_1$. The maximizing configuration for $\lambda_1(\Omega)$ corresponds to the case when $\wp_0$ occupies 'on position' in $\wp_1$. The minimizing configuration for $\lambda_1(\Omega)$ corresponds to the case when $\wp_0$ occupies 'off position' in $\wp_1$.*

In [2], a family of annular domains in $\mathbb{R}^2$ with C^2-boundary and having dihedral group symmetry is considered, and analogous result has been proved.

In §2, we state preliminary results regarding Sobolev spaces and maximum principles.

It is well known that any eigenfunction of a bounded domain U in $\mathbb{R}^d$ with smooth boundary is an element of $C^\infty(\bar{U})$ and hence it lies in the second Sobolev space $H^2(U)$. On the contrary, for $\Omega \in \mathcal{F}_\kappa$, any eigenfunction $y_1(\Omega)$ corresponding to the first eigenvalue lies in the Sobolev space $H_0^{1+\delta}(\Omega)$ where δ is any constant in the open interval $]\frac{1}{2}, \frac{3}{5}[$ (see Proposition 3.7). In §3, we develop the Green's identities for domains Ω with polygonal boundaries in $\mathbb{R}^2$ involving functions such as $y_1(\Omega)$.

The first eigenfunction is differentiable with respect to the variation of a convex or C^2-domain and expressions for the derivative is given by the Hadamard formula [3,9,13]. In §4, we develop these techniques for $\Omega \in \mathcal{F}_0$ using the Green's identities proved in §3.

In §5, we prove the main Theorem 1.2. The proof relies on the Hadamard formula proved in §4 and the moving plane method [5].

2. Preliminaries

We introduce some preliminary notations and state some known results needed in the sequel.

Let Ω be any nonempty open set in $\mathbb{R}^d$. We denote by $L^2(\Omega)$ the class of all real valued measurable functions u, defined on Ω, for which $\int_\Omega |u(x)|^2 \mathrm{d}x < \infty$. We identify in $L^2(\Omega)$ functions which are equal almost everywhere on Ω. For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in (\mathbb{Z}^+)^d$, we put $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$ and set $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_d^{\alpha_d}$. A function $v \in L^2(\Omega)$ is said to be an '*α-th weak partial derivative*' of $u \in L^2(\Omega)$ if $\int_\Omega u D^\alpha \phi \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega v\phi \mathrm{d}x$, $\forall\, \phi \in \mathcal{D}(\Omega)$.

We define the Sobolev space $H^s(\Omega)$ $(s > 0)$ as follows:

(i) When $s \in \mathbb{N}$,
$H^s(\Omega) = \{u \in L^2(\Omega) \mid \alpha\text{-th weak partial derivative } \partial^\alpha u \in L^2(\Omega), \forall\, |\alpha| \le s\}$.

(ii) When $s = m + \sigma$, where $m \in \mathbb{Z}^+$ and $0 < \sigma < 1$ in $\mathbb{R}$,

$$H^s(\Omega) = \left\{ u \in H^m(\Omega) \,\middle|\, \sum_{|\alpha|=m} \iint_{\Omega\times\Omega} \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{||x-y||^{d+2\sigma}} \mathrm{d}x\mathrm{d}y < \infty \right\}.$$

We define the Sobolev norm on $H^s(\Omega)$ $(s > 0)$ as follows:

(i) When $s \in \mathbb{N}$,

$$||u||_{H^s(\Omega)} = \left(\sum_{|\alpha|\le s} ||D^\alpha u||^2_{L^2(\Omega)} \right)^{\frac{1}{2}}.$$

(ii) When $s = m + \sigma$, where $m \in \mathbb{Z}^+$ and $0 < \sigma < 1$ in $\mathbb{R}$,

$$||u||_{H^s(\Omega)} = \left(||u||^2_{H^m(\Omega)} + \sum_{|\alpha|=m} \iint_{\Omega\times\Omega} \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{||x-y||^{d+2\sigma}} \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{2}}.$$

The closure of $\mathcal{D}(\Omega)$ in $H^s(\Omega)$ is denoted by $H^s_0(\Omega)$.

We state the Poincaré's inequality.

PROPOSITION 2.1 (cf. Theorem 1, p. 182 of [12])

Let Ω be a bounded open set in $\mathbb{R}^d$. Then there exists a constant $C = C(\Omega) > 0$ such that $||u||_{L^2(\Omega)} \leq C||\nabla u||_{L^2(\Omega)}$, $\forall u \in H^1_0(\Omega)$.

We define $H^{-s}(\Omega) = H^s_0(\Omega)^*$ $(s > 0)$. Note that for $\Omega = \mathbb{R}^d$, $H^{-s}(\mathbb{R}^d) = H^s(\mathbb{R}^d)^*$ since $H^s_0(\mathbb{R}^d) = H^s(\mathbb{R}^d)$ (cf. Theorem 7.38, p. 205 of [1]). For a Lebesgue measurable function g defined on Ω, define

$$\tilde{g}(x) = \begin{cases} g(x), & \text{if } x \in \Omega, \\ 0, & \text{if } x \in \mathbb{R}^d \backslash \Omega. \end{cases}$$

We define $\tilde{H}^s(\Omega) := \{g : \tilde{g} \in H^s(\mathbb{R}^d)\}$ $(s > 0)$ with $||.||_{H^s(\mathbb{R}^d)}$ norm.

DEFINITION 2.2

Let Ω be an open subset of $\mathbb{R}^d$. We say that Ω has Lipschitz (polygonal) boundary if every point x on the boundary of Ω has a neighborhood U_x in $\mathbb{R}^d$ such that $(\partial\Omega) \cap U_x$ is the graph of a Lipschitz (continuous piecewise linear, respectively) function.

Then we have the following result regarding the space $\tilde{H}^s(\Omega)$.

PROPOSITION 2.3 (cf. Theorem 1.2.16, p. 6 of [7])

Let Ω be a bounded open subset of $\mathbb{R}^d$ which has Lipschitz boundary. Then $\tilde{H}^s(\Omega) = H^s_0(\Omega)$ when $s - \frac{1}{2} \notin \mathbb{Z}$.

Next we have a result on extension operator.

PROPOSITION 2.4 (cf. Theorem 1.2.10, p. 5 of [7])

Let Ω be a bounded open subset of $\mathbb{R}^d$ which has Lipschitz boundary. Then for every $s > 0$ there exists a continuous linear operator P_s from $H^s(\Omega)$ to $H^s(\mathbb{R}^d)$ such that $P_s(u)_{|\Omega} = u$ for every $u \in H^s(\Omega)$.

We state a density result for the space $H^s(\Omega)$.

PROPOSITION 2.5 (cf. Theorem 1.2.7, p. 4 of [7])

Let Ω be an open subset of $\mathbb{R}^d$ which has Lipschitz boundary. Then $\mathcal{D}(\bar{\Omega})$ is dense in $H^s(\Omega)$ for all $s \geq 0$.

We state below a basic *a priori* inequality.

PROPOSITION 2.6 (cf. Theorem 2.2.3, p. 45 of [7])

Let Ω be a bounded open subset of $\mathbb{R}^2$ which has polygonal boundary. Then there exists a constant $C = C(\Omega)$ such that

$$||u||_{H^2(\Omega)} \leq C||\Delta u||_{L^2(\Omega)} \quad (u \in H^2(\Omega) \cap H_0^1(\Omega)).$$

We state the maximum principle (cf. Theorem 5, p. 61 of [11]).

PROPOSITION 2.7

Let $u(x_1, x_2, \ldots, x_n)$ satisfy the differential inequality

$$L[u] \equiv \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} \geq 0$$

in a domain D where L is uniformly elliptic. Suppose the coefficients a_{ij} and b_i are uniformly bounded. If u attains a maximum M at a point of D, then $u \equiv M$ in D.

Finally we state the maximum principle of Hopf (cf. Theorem 7, p. 65 of [11]).

PROPOSITION 2.8

Let u satisfy the inequality

$$L[u] \equiv \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} \geq 0$$

in a domain D where L is uniformly elliptic. Suppose that $u \leq M$ in D and that $u = M$ at a boundary point P. Assume that P lies on the boundary of a ball K_1 in D. If u is continuous in $D \cup P$ and an outward directional derivative $\partial u/\partial \nu$ exists at P, then $\partial u/\partial \nu > 0$ at P unless $u \equiv M$.

3. Green's identities for domains with polygonal boundaries

Following [7], $\forall\, s > 0$ consider $E^s(\Delta, L^2(\Omega)) := \{u \in H^s(\Omega) \mid \Delta u \in L^2(\Omega)\}$ with the norm function $||u||_{E^s}^2 = ||u||_{H^s(\Omega)}^2 + ||\Delta u||_{L^2(\Omega)}^2$, $\forall\, u \in E^s(\Delta, L^2(\Omega))$. And $\forall\, s > 0$ consider $E_0^s(\Delta, L^2(\Omega)) := E^s(\Delta, L^2(\Omega)) \cap H_0^1(\Omega)$ with $||.||_{E^s}$ norm. Then $E_0^s(\Delta, L^2(\Omega))$ is a Banach space. It is proved in [6] that $\mathcal{D}(\bar{\Omega})$ is dense in $E^1(\Delta, L^2(\Omega))$ (cf. Lemma 1.5.3.9, p. 59 of [6]). On similar lines we state and prove the following lemma.

Lemma 3.1. *Let Ω be a bounded open subset of $\mathbb{R}^d$ which has Lipschitz boundary. Then*

(i) *$\mathcal{D}(\bar{\Omega})$ is dense in $E^s(\Delta, L^2(\Omega))$ for any $0 \leq s < 2$ with $s \neq \frac{3}{2}, \frac{1}{2}$.*
(ii) *$H^2(\Omega) \cap H_0^1(\Omega)$ is dense in $E^{1+s}(\Delta, L^2(\Omega)) \cap H_0^1(\Omega)$ with $\frac{1}{2} < s < 1$.*

Proof.

(i) Let $l \in E^s(\Delta, L^2(\Omega))^*$ arbitrary such that $l|_{\mathcal{D}(\bar{\Omega})} = 0$. It suffices to prove that $l = 0$.

Define $T: E^s(\Delta, L^2(\Omega)) \rightarrow H^s(\Omega) \times L^2(\Omega)$ by $T(u) = (u, \Delta u)$ $\forall$ $u \in E^s(\Delta, L^2(\Omega))$. Note that T is an isometry and via T, we consider $E^s(\Delta, L^2(\Omega))$ as a closed linear subspace of $H^s(\Omega) \times L^2(\Omega)$. Then by Hahn-Banach theorem there exists $l_1 \in (H^s(\Omega) \times L^2(\Omega))^*$ such that $l_1 \circ T = l$ and $||l_1||_{\text{op}} = ||l||_{\text{op}}$ where $||l||_{\text{op}}$ denotes the operator norm of l.

Consider $P_s \times Id : H^s(\Omega) \times L^2(\Omega) \rightarrow H^s(\mathbb{R}^d) \times L^2(\Omega)$. We can write $H^s(\mathbb{R}^d) = P_s(H^s(\Omega)) \oplus Z$ for some closed linear subspace Z of $H^s(\mathbb{R}^d)$. Then

$$H^s(\mathbb{R}^d) \times L^2(\Omega) = (P_s(H^s(\Omega)) \times L^2(\Omega)) \oplus (Z \times L^2(\Omega)). \tag{3.1.1}$$

There exists unique $\tilde{l}_1 \in (H^s(\mathbb{R}^d) \times L^2(\Omega))^*$ such that $\tilde{l}_1|_{Z \times L^2(\Omega)} = 0$ and $\tilde{l}_1 \circ P_s \times Id = l_1$.

