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Absolutely continuous spectrum and spectral transition for some continuous random operators

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Dedicated to Barry Simon for his 65th birthday

Abstract. In this paper we consider two classes of random Hamiltonians on $L^2(\mathbb{R}^d)$: one that imitates the lattice case and the other a Schrödinger operator with non-decaying, non-sparse potential both of which exhibit a.c. spectrum. In the former case we also know the existence of dense pure point spectrum for some disorder thus exhibiting spectral transition valid for the Bethe lattice and expected for the Anderson model in higher dimension.

Keywords. Random operators; a.c. spectrum.

1. Introduction and main theorems

In this paper we consider a two classes of random potentials and show the absence of point spectrum for the corresponding random Schrödinger operators for large energies. We are motivated by the models considered by Rodnianski and Schlag [23] and those by Hislop *et al.* [12].

Surprisingly the methods of proof in both the models are well known, one being the use of commutators and the other wave operators.

Commutators have played a significant role in spectral and scattering theory with the Kato-Putnam theorem [22] and the Mourre theory [20,21] addressing the presence of absolutely continuous spectrum. Positive commutators also have been used in the spectral theory of random operators by Howland [11] and by Combes *et al.* [5], Krishna and Stollmann [18] even to show the continuity of density of states.

On the other extreme non-zero commutators imply the absence of point spectrum, an indirect fact well known as the 'virial theorem'. In the literature mostly this fact was used to conclude the absence of positive eigenvalues in the scattering theoretic models (see, for example, [15], [25], [22] and [3]).

A very general discussion on the 'virial theorem' is given in Georgescu and Gerard [10] who gave a collection of conditions under which the above theorem is valid when A is an unbounded self-adjoint operator.

This theorem is often used to show that there are no eigenvalues in some set or there are no eigenvalues at all, see for example, [25], Theorem VIII.59 of [22], Proposition II.4 of [20] (see also [3]).

We apply the 'virial' theorem to models of random potentials 'living on large islands' an extension of a class of models considered by Rodnianski and Schlag [23]. As far as we know, this result is not known in the literature and includes random potentials which are neither ‘decaying’ nor are ‘sparse’ as we later exhibit in Example 3.1.

Exhibiting a.c. spectrum for stationary random potentials is a hard and interesting problem and this was shown on the Bethe lattice by Klein [16] first and then by Froese *et al.* [9] by alternative methods.

On the other hand, for non-stationary random potentials such as those that decay at infinity at some rate, there has been more progress and we refer to the review of Denisov and Kiselev [7] for a more thorough exposition. We also refer to the work of Safranov [24] for a more current work in this area.

Let $\beta \geq 0$ and let $r_\beta(x)$ be a positive function on $\mathbb{R}^d$ satisfying

$$c_1|x|^\beta \leq r_\beta(x) \leq c_2|x|^\beta, \quad \text{for some } 0 < c_1 \leq c_2 < \infty.$$

Let $\mathcal{N}_{\beta,\gamma}$ be a discrete subset of $\mathbb{R}^d$ such that for points $x, y \in \mathcal{N}_{\beta,\gamma},\ x \neq y$, we have

$$\{w : |x - w| \leq \gamma r_\beta(x)\} \cap \{w : |y - w| \leq \gamma r_\beta(y)\} = \emptyset,$$

for some $0 < \gamma \leq 1$.

(It is easy to think of the case $r_\beta(x) = |x|^\beta$.)

Let $\{\omega_n, n \in \mathcal{N}_{\beta,\gamma}\}$ be independent real valued random variables and let $\alpha \geq 0$. We define random functions $V^\omega_{\beta,\gamma,\alpha}$ on $\mathbb{R}^d$ as follows:

$$V^\omega_{\beta,\gamma,\alpha}(x) = \sum_{n\in\mathcal{N}_{\beta,\gamma}} \omega_n |n|^{-\alpha} \phi\left(\frac{x-n}{r_\beta(n)}\right), \tag{1}$$

where ϕ is a smooth bump function supported in the unit ball in $\mathbb{R}^d$ (so that the n-th summand is a function centered at n and supported in a ball of radius $r_\beta(n)$ which is roughly $|n|^\beta$). We will denote the operator on $L^2(\mathbb{R}^d)$ of multiplication by the function $V^\omega_{\beta,\gamma,\alpha}$ by the same symbol.