Then by Riesz representation theorem, there exist $f \in (H^s(\mathbb{R}^d))^*$ and $g \in L^2(\Omega)$ such that

$$\tilde{l}_1(y, z) = \langle f; y \rangle + \int_\Omega gz\mathrm{d}x, \qquad \forall\ (y, z) \in H^s(\mathbb{R}^d) \times L^2(\Omega). \tag{3.1.2}$$

Let $\phi \in \mathcal{D}(\mathbb{R}^d)$ arbitrary and denote $\phi_{|_{\bar{\Omega}}}$ by φ. Then $P_s(\varphi) = \phi$. Further, using (3.1.2),

$$\begin{aligned}
0 &= l(\varphi) \\
&= (l_1 \circ T)(\varphi) \\
&= l_1(\varphi, \Delta\varphi) \\
&= \tilde{l}_1(P_s(\varphi), \Delta\varphi) \\
&= \langle f; P_s(\varphi) \rangle + \int_\Omega g\Delta\varphi\mathrm{d}x \\
&= \langle f; P_s(\varphi) \rangle + \langle \tilde{g}; \Delta\phi \rangle \\
&= f(\phi) + \langle \Delta\tilde{g}; \phi \rangle.
\end{aligned}$$

Hence

$$\Delta\tilde{g} = -f \tag{3.1.3}$$

in the sense of distributions.

Now $f \in H^{-s}(\mathbb{R}^d)$. By (3.1.3) and the elliptic regularity theorem (cf. Theorem 6.33, p. 214 of [10]), $\tilde{g} \in H^{2-s}(\mathbb{R}^d)$. Hence by Proposition 2.3, $g \in \tilde{H}^{2-s}(\Omega) = H_0^{2-s}(\Omega)$. Let $(\varphi_m)_{m \in \mathbb{N}}$ be a sequence in $\mathcal{D}(\Omega)$ such that $\varphi_m \rightarrow g$ as $m \rightarrow \infty$ in $H_0^{2-s}(\Omega)$. Then $\phi_m \rightarrow \tilde{g}$ in $L^2(\mathbb{R}^d)$ and $\Delta\phi_m \rightarrow \Delta\tilde{g}$ as $m \rightarrow \infty$ in the sense of distributions.

Let $u \in E^s(\Delta, L^2(\Omega))$ be any element. Then by Definition 1.1 we get

$$\int_\Omega (\Delta\varphi_m)u\mathrm{d}x = \int_\Omega \varphi_m(\Delta u)\mathrm{d}x, \quad \forall m \in \mathbb{N}. \tag{3.1.4}$$

Then

$$
\begin{aligned}
l(u) &= (l_1 \circ T)(u) \\
&= \tilde{l}_1(P_s(u), \Delta u) \\
&= \langle f; P_s(u)\rangle + \int_\Omega g\Delta u \mathrm{d}x \\
&= \langle -\Delta\tilde{g}; P_s(u)\rangle + \int_{\mathbb{R}^d} \tilde{g}(\widetilde{\Delta u})\mathrm{d}x \text{ [by (3.1.3)]} \\
&= \lim_{m\to\infty}\left(\langle -\Delta\phi_m; P_s(u)\rangle + \int_{\mathbb{R}^d} \phi_m(\widetilde{\Delta u})\mathrm{d}x\right) \\
&= \lim_{m\to\infty}\left(\langle -\Delta\varphi_m; u\rangle + \int_\Omega \varphi_m \Delta u \mathrm{d}x\right) \\
&= \lim_{m\to\infty}\left(\langle -\Delta\varphi_m; u\rangle + \int_\Omega (\Delta\varphi_m) u \mathrm{d}x\right) \quad \text{[by (3.1.4)]} \\
&= 0.
\end{aligned}
$$

Thus proof of (i) is complete.

(ii) Let $l \in E_0^{1+s}(\Delta, L^2(\Omega))^*$ arbitrary such that $l|_{H^2(\Omega)\cap H_0^1(\Omega)} = 0$. It suffices to prove that $l = 0$.

Define $T : E_0^{1+s}(\Delta, L^2(\Omega)) \to (H^{1+s}(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega)$ by $T(u) = (u, \Delta u)$, $\forall u \in E_0^{1+s}(\Delta, L^2(\Omega))$. Then T is an isometry. Consider

$$P_{1+s} \times Id : (H^{1+s}(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega) \to H^{1+s}(\mathbb{R}^d) \times L^2(\Omega).$$

Note that $P_{1+s}(H^{1+s}(\Omega) \cap H_0^1(\Omega))$ is a closed linear subspace of $H^{1+s}(\mathbb{R}^d)$.

The rest of the proof can be continued as done in the proof of (i). □

In the rest of this section, Ω denotes a bounded open set in $\mathbb{R}^2$ which has polygonal boundary and $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ be all the sides of Ω. Also n_j denotes the outward unit normal to Ω on the side Γ_j $(1 \le j \le n)$.

We state a result on the '*trace operator*' for Ω (cf. Theorem 1.4.2, p. 12 of [7]).

PROPOSITION 3.2

Fix $s > \frac{1}{2}$. The map $u \mapsto u_{|\Gamma_j}$ from $\mathcal{D}(\bar{\Omega})$ extends as bounded linear operator from $H^s(\Omega)$ into $H^{s-\frac{1}{2}}(\Gamma_j)$.

Now we develop various versions of Green's identities for Ω which are needed in the sequel. To begin with let $u \in H^1(\Omega)$ and $v \in H^2(\Omega)$ be arbitrary. Then by Proposition 2.5 and the classical Green's identities, we get

$$\int_\Omega u\Delta v \mathrm{d}x = -\int_\Omega \langle\nabla u, \nabla v\rangle \mathrm{d}x + \sum_{j=1}^n \int_{\Gamma_j} u_{|\Gamma_j} \frac{\partial v}{\partial n_j}\mathrm{d}s. \tag{3.2.1}$$

PROPOSITION 3.3 (cf. Theorem 1.5.4, p. 27 of [7])

Let $v \in E^1(\Delta, L^2(\Omega))$, *and* $u \in H^1(\Omega)$ *be any element such that* $u_{|\Gamma_j} \in \tilde{H}^{\frac{1}{2}}(\Gamma_j)$, $\forall 1 \leq j \leq n$. *Then*

$$\int_\Omega u\Delta v \mathrm{d}x = -\int_\Omega \langle \nabla u, \nabla v\rangle \mathrm{d}x + \sum_{j=1}^n \int_{\Gamma_j} u_{|\Gamma_j} \frac{\partial v}{\partial n_j} \mathrm{d}s.$$

PROPOSITION 3.4

Let $s > \frac{1}{2}$. *Let* $v \in E^{1+s}(\Delta, L^2(\Omega))$ *and* $u \in H^1(\Omega)$ *be arbitrary. Then*

$$\int_\Omega u\Delta v \mathrm{d}x = -\int_\Omega \langle \nabla u, \nabla v\rangle \mathrm{d}x + \sum_{j=1}^n \int_{\Gamma_j} u_{|\Gamma_j} \frac{\partial v}{\partial n_j} \mathrm{d}s.$$

Proof. By Lemma 3.1, there exists a sequence $(v_m)_{m\in\mathbb{N}}$ in $\mathcal{D}(\bar{\Omega})$ such that $v_m \to v$ as $m \to \infty$ in $E^{1+s}(\Delta, L^2(\Omega))$. Then by (3.2.1),

$$\int_\Omega u\Delta v_m \mathrm{d}x = -\int_\Omega \langle \nabla u, \nabla v_m\rangle \mathrm{d}x + \sum_{j=1}^n \int_{\Gamma_j} u_{|\Gamma_j} \frac{\partial v_m}{\partial n_j} \mathrm{d}s. \tag{3.4.1}$$

Note that $\Delta v_m \to \Delta v$ in $L^2(\Omega)$ and $\nabla v_m \to \nabla v$ in $L^2(\Omega, \mathbb{R}^2)$ as $m \to \infty$. Hence by Hölder's inequality,

$$\left.\begin{aligned} \lim_{m\to\infty} \int_\Omega u\Delta v_m \mathrm{d}x &= \int_\Omega u\Delta v \mathrm{d}x, \\ \lim_{m\to\infty} \int_\Omega \langle \nabla u, \nabla v_m\rangle \mathrm{d}x &= \int_\Omega \langle \nabla u, \nabla v\rangle \mathrm{d}x. \end{aligned}\right\} \tag{3.4.2}$$

Further, by Proposition 3.2, $\frac{\partial v_m}{\partial n_j} \to \frac{\partial v}{\partial n_j}$ as $m \to \infty$ in $L^2(\Gamma_j)$ $(j = 1, 2, \ldots, n)$. Hence

$$\lim_{m\to\infty} \int_{\Gamma_j} u_{|\Gamma_j} \frac{\partial v_m}{\partial n_j} \mathrm{d}s = \int_{\Gamma_j} u_{|\Gamma_j} \frac{\partial v}{\partial n_j} \mathrm{d}s. \tag{3.4.3}$$

The result now follows by (3.4.1), (3.4.2) and (3.4.3). □

PROPOSITION 3.5

Let $s > \frac{1}{2}$. *Let* $u \in E_0^{1+s}(\Delta, L^2(\Omega))$ *and* $v \in E^s(\Delta, L^2(\Omega))$ *be arbitrary. Then*

$$\int_\Omega (u\Delta v - v\Delta u) \mathrm{d}x = -\sum_{j=1}^n \int_{\Gamma_j} v_{|\Gamma_j} \frac{\partial u}{\partial n_j} \mathrm{d}s. \tag{3.5.1}$$

Proof. By Lemma 3.1, there exists a sequence $(u_m)_{m\in\mathbb{N}}$ in $H^2(\Omega)\cap H_0^1(\Omega)$ such that $u_m \to u$ as $m\to\infty$ in $E_0^{1+s}(\Delta, L^2(\Omega))$ and there exists a sequence $(v_m)_{m\in\mathbb{N}}$ in $\mathcal{D}(\bar{\Omega})$ such that $v_m \to v$ as $m\to\infty$ in $E^s(\Delta, L^2(\Omega))$. We have by (3.2.1)

$$\int_\Omega (u_m\Delta v_m - v_m\Delta u_m)\mathrm{d}x = \sum_{j=1}^n \int_{\Gamma_j}\left(u_{m|\Gamma_j}\frac{\partial v_m}{\partial n_j} - v_{m|\Gamma_j}\frac{\partial u_m}{\partial n_j}\right)\mathrm{d}s,$$

$$\text{i.e.}\quad \int_\Omega (u_m\Delta v_m - v_m\Delta u_m)\mathrm{d}x = -\sum_{j=1}^n \int_{\Gamma_j}\left(v_{m|\Gamma_j}\frac{\partial u_m}{\partial n_j}\right)\mathrm{d}s. \tag{3.5.2}$$

As seen in the proof of Proposition 3.4, we get

$$\left.\begin{aligned} \lim_{m\to\infty}\int_\Omega u_m\Delta v_m\mathrm{d}x &= \int_\Omega u\Delta v\mathrm{d}x,\\ \lim_{m\to\infty}\int_\Omega v_m\Delta u_m\mathrm{d}x &= \int_\Omega v\Delta u\mathrm{d}x,\\ \lim_{m\to\infty}\int_{\Gamma_j}\left(v_{m|\Gamma_j}\frac{\partial u_m}{\partial n_j}\right)\mathrm{d}s &= \sum_{j=1}^n\int_{\Gamma_j}\left(v_{|\Gamma_j}\frac{\partial u}{\partial n_j}\right)\mathrm{d}s. \end{aligned}\right\} \tag{3.5.3}$$

Then by (3.5.2) and (3.5.3) we get

$$\int_\Omega (u\Delta v - v\Delta u)\mathrm{d}x = -\sum_{j=1}^n\int_{\Gamma_j} v_{|\Gamma_j}\frac{\partial u}{\partial n_j}\mathrm{d}s.$$

□

Finally we include a proof of the following known result on the existence and uniqueness of the solution of the Dirichlet's boundary value problem. This result will be used in §4.

PROPOSITION 3.6

For every $f\in L^2(\Omega)$ there exists an unique solution $u\in H_0^1(\Omega)$ of the Dirichlet boundary value problem:

$$\left.\begin{aligned} \Delta u &= f \text{ on } \Omega \text{ in the sense of distributions},\\ u &= 0 \text{ on } \partial\Omega. \end{aligned}\right\} \tag{3.6.1}$$

Proof. For all $f, g\in L^2(\Omega)$, we define the inner product $\langle f, g\rangle_{L^2(\Omega)} = \int_\Omega fg\mathrm{d}x$. For all $v, w\in H_0^1(\Omega)$, we define $\langle v, w\rangle_\sim = \int_\Omega\langle\nabla v, \nabla w\rangle\mathrm{d}x$ and $\|v\|_\sim = \langle v, v\rangle_\sim^{\frac{1}{2}}$. Then by Poincare inequality (Proposition 2.1), $\|.\|_\sim$ and $\|.\|_{H^1(\Omega)}$ are equivalent norms on $H_0^1(\Omega)$, and $(H_0^1(\Omega), \langle,\rangle_\sim)$ is a Hilbert space.

Now given $f \in L^2(\Omega)$, we define a linear map $L : H_0^1(\Omega) \to \mathbb{R}$ by $L(w) = \langle f, w\rangle_{L^2(\Omega)}$. Then

$$\begin{aligned} |L(w)| &= |\langle f, w\rangle_{L^2(\Omega)}| \\ &\leq ||f||_{L^2(\Omega)}||w||_{L^2(\Omega)} \\ &\leq C||f||_{L^2(\Omega)}||\nabla w||_{L^2(\Omega)} \quad \text{(by Proposition 2.1)} \\ &= C||f||_{L^2(\Omega)}||w||_{\sim}. \end{aligned}$$

So L is a bounded linear functional on the Hilbert space $(H_0^1(\Omega), \langle , \rangle_{\sim})$. By the Riesz representation theorem, there exists a unique $u' = u'(f) \in H_0^1(\Omega)$ such that

$$L(w) = \langle u', w\rangle_{\sim}, \quad \forall\ w \in H_0^1(\Omega). \tag{3.6.2}$$

Then by (3.6.2) we get

$$\begin{aligned} \langle f, \phi\rangle_{L^2(\Omega)} = L(\phi) &= \langle u', \phi\rangle_{\sim} = \langle \nabla u', \nabla\phi\rangle_{L^2(\Omega)} \\ &= -\langle u', \Delta\phi\rangle_{L^2(\Omega)}, \quad \forall\phi \in \mathcal{D}(\Omega). \end{aligned}$$

Hence $\Delta u' = -f$ on Ω in the sense of distributions. Thus $u = -u'$ is a desired solution to problem (3.6.1).

To prove the uniqueness of solution of Problem (3.6.1), let $u_1, u_2 \in H_0^1(\Omega)$ both satisfy (3.6.1). Then $\Delta(u_1 - u_2) = 0$ on Ω in the sense of distributions and hence $u_1 - u_2 \in E^1(\Delta, L^2(\Omega))$. Then

$$\begin{aligned} 0 &= \langle \Delta(u_1 - u_2), u_1 - u_2\rangle_{L^2(\Omega)} \\ &= \langle \nabla(u_1 - u_2), \nabla(u_1 - u_2)\rangle_{L^2(\Omega)} \quad \text{[by Proposition 3.3]} \\ &= \langle u_1 - u_2, u_1 - u_2\rangle_{\sim} \\ &= ||u_1 - u_2||_{\sim}^2. \end{aligned}$$

This implies that $u_1 = u_2$ in $H_0^1(\Omega)$. □

Recall that Ω denotes a bounded open set in $\mathbb{R}^2$ which has polygonal boundary. We denote by w_j the interior angle at jth vertex of Ω. Put $\Lambda = \dfrac{\pi}{\max_j w_j}$. Then $\Lambda > \frac{1}{2}$. We state the following important regularity result which is applicable for the first eigenfunction $y_1(\Omega)$.

PROPOSITION 3.7 (cf. Corollary 2.4.4 and Remark 2.4.6, pp. 58–59 of [7])

Let $s_0 = 1 + \min\{1, \Lambda\} > \frac{3}{2}$. *If* $u \in H_0^1(\Omega)$ *such that* $\Delta u \in L^2(\Omega)$, *then* $u \in H_0^s(\Omega)$, $\forall\, s < s_0$.

We make the following two remarks on Proposition 3.7.

Remark 3.8 (cf. Theorem 2.3.7, p. 54 of [7]). Let Ω be a non-convex bounded open set in $\mathbb{R}^2$ with polygonal boundary. Then $\frac{1}{2} < \Lambda < 1$. Hence $s_0 = 1 + \min\{1, \Lambda\} < 2$.

Let m be the number of vertices of Ω whose interior angle is greater than π. Let $W := \{f \in L^2(\Omega) \mid \Delta f = 0 \text{ on } \Omega \text{ in the sense of distributions}\}$. Then W is a vector space of dimension m over $\mathbb{R}$. Let $f = f_0 + g$ where $f_0 \in W \backslash \{0\}$ and $g \in \{\Delta v \mid v \in H^2(\Omega) \cap H_0^1(\Omega)\}$ are arbitrary. Then the solution $u = u(f) \in H_0^1(\Omega)$ of Problem (3.6.1) does not belong to $H^2(\Omega)$.

Remark 3.9. Let Ω be a convex bounded open set in $\mathbb{R}^2$ with polygonal boundary. Then $\Lambda > 1$. Let $u \in H_0^1(\Omega)$ be such that $\Delta u \in L^2(\Omega)$. Then $u \in H^2(\Omega)$, by Proposition 3.7.

4. Derivative with respect to the domain, and Hadamard formula

Throughout this section Ω denotes an arbitrary element of the family $\mathcal{F}_0$ which was introduced in §1.

Let δ be any real constant in $]\frac{1}{2}, \frac{3}{5}[$. For any $u \in E^{1+\delta}(\Delta, L^2(\Omega))$, $\Delta u \in L^2(\Omega)$. So we have an operator $\Delta : E_0^{1+\delta}(\Delta, L^2(\Omega)) \to L^2(\Omega)$.

PROPOSITION 4.1

$\Delta : E_0^{1+\delta}(\Delta, L^2(\Omega)) \to L^2(\Omega)$ *is an invertible bounded operator.*

Proof. Clearly $||\Delta u||_{L^2(\Omega)} \leq ||u||_{E^{1+\delta}}$. Hence $\Delta : E_0^{1+\delta}(\Delta, L^2(\Omega)) \to L^2(\Omega)$ is a bounded operator.

Let $f \in L^2(\Omega)$ be arbitrary. By Proposition 3.6, there exists unique $u = u(f) \in H_0^1(\Omega)$ such that $\Delta u = f$ on Ω in the sense of distributions.

The boundary $\partial\Omega$ is the union of two regular polygons each having n sides and the angle at any vertex of the boundary polygon of Ω lies in the set $\{\left(\frac{n-2}{n}\right)\pi, \left(\frac{n+2}{n}\right)\pi\}$. Put $\Lambda_n = \dfrac{\pi}{\left(\frac{n+2}{n}\right)\pi}$. Then $\Lambda_n = \frac{n}{n+2}$. Thus $\min\{1, \Lambda_n\} \geq \frac{3}{5}$, $\forall\, n \geq 3$. By Proposition 3.7, $u(f) \in H_0^s(\Omega)$, $\forall\, s < 1 + \frac{3}{5}$. In particular, $u(f) \in H_0^{1+\delta}(\Omega)$ and hence $u(f) \in E_0^{1+\delta}(\Delta, L^2(\Omega))$. Thus $\Delta : E_0^{1+\delta}(\Delta, L^2(\Omega)) \to L^2(\Omega)$ is onto.

By Proposition 3.6, $\Delta : E_0^{1+\delta}(\Delta, L^2(\Omega)) \to L^2(\Omega)$ is an injective operator. Thus $\Delta : E_0^{1+\delta}(\Delta, L^2(\Omega)) \to L^2(\Omega)$ is a bijective bounded operator. So by the open mapping theorem (cf. Theorem 4.12-2, p. 286 of [10]), $\Delta : E_0^{1+\delta}(\Delta, L^2(\Omega)) \to L^2(\Omega)$ is an open map. Hence the result. □

Recall that $y_1(\Omega) \in H_0^1(\Omega)$ is the first eigenfunction of Ω characterized by $y_1(\Omega) > 0$ and $||y_1(\Omega)||_{L^2(\Omega)} = 1$.

PROPOSITION 4.2

Let $L := (\Delta + \lambda_1 Id) : E_0^{1+\delta}(\Delta, L^2(\Omega)) \to L^2(\Omega)$. *Then for* $v \in L^2(\Omega), \exists\ u_0 \in E_0^{1+\delta}(\Delta, L^2(\Omega))$ *such that* $L(u_0) = v$ *if and only if* $\langle v, y_1\rangle_{L^2(\Omega)} = 0$. *Moreover, when* $\langle v, y_1\rangle_{L^2(\Omega)} = 0$ *any other* $u \in E_0^{1+\delta}(\Delta, L^2(\Omega))$ *satisfying* $L(u) = v$ *is of the form* $u_0 + ay_1$ $(a \in \mathbb{R})$.

Proof. Fix $v \in L^2(\Omega)$. Suppose there exists $u_0 \in E_0^{1+\delta}(\Delta, L^2(\Omega))$ such that $L(u_0) = v$. Then we show that $\langle v, y_1\rangle_{L^2(\Omega)} = 0$.

By Proposition 3.5, $\langle \Delta u_0, y_1 \rangle_{L^2(\Omega)} = \langle u_0, \Delta y_1 \rangle_{L^2(\Omega)}$. Therefore

$$\begin{aligned}\langle v, y_1 \rangle_{L^2(\Omega)} &= \langle \Delta u_0 + \lambda_1 u_0, y_1 \rangle_{L^2(\Omega)} \\ &= \langle \Delta u_0, y_1 \rangle_{L^2(\Omega)} + \lambda_1 \langle u_0, y_1 \rangle_{L^2(\Omega)} \\ &= \langle u_0, (\Delta + \lambda_1 Id) y_1 \rangle_{L^2(\Omega)} \\ &= 0.\end{aligned}$$

Conversely, suppose $v \in L^2(\Omega)$ such that $\langle v, y_1 \rangle_{L^2(\Omega)} = 0$. We prove that there exists $u_0 \in E_0^{1+\delta}(\Delta, L^2(\Omega))$ such that $L(u_0) = v$.