Theorem 1.1. *Consider the sets $\mathcal{N}_{\beta,\gamma}$, i.i.d random variables $\{\omega_n, n \in \mathcal{N}_{\beta,\gamma}\}$ with compactly supported distribution μ and consider the random Schrödinger operators*

$$H^\omega_{\beta,\gamma,\alpha} = -\Delta + V^\omega_{\beta,\gamma,\alpha}$$

on $L^2(\mathbb{R}^d)$. Then for all ω,

1. *Let $\alpha, \beta \geq 0,\ \alpha + \beta \geq 1$. Then there is a $E_0 < \infty$ such that*

$$\sigma_{pp}(H^\omega_{\beta,\gamma,\alpha}) \cap (E_0, \infty) = \emptyset;$$

2. *Suppose $\alpha + 2\beta \geq 2$. Then*

$$\sigma_s(H^\omega_{\beta,\gamma,\alpha}) \cap (E_0, \infty) = \emptyset.$$

Remark 1.2.

1. In the case when $d \geq 2$ and $\beta > \frac{1}{2}$, $\alpha = 3/4$, Rodnianski and Schlag [23] showed the existence of modified wave operators for the pair $H^\omega_{\beta,\gamma,\alpha}, -\Delta$ and thus showed that $\sigma_{ac}(H^\omega_{\beta,\gamma,\alpha}) = [0, \infty)$. We consider weaker conditions on V^ω but also weaker conclusions.

2. In the above theorem all we need is that $[V^{\omega}_{\beta,\gamma,\alpha}, A]$ extends to a bounded operator from $\mathcal{S}(\mathbb{R}^d)$, say, to $L^2(\mathbb{R}^d)$, where A is the generator of dilation group given below.
3. In the case $\beta = 1$, the 'thickest' possible sets $\mathcal{N}_{\beta,\gamma}$ are in some sense opposite of the Bethe lattice. The number of points $N(R)$ at a distance R from the origin here grows logarithmically in R asymptotically, while on the Bethe lattice $N(R)$ grows exponentially. When β varies from 1 to 0, the growth behaviour of $N(R)$ changes from logarithmic to polynomial.

Taking the case $\alpha = 0, \beta = 1$ in the above theorem we have as follows:

COROLLARY 1.3

Let $0 < \gamma < 1$ and let ϕ be a smooth function supported in a ball of radius γ centred at the origin in $\mathbb{R}^d$. Let $\{\omega_n, n \in \mathcal{N}_{1,\gamma}\}$ be i.i.d. random variables distributed according to a compactly supported distribution μ. Consider the random operators

$$H^{\omega} = -\Delta + V^{\omega}, \quad V^{\omega}(x) = \sum_{n \in \mathcal{N}_{1,\gamma}} \omega_n \phi\left(\frac{x-n}{r_1(n)}\right), \quad \gamma < 1$$

on $L^2(\mathbb{R}^d)$. Then there is a $E_0 < \infty$ such that the spectrum of H^{ω} in (E_0, ∞) is purely absolutely continuous.

Remark 1.4. Since $r_1(n) \approx |n|$, the corollary gives non-decaying, non-sparse potentials with a.c. spectrum and is also valid in one dimension. The potential configurations consist of independent barriers or wells whose supports together 'cover' a fraction of $\mathbb{R}^d$.

We want to make sure that there is spectrum in the region of energies we are interested in and this is guaranteed by the following theorem.