Let $i : E_0^{1+\delta}(\Delta, L^2(\Omega)) \to H_0^1(\Omega)$ be an inclusion map. It is a bounded operator. By Rellich compactness theorem (cf. Theorem 1.2.11, p. 5 of [7]), the inclusion map $j : H_0^1(\Omega) \to L^2(\Omega)$ is a compact embedding. Also by Proposition 4.1, $\Delta : E_0^{1+\delta}(\Delta, L^2(\Omega)) \to L^2(\Omega)$ is an invertible bounded operator. Define $A : H_0^1(\Omega) \to H_0^1(\Omega)$ by $A = i \circ \Delta^{-1} \circ j$. Then A is a compact operator.

For all $v, w \in H_0^1(\Omega)$, recall that $\langle v, w \rangle_\sim := \int_\Omega \langle \nabla v, \nabla w \rangle \mathrm{d}x$ and $\|v\|_\sim = \langle v, v \rangle_\sim^{\frac{1}{2}}$. Then $(H_0^1(\Omega), \langle , \rangle_\sim)$ is a Hilbert space as mentioned in the proof of Proposition 3.

Claim. A:$(H_0^1(\Omega), \langle , \rangle_\sim) \to (H_0^1(\Omega), \langle , \rangle_\sim)$ is self-adjoint operator. Let $f_1, f_2 \in H_0^1(\Omega)$ be arbitrary. Since $H_0^1(\Omega) \subseteq L^2(\Omega)$, by Proposition 4.1, $\exists\, u_1, u_2 \in E_0^{1+\delta}(\Delta, L^2(\Omega))$ such that $\Delta u_i = f_i (i = 1, 2)$. Hence $Af_i = u_i \;\; \forall i = 1, 2$. Then

$$\begin{aligned}\langle Af_1, f_2 \rangle_\sim &= \langle u_1, f_2 \rangle_\sim \\ &= \langle \nabla u_1, \nabla f_2 \rangle_{L^2(\Omega)}.\end{aligned}$$

By Proposition 3.4,

$$\begin{aligned}\langle \nabla u_1, \nabla f_2 \rangle_{L^2(\Omega)} &= \langle -\Delta u_1, f_2 \rangle_{L^2(\Omega)} \\ &= -\langle f_1, f_2 \rangle_{L^2(\Omega)}.\end{aligned}$$

Therefore $\langle Af_1, f_2 \rangle_\sim = -\langle f_1, f_2 \rangle_{L^2(\Omega)}$.

Similarly $\langle f_1, Af_2 \rangle_\sim = -\langle f_1, f_2 \rangle_{L^2(\Omega)}$. Hence the claim.

Put $Y := \{u \in H_0^1(\Omega) \mid \langle u, y_1 \rangle_\sim = 0\}$. Now $Ay_1 = -\frac{1}{\lambda_1} y_1$, and A is a compact self-adjoint operator. So $A_{|Y} : Y \to Y$. Also $\ker((A + \frac{1}{\lambda_1} Id)_{|Y}) = 0$. Hence by the Fredholm's theorem (cf. Corollary (0.42), p. 27 of [4]),

$$A + \frac{1}{\lambda_1} Id : Y \to Y \quad \text{is an isomorphism.} \tag{4.2.1}$$

Since $v \in L^2(\Omega)$ by Proposition 4.1, there exists unique $w \in E_0^{1+\delta}(\Delta, L^2(\Omega))$ such that $\Delta w = v$. Now

$$\begin{aligned}\langle w, y_1 \rangle_\sim &= \langle \nabla w, \nabla y_1 \rangle_{L^2(\Omega)} \\ &= -\langle \Delta w, y_1 \rangle_{L^2(\Omega)} \quad \text{(by Proposition 3.4)} \\ &= -\langle v, y_1 \rangle_{L^2(\Omega)} \\ &= 0.\end{aligned}$$

Hence $w \in Y$ and by (4.2.1), there exists a unique $u_0 \in Y$ such that $(A + \frac{1}{\lambda_1} Id)(u_0) = \frac{1}{\lambda_1} w$,

$$\text{i.e. } \frac{1}{\lambda_1} u_0 = -\Delta^{-1} u_0 + \frac{1}{\lambda_1} w. \tag{4.2.2}$$

Now $w \in E_0^{1+\delta}(\Delta, L^2(\Omega))$, and by Proposition 4.1, $\Delta^{-1} u_0 \in E_0^{1+\delta}(\Delta, L^2(\Omega))$. Hence by (4.2.2) $u_0 \in E_0^{1+\delta}(\Delta, L^2(\Omega))$. Now apply operator Δ of Proposition 4.1 to both the sides of (4.2.2) and we get

$$\tfrac{1}{\lambda_1} \Delta u_0 \ = \ -u_0 + \tfrac{1}{\lambda_1} \Delta w,$$

i.e. $(\Delta + \lambda_1 Id) u_0 = v$ and $L(u_0) = v$. Thus given $v \in L^2(\Omega)$ such that $\langle v, y_1 \rangle_{L^2(\Omega)} = 0$ there exists $u_0 \in E_0^{1+\delta}(\Delta, L^2(\Omega))$ such that $L(u_0) = v$.

Finally, if $u \in E_0^{1+\delta}(\Delta, L^2(\Omega))$ is such that $L(u) = v$, then $(u - u_0) \in E_0^{1+\delta}(\Delta, L^2(\Omega))$ and $L(u - u_0) = 0$. Since λ_1 is simple, $u - u_0 = a y_1$, i.e., $u = u_0 + a y_1$. The proof of Theorem is now complete. □

Let $L(\mathbb{R}^2, \mathbb{R}^2)$ be the space of all linear maps from $\mathbb{R}^2$ to $\mathbb{R}^2$. Let $\{A(x)\}_{x \in \mathbb{R}^2}$ be a compactly supported smooth family of linear maps from $\mathbb{R}^2$ to $\mathbb{R}^2$. Fix $A = A(x)$ in the family $\{A(x)\}_{x \in \mathbb{R}^2}$.

Lemma 4.3. *Consider* $\delta_A : (H^2(\Omega) \cap H_0^1(\Omega), \|.\|_{E^{1+\delta}}) \to L^2(\Omega)$ *defined by*

$$\delta_A(u) := -\mathrm{div}(A(\nabla u)), \quad \forall\, u \in (H^2(\Omega) \cap H_0^1(\Omega), \|.\|_{E^{1+\delta}}).$$

Then

(1) $\delta_A : (H^2(\Omega) \cap H_0^1(\Omega), \|.\|_{E^{1+\delta}}) \to L^2(\Omega)$ *is a bounded linear operator.*
(2) δ_A *extends as a bounded operator*, *again called* δ_A,

$$\delta_A : (E_0^{1+\delta}(\Delta, L^2(\Omega)), \|.\|_{E^{1+\delta}}) \to L^2(\Omega).$$

(3) *For* δ_A *defined in* (2), *the following holds*:

$$\int_\Omega \langle A(\nabla u), \nabla \psi \rangle \mathrm{d}x$$
$$= \int_\Omega \delta_A(u) \psi \,\mathrm{d}x \ \ (\psi \in \mathcal{D}(\Omega), \quad u \in E_0^{1+\delta}(\Delta, L^2(\Omega))).$$

Proof. Clearly δ_A is a linear operator. We prove that δ is a bounded operator. Take any $u \in H^2(\Omega) \cap H_0^1(\Omega)$. Let $A(x) = (a_{ij}(x))_{2\times 2}$ $(x \in \mathbb{R}^2)$. Then

$$\delta_A(u) = -\sum_{i,j=1}^{d} \left(\frac{\partial a_{ij}}{\partial x_i} \frac{\partial u}{\partial x_j} + a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} \right).$$

Put $M := \max_{1\le i,j\le d} \max_{x\in D} \left\{ \left| \frac{\partial a_{ij}(x)}{\partial x_i} \right|, |a_{ij}(x)| \right\}$. Then from the above equality we get,

$$||\delta_A(u)||_{L^2(\Omega)} \le 2M||u||_{H^2(\Omega)}. \tag{4.3.1}$$

By Proposition 2.6, there exists a constant C such that

$$||u||_{H^2(\Omega)} \le C||\Delta u||_{L^2(\Omega)}, \ \ \forall \ u \in H^2(\Omega) \cap H_0^1(\Omega). \tag{4.3.2}$$

Then by (4.3.1) and (4.3.2),

$$||\delta_A(u)||_{L^2(\Omega)} \le 2MC||\Delta u||_{L^2(\Omega)} \le 2MC||u||_{E^{1+\delta}}, \ \ \forall \ u \in H^2(\Omega) \cap H_0^1(\Omega).$$

Thus $\delta_A : (H^2(\Omega) \cap H_0^1(\Omega), \|.\|_{E^{1+\delta}}) \to L^2(\Omega)$ is a bounded operator. Hence (1) is proved. Then by Lemma 3.1, δ_A extends as a bounded operator from $E_0^{1+\delta}(\Delta, L^2(\Omega))$ to $L^2(\Omega)$. Thus (2) is proved.

Let $u \in E_0^{1+\delta}(\Delta, L^2(\Omega))$. By Lemma 3.1 there exists a sequence $(u_m)_{m\in\mathbb{N}}$ in $H^2(\Omega) \cap H_0^1(\Omega)$ such that $u_m \to u$ with respect to the norm $||.||_{E^{1+\delta}}$. Hence $\forall \psi \in \mathcal{D}(\Omega)$,

$$\left.\begin{aligned} \lim_{m\to\infty} \int_\Omega \langle A(\nabla u_m), \nabla\psi\rangle \mathrm{d}x &= \int_\Omega \langle A(\nabla u), \nabla\psi\rangle \mathrm{d}x, \\ \lim_{m\to\infty} \int_\Omega \delta_A(u_m)\psi \mathrm{d}x &= \int_\Omega \delta_A(u)\psi \mathrm{d}x. \end{aligned}\right\} \tag{4.3.3}$$

Fix $u_m \in H^2(\Omega) \cap H_0^1(\Omega)$ and $\psi \in \mathcal{D}(\Omega)$. Let $\Omega' \subset \Omega_0$ relatively compact domain with C^∞-boundary and $\mathrm{supp}(\psi) \subset \Omega'$. Now by the Divergence theorem (cf. Theorem 0.4, p. 6 of [4]),

$$\int_{\Omega'} \langle A(\nabla u_m), \nabla\psi\rangle \mathrm{d}x = \int_{\Omega'} \delta_A(u_m)\psi \mathrm{d}x.$$

Then

$$\int_\Omega \langle A(\nabla u_m), \nabla\psi\rangle \mathrm{d}x = \int_\Omega \delta_A(u_m)\psi \mathrm{d}x, \ \ \forall\psi \in \mathcal{D}(\Omega). \tag{4.3.4}$$

Then (3) follows from (4.3.3) and (4.3.4). □

In §1, we have fixed objects $p \in \mathbb{R}^2$, $R > 0$ and a regular polygon $\wp_1$ of n sides in $\mathbb{R}^2$ circumscribed by the circle $C(p, R)$ with center at p and radius R. Let $B(p, s)$ denote the open ball with center at p and radius $s > 0$ in $\mathbb{R}^2$. Also we have fixed a number $r > 0$ such that the circle $C(p, r)$ is contained in the interior of $\wp_1$. Let $\wp_2$ denote the regular polygon of n sides in $\mathbb{R}^2$ circumscribed by the circle $C(p, r)$ such that $\wp_2$ occupies the 'on position' in $\wp_1$, i.e. two vertices of $\wp_2$ and $\wp_1$ lie on the same half-axes of symmetry emanating from p.

We identify $\mathbb{R}^2$ with $\mathbb{C}$. Define $\rho_t(z) = \mathrm{e}^{it}(z - p) + p, \forall\, z \in \mathbb{C} = \mathbb{R}^2$ $(t \in \mathbb{R})$. Note that the map ρ_t fixes p and its total derivative $(D\rho_t)_p$ is a rotation by angle t on the tangent space $T_p\mathbb{R}^2$.

Let $\wp_2^{(t)} = \rho_t(\wp_2)$, $\Omega_t = (\wp_1 \backslash \wp_2^{(t)})^0$ $(t \in \mathbb{R})$, the interior of $\wp_1 \backslash \wp_2^{(t)}$. Throughout this section the family of punctured sets $(\wp_1 \backslash \wp_2^{(t)})^0$ $(t \in \mathbb{R})$ is denoted by $\mathcal{F}_0$. Then the regular polygon $\wp_2^{(\pi/n)}$ occupies the 'off position' in $\wp_1$, i.e. a vertex of $\wp_2$ lies on the half-axes of symmetry emanating from p and passing through the mid-point of a side of $\wp_1$. Since $\rho_{t+\frac{2\pi}{n}}(\wp_2) = \rho_t(\wp_2)$, $\forall\, t \in \mathbb{R}$, we have $\mathcal{F}_0 = \{\Omega_t \mid t \in (-\frac{\pi}{n}, \frac{\pi}{n}]\}$.

The in-radius of $\wp_1$ is defined as $\sup\{s > 0 \mid C(p, s) \subset \wp_1\}$. We denote the in-radius of $\wp_1$ by $\mathrm{inrad}(\wp_1)$. Let $r_1 = \dfrac{r + \mathrm{inrad}(\wp_1)}{2}$. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be a C^∞-function such that

$$\varphi(z) \geq 0,\ \varphi(z) = 1,\ \forall\, z \in B(p, r) \quad \text{and} \quad \varphi(z) = 0,\ \forall\, z \in \mathbb{R}^2 \backslash B(p, r_1).$$

Consider the vector field V of $\mathbb{R}^2$ defined by

$$V(z) = \varphi(z) i (z - p)\ (z \in \mathbb{C}). \tag{4.3.5}$$

Let $\{\psi_t\}_{t \in \mathbb{R}}$ be the 1-parameter group of diffeomorphisms of $\mathbb{R}^2$ associated with the vector field V. Note that $\psi_t(z) = \mathrm{e}^{it}(z - p) + p$, $\forall\, z \in \wp_2$ and hence $\psi_t(\wp_2) = \wp_2^t$. Also $\psi_t(z) = z$, $\forall\, z \in \partial\wp_1$. Then $\Omega_t = \psi_t(\Omega_0)$ holds $\forall\, t \in \mathbb{R}$.

Let $\lambda_1(t)$ be the first eigenvalue of Ω_t and $y_1(t) \in H_0^1(\Omega_t)$ be the first eigenfunction of Ω_t characterized by $y_1(t) > 0$ and $\int_{\Omega_t} y_1^2(t)\mathrm{d}x = 1$, i.e.

$$\left.\begin{aligned} -\Delta y_1(t) &= \lambda_1(t) y_1(t) \quad \text{on } \Omega_t \text{ in the sense of distributions,} \\ y_1(t) &= 0 \qquad\qquad\ \text{on } \partial\Omega_t, \\ y_1(t) &> 0, \\ \textstyle\int_{\Omega_t} y_1^2(t)\mathrm{d}x &= 1. \end{aligned}\right\} \tag{4.3.6}$$

Throughout this section $y_1 = y_1(\Omega_0)$ denotes the first eigenfunction of Ω_0.

Remark 4.4. By the elliptic regularity theorem (cf. Theorem 6.33, p. 214 of [4]) $y_1 \in C^\infty(\Omega_0)$.

We denote $y_1(t) \circ \psi_{t|_{\Omega_0}}$ by y_1^t $(t \in \mathbb{R})$. By Proposition 3, $y_1(t) \in H^{1+\delta}(\Omega_t)$ and hence $y_1^t \in E_0^{1+\delta}(\Delta, L^2(\Omega_0))$, $\forall\, t \in \mathbb{R}$. Following [8], we prove the following result.

PROPOSITION 4.5

The map $t \mapsto (\lambda_1(t), y_1^t)$ *is* C^1*-curve in* $\mathbb{R} \times E_0^{1+\delta}(\Delta, L^2(\Omega_0))$ *from a neighborhood of* 0 *in* $\mathbb{R}$.

Proof. By Proposition 3.4 for all $t \in \mathbb{R}$, $y_1(t)$ satisfies the equation

$$\int_{\Omega_t} \lambda_1(t) y_1(t) \eta \,\mathrm{d}x = \int_{\Omega_t} \langle \nabla y_1(t), \nabla \eta \rangle \mathrm{d}x, \quad \forall \eta \in \mathcal{D}(\Omega_t). \tag{4.5.1}$$

Let $(D\psi_t)_x$ denote the total derivative of the map ψ_t at $x \in \mathbb{R}^2$. Let $\gamma_t : \mathbb{R}^2 \to (0, \infty)$ be the function defined by $\gamma_t(x) = \det(D\psi_t)_x$ $\forall x \in \mathbb{R}^2$. Put

$$B_t = (D\psi_t)^{-1}, \quad B_t^* = \text{ transpose of } B_t$$

(i.e. let $x \in \Omega_t$, $x' = \psi_t^{-1}(x)$ and $\langle B_t(x)v, \omega\rangle = \langle v, B_t^*(x)\omega\rangle$, $\forall\, v \in T_x\Omega_t$ and $\forall\, \omega \in T_{x'}\Omega_0$). Put $A_t = \gamma_t B_t B_t^*$. By the change of variable $\psi_t : \Omega_0 \to \Omega_t$, equation (4.5.1) can be re-written as

$$\int_{\Omega_0} \lambda_1(t) y_1^t \gamma_t \phi \mathrm{d}x = \int_{\Omega_0} \langle A_t \nabla y_1^t, \nabla\phi\rangle \gamma_t \mathrm{d}x, \tag{4.5.2}$$

where $\phi = \eta \circ \psi_t$. By Lemma 4.3, equation (4.5.2) can be re-written as

$$\int_{\Omega_0} \lambda_1(t) y_1^t \gamma_t \phi \mathrm{d}x = \int_{\Omega_0} \delta_{A_t}(y_1^t)\phi \mathrm{d}x. \tag{4.5.3}$$

Therefore $t \mapsto (\lambda_1(t), y_1^t)$ satisfies

$$\left.\begin{aligned} -\delta_{A_t}(y_1^t) + \lambda_1(t)\gamma_t y_1^t &= 0 \quad \text{in } L^2(\Omega_0),\\ y^t_{1|\partial\Omega_0} &= 0 \quad \text{in } H^{\frac{1}{2}+\delta}(\partial\Omega_0),\\ \textstyle\int_{\Omega_0} (y_1^t)^2 \gamma_t \mathrm{d}x &= 1. \end{aligned}\right\} \tag{4.5.4}$$

Put $X := \mathbb{R} \times E_0^{1+\delta}(\Delta, L^2(\Omega_0))$. Define $F : \mathbb{R} \times X \to L^2(\Omega_0) \times \mathbb{R}$ by

$$F(t, \mu, u) = \left(-\delta_{A_t}(u) + \mu\gamma_t u, 1 - \int_{\Omega_0} u^2 \gamma_t \mathrm{d}x\right).$$

Then F is a C^1-map. Further, since $A_0 = Id$ and $\gamma_0 = 1$,

$$\begin{aligned} (D_2F)_{(0,\lambda_1,y_1)}(0, \mu, u) &= (-\delta_{A_0}(u) + \lambda_1 u + \mu y_1, -2 \textstyle\int_{\Omega_0} y_1 u \mathrm{d}x)\\ &= (\Delta u + \lambda_1 u + \mu y_1, -2 \textstyle\int_{\Omega_0} y_1 u \mathrm{d}x). \end{aligned}$$

Claim. $(D_2F)_{(0,\lambda_1,y_1)} : X \to L^2(\Omega_0) \times \mathbb{R}$ is an isomorphism. Let $(v, b) \in L^2(\Omega_0) \times \mathbb{R}$ be arbitrary. Consider the following problem:

$$\left.\begin{aligned} \Delta u + \lambda_1 u + \mu y_1 &= v \quad \text{on } \Omega_0,\\ 2 \textstyle\int_{\Omega_0} y_1 u \mathrm{d}x &= b \end{aligned}\right\} \tag{4.5.5}$$

By Proposition 4.2, $\Delta u + \lambda_1 u = v - \mu y_1$ has a solution in $E_0^{1+\delta}(\Delta, L^2(\Omega_0))$ if and only if $(v - \mu y_1) \perp y_1$ in $L^2(\Omega_0)$. So, applying Proposition 4.2 for $\mu_0 := \int_{\Omega_0} v y_1 \mathrm{d}x$, there exists a unique $u_1 \in E_0^{1+\delta}(\Delta, L^2(\Omega_0))$ such that $\Delta u_1 + \lambda_1 u_1 + \mu_0 y_1 = v$. Given $b \in \mathbb{R}$, let $a_0 := \frac{b}{2} - \int_{\Omega_0} y_1 u_1 \mathrm{d}x$ and let $u_0 = u_1 + a_0 y_1$ Then $2\int_\Omega y_1 u_0 = b$. Thus for $(v, b) \in L^2(\Omega_0) \times \mathbb{R}$ there exists unique $(\mu_0, u_0) \in \mathbb{R} \times E_0^{1+\delta}(\Delta, L^2(\Omega_0))$ such that $(D_2F)_{(0,\lambda_1,y_1)}(0, \mu_0, u_0) = (v, b)$. This proves the claim.

By (4.5.4), $F(t, \lambda_1(t), y_1^t) = 0\, \forall t$. Proposition 4.5 now follows by the claim above and the implicit function theorem. □

DEFINITION 4.6

$\dot{y}_1(\Omega_0, V) := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} y_1^t \in E_0^{1+\delta}(\Delta, L^2(\Omega_0))$ is called the material derivative of y_1 in the direction of V.

Lemma 4.7. *Let $t \mapsto f(t)$ be a differentiable curve in $L^2(\Omega_0)$ with $f(0) \in H^1(\Omega_0)$. Let $\Omega' \subset \Omega_0$ be an open set with C^∞-boundary such that $\overline{\Omega'} \subseteq \Omega_0$. Then for t sufficiently close to 0, $t \mapsto f(t) \circ \phi_t$ is differentiable in $L^2(\Omega')$ at $t = 0$ and*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (f(t) \circ \phi_t) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(t) + \langle \nabla f(t), v\rangle \ \ in \ \ L^2(\Omega').$$

By Proposition 4.5 and Lemma 4.7, we have following corollary.

COROLLARY 4.8

Let $\Omega' \subset \Omega_0$ be an open set with C^∞-boundary such that $\overline{\Omega'} \subseteq \Omega_0$. Then the map $t \mapsto y_1(t)_{|\Omega'}$ is a C^1-curve in $L^2(\Omega')$ from a neighborhood of 0 in $\mathbb{R}$ and

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(y_1(t)_{|\Omega'}\right) = (\dot{y}_1 - \langle \nabla y_1, V\rangle)_{|\Omega'} \ \ in \ \ L^2(\Omega').$$

DEFINITION 4.9

The shape derivative y_1' of y_1 in the direction of V is defined by

$$y_1' = \dot{y}_1 - \langle \nabla y_1, V\rangle.$$

Remark 4.10. Note that $y_1' \in H^\delta(\Omega_0)$. By Remark 4.4, $y_1 \in C^\infty(\Omega_0)$. By Proposition 4.8, $\dot{y}_1 \in H^{1+\delta}(\Omega_0)$. So for any open set $\Omega' \subset \Omega_0$ with C^∞-boundary such that $\overline{\Omega'} \subseteq \Omega_0$,

$$\dot{y}_1 - \langle \nabla y_1, V\rangle \in H^{1+\delta}(\Omega'), \quad \text{i.e. } y'_{1|_{\Omega'}} \in H^{1+\delta}(\Omega').$$

PROPOSITION 4.11

Let n be the outward unit normal field of Ω_0 on $\partial\Omega_0$. Then

$$y'_{1|_{\partial\Omega_0}} = -\frac{\partial y_1}{\partial n} \ \langle V, n\rangle.$$

Proof. $y_1' \in H^\delta(\Omega_0)$ and $\delta > \frac{1}{2}$. By Proposition 2.4, $y'_{1|_{\partial\Omega_0}} \in H^{\delta-\frac{1}{2}}(\partial\Omega_0) \subseteq L^2(\partial\Omega_0)$. Then

$$\begin{aligned} y'_{1|_{\partial\Omega_0}} &= \dot{y}_{1\,|_{\partial\Omega_0}} - (\langle \nabla y_1, V\rangle)_{|\partial\Omega_0} \\ &= -(\langle \nabla y_1, V\rangle)_{|\partial\Omega_0} \qquad (\text{since } \ \dot{y}_{1\,|_{\partial\Omega_0}} = 0) \\ &= -\tfrac{\partial y_1}{\partial n} \ \langle V, n\rangle \qquad (\text{since } \ y_1 \in H^\delta(\Omega_0)). \end{aligned}$$

□

Notation 4.12. $\lambda_1' := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \lambda_1(t)$.

PROPOSITION 4.13

The shape derivative y_1' satisfies the non homogeneous elliptic equation $\Delta y_1' = -(\lambda_1 y_1' + \lambda_1' y_1)$ in the sense of distributions on Ω_0.

Proof. Fix $\psi \in \mathcal{D}(\Omega_0)$ arbitrary. Let $\Omega' \subset \Omega_0$ be an open set with C^∞-boundary such that $\overline{\Omega'} \subseteq \Omega_0$ and $\mathrm{supp}(\psi) \subset \Omega'$. Then there exists $a > 0$ such that $\Omega' \subset \Omega_t \ \forall \ |t| < a$. By Remark 4.4, $y_1(t)_{|\Omega'} \in C^\infty(\Omega_0)$. Hence

$$\int_{\Omega'} (\Delta y_1(t)\psi - y_1(t)\Delta\psi)\mathrm{d}x = \int_{\partial\Omega'} \left(\psi_{|\partial\Omega'} \frac{\partial y_1(t)}{\partial n} - y_1(t)_{|\partial\Omega'} \frac{\partial \psi}{\partial n}\right)\mathrm{d}\sigma.$$

Since $\mathrm{supp}(\psi) \subset \Omega'$ and $\Delta y_1(t) = -\lambda_1(t) y_1(t)$ in Ω, the above equation reduces to

$$-\int_{\Omega'} (\lambda_1(t) y_1(t))\mathrm{d}x = \int_{\Omega'} (y_1(t)\Delta\psi)\mathrm{d}x, \quad \forall \ |t| < \delta.$$

Hence

$$-\int_{\Omega'} \left(\frac{\lambda_1(t) y_1(t) - \lambda_1 y_1}{t}\right) \psi \mathrm{d}x = \int_{\Omega'} \left(\frac{y_1(t) - y_1}{t}\right) \Delta\psi \mathrm{d}x, \quad \forall \ |t| < \delta. \tag{4.13.1}$$

Then by Corollary 4.8, it follows that

$$\left.\begin{aligned} &\lim_{t\to 0} \int_{\Omega'} \left(\frac{\lambda_1(t) y_1(t) - \lambda_1 y_1}{t}\right) \psi \mathrm{d}x = \int_{\Omega'} (\lambda_1 y_1' + \lambda_1' y_1)\psi \mathrm{d}x,\\ &\lim_{t\to 0} \int_{\Omega'} \left(\frac{y_1(t) - y_1}{t}\right) \Delta\psi \mathrm{d}x = \int_{\Omega'} y_1' \Delta\psi \mathrm{d}x. \end{aligned}\right\} \tag{4.13.2}$$

Now from (4.13.1) and (4.13.2) we get

$$-\int_{\Omega'} (\lambda_1 y_1' + \lambda_1' y_1)\psi \mathrm{d}x = \int_{\Omega'} y_1' \Delta\psi \mathrm{d}x.$$

Then $\mathrm{supp}(\psi) \subseteq \Omega'$ implies that

$$-\int_{\Omega_0} (\lambda_1 y_1' + \lambda_1' y_1)\psi \mathrm{d}x = \int_{\Omega_0} y_1' \Delta\psi \mathrm{d}x.$$

Since ψ is arbitrary, $\Delta y_1' = -(\lambda_1 y_1' + \lambda_1' y_1)$ in the sense of distributions on Ω_0. □

COROLLARY 4.14

$\Delta y_1' \in H^\delta(\Omega_0) \subseteq L^2(\Omega_0)$.

Proof. $\lambda_1 y_1' + \lambda_1' y_1 \in L^2(\Omega_0)$. By Remark 4.10, $y_1' \in H^\delta(\Omega_0)$ and hence $\lambda_1 y_1' + \lambda_1' y_1 \in H^\delta(\Omega_0)$. Then by Proposition 4.13, we get $\Delta y_1' \in H^\delta(\Omega_0)$. □

COROLLARY 4.15

$y_1' \in E^\delta(\Delta, L^2(\Omega_0))$.

PROPOSITION 4.16 (Hadamard formula)

$$\lambda_1' = -\int_{\partial\Omega_0} \left(\frac{\partial y_1}{\partial n}\right)^2 \langle V, n\rangle \mathrm{d}s.$$

Proof. $y_1 \in H_0^{1+\delta}(\Omega_0)$ and $\delta > \frac{1}{2}$. By Proposition 2.4, $\frac{\partial y_1}{\partial n} \in H^{\delta - \frac{1}{2}}(\Omega_0) \subseteq L^2(\partial\Omega_0)$. Also $\langle V, n\rangle \in L^\infty(\partial\Omega_0)$. So $\left(\frac{\partial y_1}{\partial n}\right)^2 \langle V, n\rangle \in L^1(\partial\Omega_0)$. By Proposition 4.13,

$$\lambda_1' = \lambda_1' \int_{\Omega_0} y_1^2 \mathrm{d}x = -\int_{\Omega_0} (y_1 \Delta y_1' + \lambda_1 y_1 y_1')\mathrm{d}x,$$

i.e.

$$\lambda_1' = -\int_{\Omega_0} (y_1 \Delta y_1' - y_1' \Delta y_1)\mathrm{d}x. \tag{4.16.1}$$

Recall $y_1 \in E_0^{1+\delta}(\Delta, L^2(\Omega_0))$, $y_1' \in E^\delta(\Delta, L^2(\Omega_0))$. Then by Proposition 3.5 and (4.16.1), we get

$$\lambda_1' = \int_{\partial\Omega_0} y_1'|_{\partial\Omega_0} \frac{\partial y_1}{\partial n} \mathrm{d}s. \tag{4.16.2}$$

By Proposition 4.11, $y_1'|_{\partial\Omega_0} = -\frac{\partial y_1}{\partial n} \langle V, n\rangle$. So by (4.16.2) we get the Hadamard formula. □

5. Proof of Theorem 1.2

In §1, we have fixed objects $p \in M^2(\kappa)$, $R = R(\kappa) > 0$ and a regular polygon $\wp_1$ of n sides in $M^2(\kappa)$ circumscribed by the distance circle $C(p, R)$ with center at p and radius R. Recall that $R(\kappa) > 0$ is an arbitrary constant for $\kappa = -1, 0$. And for $\kappa = 1$, $R(\kappa)$ is any arbitrary constant with $0 < R(\kappa) < \pi/2$. Also we have fixed a number $r = r(\kappa, R) > 0$ such that the circle $C(p, r)$ is contained in the interior of $\wp_1$. Let $\wp_2$ denote the regular polygon of n sides in $M^2(\kappa)$ circumscribed by the circle $C(p, r)$ such that $\wp_2$ occupies the 'on position' in $\wp_1$, i.e. two vertices of $\wp_2$ and $\wp_1$ lie on the same half-axes of symmetry emanating from p.

Every open set of $\mathbb{R}^d$ has a canonical orientation associated with natural basis of $\mathbb{R}^d$. Also S^2, being the boundary of the unit ball in $\mathbb{R}^3$, is oriented by the boundary orientation. Thus for any $\theta \in \mathbb{R}$, the rotation R_θ by angle θ is defined on the tangent space $T_p M^2(\kappa)$. There exists a unique 1-parameter group $\{\rho_t\}_{t\in\mathbb{R}}$ of orientation preserving isometries of $M^2(\kappa)$ such that $\rho_t(p) = p$, and the total derivative $(D\rho_t)_p$ is the rotation by angle t on the tangent space $T_p M^2(\kappa)$. Here $(D\rho_t)_p$ denotes the Frechet derivative of the map ρ_t at $p \in M^2(\kappa)$.

Let $\wp_2^{(t)} = \rho_t(\wp_2)$, $\Omega_t = (\wp_1 \backslash \wp_2^{(t)})^0$ $(t \in \mathbb{R})$, the interior of $\wp_1 \backslash \wp_2^{(t)}$ in $M^2(\kappa)$. Throughout this section $\mathcal{F}_\kappa = \mathcal{F}(\kappa, R, r)$ denotes the family of punctured sets $(\wp_1 \backslash \wp_2^{(t)})^0$ $(t \in \mathbb{R})$. The regular polygon $\wp_2^{(\pi/n)}$ occupies the 'off position' in $\wp_1$, i.e. a vertex of $\wp_2^{(\pi/n)}$ lies on the half-axes of symmetry emanating from p and passing through the mid-point of a side of $\wp_1$. Since $\rho_{t+\frac{2\pi}{n}}(\wp_2) = \rho_t(\wp_2)$, $\forall\, t \in \mathbb{R}$, we have $\mathcal{F}_\kappa = \{\Omega_t \mid t \in (-\frac{\pi}{n}, \frac{\pi}{n}]\}$.

Let $\lambda_1(t)$ be the first eigenvalue of Ω_t and $y_1(t) \in H_0^1(\Omega_t)$ be the first eigenfunction of Ω_t characterized by $y_1(t) > 0$ and $\int_{\Omega_t} y_1^2(t)\, \mathrm{d}x = 1$ i.e.,

$$\left.\begin{aligned} -\Delta y_1(t) &= \lambda_1(t) y_1(t) \text{ on } \Omega_t \text{ in the sense of distributions,}\\ y_1(t) &= 0 \quad \text{on } \partial\Omega_t,\\ y_1(t) &> 0,\\ \textstyle\int_{\Omega_t} y_1^2(t)\mathrm{d}x &= 1. \end{aligned}\right\} \tag{5.0.1}$$

Since $\rho_{t+\frac{2\pi}{n}}(\wp_i) = \rho_t(\wp_i)$ $(i = 1, 2;\ t \in \mathbb{R})$, Ω_t is congruent to $\Omega_{t+\frac{2\pi}{n}}$ and hence $\lambda_1(t)$ is a $(2\pi/n)$-periodic function. Let $\sigma : M^2(\kappa) \to M^2(\kappa)$ be the reflection map about any fixed axis of symmetry of $\wp_1$. Then $\sigma \circ \rho_t \circ \sigma = \rho_{-t} (t \in \mathbb{R})$ and $\sigma(\Omega_0) = \Omega_0$. Hence $\sigma(\Omega_t) = \sigma \circ \rho_t \circ \sigma(\Omega_0) = \rho_{-t}(\Omega_0) = \Omega_{-t} (t \in \mathbb{R})$. So Ω_t is congruent to Ω_{-t} and hence $\lambda_1(t) = \lambda_1(-t)\ \forall\, t \in \mathbb{R}$. Thus $\lambda_1(t)$ is an even function and we get $\lambda_1'(0) = 0$. Also $\forall t \in \mathbb{R}$, $\lambda_1(\frac{\pi}{n} + t) = \lambda_1(-\frac{\pi}{n} - t + \frac{2\pi}{n}) = \lambda_1(\frac{\pi}{n} - t)$ and hence $\lambda_1'(\pi/n) = 0$. We prove that $\lambda_1'(t) < 0$, $\forall\ 0 < t < \pi/n$. It follows that, on the interval $(-\frac{\pi}{n}, \frac{\pi}{n}]$ $\lambda_1(t)$ attains maximum when $t = 0$ and minimum when $t = \pi/n$. Then Theorem 1.2 follows.

The in-radius of $\wp_1$ is defined as the $\sup\{s \in \mathbb{R} \mid C(p, s) \subset \wp_1\}$ and is denoted by $\mathrm{inrad}(\wp_1)$. Let $r_1 = (r + \mathrm{inrad}(\wp_1))/2$. Let $B(p, s)$ denote the open ball of radius $s > 0$ in $M^2(\kappa)$. Let $\varphi : M^2(\kappa) \to \mathbb{R}$ be a C^∞ function such that

$$\varphi(x) \geq 0,\ \varphi(x) = 1,\ \forall\, x \in B(p, r) \quad \text{and} \quad \varphi(x) = 0,\ \forall\, x \in M^2(\kappa) \setminus B(p, r_1).$$

Consider the vector field v of $M^2(\kappa)$ defined by

$$v(x) = \varphi(x) \left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \rho_t(x) \right) \ (x \in M^2(\kappa)). \tag{5.0.2}$$

Let $\{\psi_t\}_{t \in \mathbb{R}}$ be the 1-parameter group of diffeomorphisms of $M^2(\kappa)$ associated with the vector field v. Note that

$$\psi_t(x) = \rho_t(x),\ \forall\, x \in \wp_2 \quad \text{and} \quad \psi_t(x) = x,\ \forall\, x \in \partial\wp_1.$$

Hence $\psi_t(\Omega_0) = \Omega_t,\ \forall\, t \in \mathbb{R}$.

Since $v(x) = 0$, $\forall\, x \in \partial\wp_1$, by the Hadamard formula (Proposition 4.16) we get

$$\lambda_1'(t) = -\int_{\partial\wp_2^{(t)}} \left(\frac{\partial y_1(t)}{\partial n_t} \right)^2 g(n_t, v)\mathrm{d}s,$$

where n_t denotes the outward unit normal field of Ω_t defined at its smooth points of $\partial\Omega_t$.

Fix any side $[b, c]$ of the polygon $\wp_2$ (see Appendix B). Put $b_t := \rho_t(b)$ and $c_t := \rho_t(c)$. Then $[b_t, c_t]$ is a side of the polygon $\wp_2^t$. Since the rotation map $\rho_{\frac{2\pi}{n}}$ is an orientation preserving isometry of Ω_t, $n_t(x) = n_t(\rho_{\frac{2\pi}{n}}(x))$, $\forall\ x \in \partial\wp_2^{(t)}$. Also by the construction of v, $(D\rho_t)_x(v(x)) = v(\rho_t(x))$, $\forall\ t \in \mathbb{R}$, $\forall\ x \in \partial\wp_2^{(t)}$. It follows that the functions $\frac{\partial y_1(t)}{\partial n_t}$, $g(n_t, v)$ are invariant under the rotation map $\rho_{\frac{2\pi}{n}}$ and hence

$$\lambda_1'(t) = -n \int_{(b_t, c_t)} \left(\frac{\partial y_1(t)}{\partial n_t} \right)^2 g(n_t, v)\mathrm{d}s,$$

where $(b_t, c_t) := [b_t, c_t] \setminus \{b_t, c_t\}$ and n is the number of sides of the polygon $\wp_1$.

Let m be the mid-point of the side $[b, c]$ of $\wp_2$ and let $\sigma : M^2(\kappa) \to M^2(\kappa)$ be the reflection map about the axis of symmetry of $\wp_2$ passing through m. Then $m_t = \rho_t(m)$ is the mid-point of side $[b_t, c_t]$ of $\wp_2^{(t)}$. Since the reflection map σ is a symmetry of both $\wp_1$ and $\wp_2$, it follows that $\sigma_t := \rho_t \circ \sigma \circ \rho_{-t}$ is a symmetry of Ω_t. Note that $[m_t, c_t] = \sigma_t([b_t, m_t])$. Put $x^* = \sigma_t(x)$ $(x \in M^2(\kappa))$. Then for any $x \in (b_t, m_t) \subseteq (b_t, c_t)$,

$$n_t(x^*) = D\sigma_t(n_t(x)) \quad \text{and} \quad v(x^*) = -D\sigma_t(v(x)).$$

Hence

$$g(n_t(x^*), v(x^*)) = g\left(D\sigma_t(n_t(x)), -D\sigma_t(v(x))\right) = -g\left(n_t(x), v(x)\right).$$

Thus

$$\lambda_1'(t) = -n \int_{(b_t, m_t)} \left\{ \left| \frac{\partial y_1(t)}{\partial n_t}(x) \right|^2 - \left| \frac{\partial y_1(t)}{\partial n_t}(x^*) \right|^2 \right\} g(n_t(x), v(x)) \mathrm{d}s. \tag{5.0.3}$$

We prove later (see Lemma A.1 in Appendix A) that

$$g\left(n_t(x), v(x)\right) < 0, \quad \forall x \in (b_t, m_t). \tag{5.0.4}$$

So we only need to compare the absolute values of the normal derivatives of $y_1(t)$ at x and x^*.

Since $y_1(t)$ satisfies (5.0.1),

$$\begin{cases} \Delta(-y_1(t)) \geq 0 \text{ on } \Omega_t, \\ -y_1(t) < 0 \text{ on } \Omega_t, \\ -y_1(t) = 0 \text{ on } \partial\Omega_t. \end{cases}$$

By Proposition 2.8, $\frac{\partial(-y_1(t))}{\partial n_t}(x) > 0$ for all smooth points $x \in \partial\Omega_t$. In particular,

$$\frac{\partial y_1(t)}{\partial n_t}(x) < 0, \quad \forall\, x \in (b_t, m_t). \tag{5.0.5}$$

Fix $0 < t < \frac{\pi}{n}$. Let $a(\kappa) = \pi/2$ for $\kappa = 1$ and $a(\kappa) = \infty$ for $\kappa = -1, 0$. For $1 \leq j \leq 3$, let $\gamma_j : [0, a_\kappa) \to M^2(\kappa)$ be the unit speed geodesics emanating from p such that $\gamma_1, \gamma_2, \gamma_3$ pass through b_t, m_t, c_t respectively. Let γ_j intersect $\partial\wp_1$ at p_1, p_2, p_3 for $j = 1, 2, 3$ respectively. Then $\sigma_t \circ \gamma_1 = \gamma_3$ and hence $\sigma_t(p_1) = p_3$ and $\sigma_t([p, p_2, p_1]) = ([p, p_2, p_3])$. Let $\mathcal{O}_t = [p, p_1, p_2] \cap \Omega_t$ and $\mathcal{O}_t^* = \sigma_t(\mathcal{O}_t)$. Then $\mathcal{O}_t$ and $\mathcal{O}_t^*$ are the polygons with 4 sides having vertices m_t, b_t, p_1, p_2 and m_t, c_t, p_3, p_2 respectively. See figure 3 for the case of $n = 3$ and $\kappa = -1$.

Define $\omega : \mathcal{O}_t \to \mathbb{R}$ by $\omega(x) = y_1(t)(x) - y_1(t)(x^*), \ \forall\, x \in \mathcal{O}_t$. Since the rotation map $\rho_{\frac{2\pi}{n}}$ is a symmetry of Ω_t, $y_1(t)(x) = y_1(t)(x^*), \ \forall\, x \in [b_t, p_1]$ and hence $\omega = 0$ on $[b_t, p_1]$. Further ω satisfies

$$\left.\begin{aligned} \Delta\omega &= -\lambda_1(t)\omega \text{ on } \mathcal{O}_t \\ \omega &\leq 0 \quad\quad\quad \text{ on } \partial\mathcal{O}_t. \end{aligned}\right\} \tag{5.0.6}$$

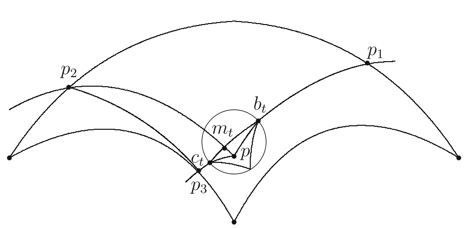

Figure 3. $\mathcal{O}_t$ and $\mathcal{O}_t^*$ for the case of $n = 3$ and $\kappa = -1$.

Put $\omega^+(x) = \max\{\omega(x), 0\}$, $\forall\, x \in \mathcal{O}_t$ and $\mathcal{O}_t^+ = \{x \in \mathcal{O}_t \mid \omega(x) \geq 0\}$. Then $\omega^+ \in H_0^1(\mathcal{O}_t)$ and by (5.4)

$$\begin{aligned} 0 &= \int_{\mathcal{O}_t} (-\Delta\omega)\omega^+ \mathrm{d}A - \lambda_1(t) \int_{\mathcal{O}_t} \omega\omega^+ \mathrm{d}A \\ &= \int_{\mathcal{O}_t^+} (-\Delta\omega)\omega^+ \mathrm{d}A - \lambda_1(t) \int_{\mathcal{O}_t^+} \omega\omega^+ \mathrm{d}A \\ &= \int_{\mathcal{O}_t^+} (-\Delta\omega^+)\omega^+ \mathrm{d}A - \lambda_1(t) \int_{\mathcal{O}_t^+} (\omega^+)^2 \mathrm{d}A \\ &= \int_{\mathcal{O}_t} |\nabla\omega^+|^2 \mathrm{d}A - \lambda_1(t) \int_{\mathcal{O}_t} (\omega^+)^2 \mathrm{d}A. \end{aligned}$$

Now $\mathcal{O}_t$ is a proper open subset of Ω_t and hence by the variation principle for eigenvalues, $\lambda_1(\mathcal{O}_t) > \lambda_1(\Omega_t)$. So if $\omega^+ \neq 0$, then

$$\lambda_1(\mathcal{O}_t) > \frac{\int_{\mathcal{O}_t} |\nabla\omega^+|^2 \mathrm{d}A}{\int_{\mathcal{O}_t} (\omega^+)^2 \mathrm{d}A}$$

which is not possible by (1.1.2). So $\omega < 0$ on $\mathcal{O}_t$, and from (5.0.6) it follows that $\Delta\omega \geq 0$ on $\mathcal{O}_t$. Since $\omega = 0$ on (b_t, m_t), by Proposition 2.8, $\frac{\partial\omega}{\partial n_t}(x) > 0$, $\forall\, x \in (b_t, m_t)$. i.e.,

$$\frac{\partial y_1(t)}{\partial n_t}(x^*) < \frac{\partial y_1(t)}{\partial n_t}(x), \quad \forall\, x \in (b_t, m_t) \text{ and } t \in \left(0, \frac{\pi}{n}\right). \tag{5.0.7}$$

Thus from (5.0.3), (5.0.4), (5.0.5) and (5.0.7), $\lambda_1'(t) < 0$, $\forall\, t \in (0, \pi/n)$. Hence $\lambda_1(t)$ attains a maximum when $t = 0$ which corresponds to the 'on position' of $\wp_0$ (i.e. $\wp_2^0$) in $\wp_1$ and $\lambda_1(t)$ attains minimum when $t = \pi/n$ which corresponds to the 'off position' of $\wp_0$ (i.e. $\wp_2^{\pi/n}$)in $\wp_1$. This proves the theorem. □

Appendix

Appendix A

The object of this appendix is to prove the inequality (5.0.4) in the proof of Theorem 1.2. The proof of this inequality is technical and the proof is given case by case. Also the proof is not obvious unless the point $p \in M^2(\kappa)$ and the side $[b, c]$ of $\wp_2$ are chosen carefully.

We continue with the notations of §5.

Lemma A.1. *Let m_t be the mid-point of the side $[b_t, c_t]$ of the polygon $\wp_2^t$ and n_t denote the outward unit normal of Ω_t defined at the smooth points of its boundary. Let v denote the vector field of $M^2(\kappa)$ described in* (4.3.5). *Then $g\,(n_t(x), v(x)) < 0$, $\forall\, x \in (b_t, m_t)$, $\forall\, t \in \mathbb{R}$.*

Proof. Put $x' = \rho_{-t}(x)$ $\forall\, x \in (b_t, m_t)$. The rotation map ρ_t is an orientation preserving isomentry of $M^2(\kappa)$ and $\rho_t(\Omega_0) = \Omega_t$. Also $D\rho_t(v(x')) = v(\rho_t(x))$ $\forall x' \in \wp_2$. So $\forall\, x \in (b_t, m_t)$,

$$g\,(n_t(x), v(x)) = g(D\rho_t(n_0(x'), D\rho_t(v(x')) = g(n_0(x'), v(x')).$$

Hence it suffices to prove that $g\,(n_0(x), v(x)) < 0\;\forall\, x \in (b, m)$.

Fix $k \in \mathbb{N}$ and let $\{x_i, y_i \mid 1 \leq i \leq k\} \subset M^2(\kappa)$ such that $d_\kappa(x_i, x_j) = d_\kappa(y_i, y_j)$, $\forall\, 1 \leq i, j \leq k$. Then $\exists$ an isometry f of $M^2(\kappa)$ such that $f(x_i) = y_i$, $\forall\, 1 \leq i \leq k$. Hence we can choose $p, b, c \in M^2(\kappa)$ so that the computations become simpler.

Case (0): $\kappa = 0$. For simplicity we assume that $p = (0, 0)$, $b = \mathrm{e}^{-\pi/n}$, $m = (1, 0)$, $c = \mathrm{e}^{\pi/n}$. It follows that, $\forall\, t \in \mathbb{R}$ and $\forall\, z \in \mathbb{R}^2$ we get

$$\begin{aligned} \rho_t(z) &= \mathrm{e}^{it} z, \\ \frac{\mathrm{d}}{\mathrm{d}t}\rho_t(z)\Big|_{t=0} &= iz, \\ v(z) &= \phi(z) iz. \end{aligned}$$

Now for all $q = (x, y) \in (b, m)$, $y < 0$ and $n_0(q) = (-1, 0)$. Hence $g(n_0(q), v(q)) = y < 0$. Thus we have proved the lemma for the case $\kappa = 0$.

Case (1): $\kappa = 1$. For simplicity we assume that

$$\begin{aligned} p &= (0, 0, 1), \\ b &= (\sin(\pi/n)\sin r, \cos(\pi/n)\sin r, \cos r), \\ c &= (-\sin(\pi/n)\sin(r), \cos(\pi/n)\sin r, \cos r). \end{aligned}$$

It follows that, $\forall\, t \in \mathbb{R}$ and $\forall\, (x_1, x_2, x_3) \in S^2$,

$$\begin{aligned} \rho_t(x_1, x_2, x_3) &= (x_1\cos(t) - x_2\sin(t), x_1\sin(t) + x_2\cos(t), x_3), \\ v(x_1, x_2, x_3) &= \varphi(x_1, x_2, x_3)(-x_2, x_1, 0). \end{aligned}$$

Let $0 \le s \le d(b, c)$ and put

$$c' = \frac{c - g(c, b)b}{\|c - g(c, b)b\|}$$

and define $\gamma : [0, d(b, c)] \to S^2$ by $\gamma(s) = \cos(s)b + \sin(s)\, c'$. Then γ is a parametrization of the side $[b, c]$ of the polygon $\wp_2$. Since $\gamma(s) \times \gamma'(s) = b \times c'$, we conclude that $b \times c'$ is perpendicular to $\gamma'(s)$ in $T_{\gamma(s)}S^2$. Now

$$\begin{aligned} b \times c' &= \tfrac{b\times c}{\|b\times c\|} \\ &= \tfrac{1}{\|b\times c\|}(0,\ -\sin(\pi/n)\sin(2r),\ \sin(2\pi/n)\sin^2(r)\,) \end{aligned}$$

and $g(b \times c', p) = \frac{\sin(2\pi/n)\sin^2(r)}{\|b\times c\|} > 0$. Hence the outward unit normal vector $n_0(q)$ of Ω_0 at any $q \in (b, c)$ is indeed $b \times c'$. Thus for all $q = (x_1, x_2, x_3) \in (b, m)$, $x_1 > 0$ and

$$g(n_0(q), v(q)) = \frac{-x_1\sin(\pi/n)\sin(2r)}{\|b \times c\|} < 0.$$

Thus we have proved the lemma for the case of $\kappa = 1$.

Case (2): $\kappa = -1$. We assume that $p = i$. Then $\forall\, z \in \mathbb{H}^2,\ t \in \mathbb{R},\ w \in T_z\mathbb{H}^2$,

$$\rho_t(z) = \frac{\cos(t/2)\, z + \sin(t/2)}{-\sin(t/2)\, z + \cos(t/2)} \quad \text{and} \quad D\rho_t(w) = \mathrm{e}^{it} w.$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \rho_t(z) = \frac{1 + z^2}{2}, \quad \forall\, z \in \mathbb{H}^2,$$

we get

$$v(z) = \varphi(z)\frac{1 + z^2}{2}, \quad \forall\, z \in \Omega_t.$$

We assume that

$$b = \rho_{-\pi/n}(i\mathrm{e}^r) \quad \text{and} \quad c = \rho_{\pi/n}(i\mathrm{e}^r).$$

Then the side $[b, c]$ of the polygon $\wp_2$ is an arc of the Euclidean circle which has center at $(0, 0)$ and which passes through b, c. Hence the outward unit normal $n_0(z)$ of Ω_0 at any $z \in (b, c)$ is

$$\frac{-\mathrm{Im}^2(z)\, z}{|z|^2}.$$

Thus $\forall\, z \in (b, m)$, we have $\mathrm{Re}(z) > 0$ and

$$g(n_0(z), v(z)) = -\frac{(1 + |z|^2)\mathrm{Re}(z)}{2|z|^2} < 0.$$

Thus the lemma is proved for the case of $\kappa = -1$. □

Appendix B: Basic notions about a regular polygon

Let $D(\kappa)$ denote the domain in $M^2(\kappa)$ such that $D(\kappa) = \mathbb{H}^2,\ \mathbb{R}^2$ for $\kappa = -1, 0$ respectively and $D(\kappa) = \{q \in S^2 \mid d_\kappa(p, q) < \pi/2\}$ for $\kappa = 1$. Then $D(\kappa)$ is a strictly convex domain in $M^2(\kappa)$.

Let q_1, q_2 be any two distinct points in $D(\kappa)$ and let $l = d_\kappa(q_1, q_2)$. Let $\gamma : [0, l] \to D(\kappa)$ be the unique unit speed geodesic such that $\gamma(0) = q_1$ and $\gamma(l) = q_2$. We denote the trace of γ by $[q_1, q_2]$. Thus $[q_1, q_2]$ is the *geodesic segment* joining q_1 and q_2 in $D(\kappa)$.

A *polygon* $\wp$ in $D(\kappa)$ is a simply connected closed region whose boundary $\partial\wp$ is a simple closed curve consisting of finitely many geodesic segments. A point q of boundary $\partial\wp$ is called a *vertex* of $\wp$ if $\partial\wp$ intersected with some open ball of $M^2(\kappa)$ with center at q consists of two geodesic segments which are not extensions of each other. A polygon $\wp$ in $D(\kappa)$ has at least three vertices since $D(\kappa)$ is a strictly convex domain. If q_1, q_2 are successive vertices of $\wp$ then the geodesic segment $[q_1, q_2]$ is called a *side* of polygon $\wp$. If all the sides of $\wp$ are equal in length then $\wp$ is said to be an *equilateral polygon*. If for any $x, y \in \wp$ the geodesic segment $[x, y]$ is contained in $\wp$ then $\wp$ is said to be *convex polygon*. If all angles at the vertices of $\wp$ are the same then $\wp$ is said to be *equiangular*. A convex, equilateral and equiangular polygon is called a *regular polygon*. We denote the polygon of 3 sides with vertices q_1, q_2, q_3 by $[q_1, q_2, q_3]$.

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