Theorem 1.5. *Consider the operators $H^{\omega}_{\beta,\gamma,\alpha}$ given in Theorem* 1.1 *with $\alpha > 0$. Then*

$$\sigma_{ess}(H^{\omega}_{\beta,\gamma,\alpha}) = [0, \infty).$$

Therefore the spectrum in $(-\infty, 0)$, if any, is discrete for each ω. If $\alpha = 0$, then for every $E > 0$,

$$\sigma(H^{\omega}_{\beta,\gamma,\alpha}) \cap (E, \infty) \neq \emptyset.$$

The second model we consider comes from the paper of Hislop *et al.* [12] (which we recollect in the Appendix almost verbatim for the reader's convenience). Let

$$\Lambda = \cup_{i=1}^{d} \{n \in \mathbb{Z}^d : |n_i| \leq M_i < \infty\}, \quad F = \{0,1\}^d \setminus \{(0,0,\ldots,0)\}$$
$$I = F \times \mathbb{Z} \times \mathbb{Z}^d, \quad I_{\Lambda} = F \times \mathbb{Z} \times \Lambda \subset I.$$

Let Ψ be a a multidimensional wavelet indexed by the set as in Hypothesis 4.1, $\{P_n\}$ be the orthogonal projections on $\ell^2(\Lambda)$ given by

$$P_{\mathbf{n}} = |\Phi_{\mathbf{n}}\rangle\langle\Phi_{\mathbf{n}}|, \quad n \in I_{\Lambda}$$

associated with the orthonormal set of functions $\{\Phi_{\mathbf{n}}\}$ defined in equation (A2). Let $\{\omega_n\}$ random variables satisfying Hypothesis 4.2 and consider the operators

$$H_\Lambda^\omega = -\Delta + \sum_{\mathbf{n}\in I_\Lambda} \omega_n P_{\mathbf{n}} \tag{2}$$

on $\ell^2(\Lambda)$.

Combining with a theorem (Theorem 3.5) of Hislop *et al.* [12] we have as follows:

Theorem 1.6. *Consider the operators H_Λ^ω given in equation* (2) *such that Hypothesis* 4.1 *and* 4.2 *are satisfied. Then*

1. $\sigma_{ac}(H_\Lambda^\omega) \supset [0, \infty)$, *for all* ω.
2. *There is a* $E(\mu) < 0$ *such that the essential spectrum of* H_Λ^ω *in* $(-\infty, E(\mu))$ *is non-empty and is a pure point.*

Remark 1.7.

- We would like to point out a subtlety involved in the proof of (2) of the above theorem. The weak/intermediate disorder case of fractional moment method of Aizenman [1] was used in the proof by Hislop *et al.* [12] in proving (2). This proof considers spectrum of the random operator in the resolvent set of the free part and so gives purity of the point spectrum even if one takes $V^\omega = \sum_{n\subset K} \omega_n P_n$, with P_n's mutually orthogonal rank one projections and $\sum_{n\subset K} P_n \neq I$. This proof should be contrasted with the method of Aizenman and Molchanov [2] which (implicitly) requires $I - \sum P_n$ to be a finite rank.
- The model presented here is similar (in spirit) to the ones studied by Jaksić and Last [13,14] for which they show the existence of pure absolutely continuous spectrum in the spectrum of the free part.

2. Proofs of the theorems

We start with recollecting a more 'practical' version of the 'virial theorem' which is given in Proposition 7.2.10 of [3], incorporating also the conditions from their Theorem 6.2.10.

Theorem 2.1 (Virial theorem). *Suppose H, A is a pair of self-adjoint operators on a separable Hilbert space $\mathcal{H}$ such that*

1. *there is a constant* $c < \infty$ *such that for all* $f \in D(H) \cap D(A)$,

   $$|\langle Hf, Af\rangle - \langle Af, Hf\rangle| \leq c(\|Hf\|^2 + \|f\|^2)$$

   and
2. *for some* $z \in \rho(H)$, *the set*

   $$\{f \in D(A) : R(z)f, R(\bar{z})f \in D(A)\}$$

   is a core for $D(A)$.

Then,

$$\langle Hf, Ag\rangle - \langle Af, Hg\rangle = 0$$

whenever f, g are eigenvectors of H with the same eigenvalue.

Proof of Theorem 1.1. In the following we drop the indices α, β, γ on both $V^{\omega}_{\beta,\gamma,\alpha}$ and $H^{\omega}_{\beta,\gamma,\alpha}$ for ease of reading.

To prove (1) we first note that since $V^{\omega}_{\beta,\gamma,\alpha}$ is a bounded operator, $D(H^{\omega}_{\beta,\gamma,\alpha}) = D(-\Delta)$. We consider the generator of dilation group $A = -i\sum_{i=1}^{d}\left(x_i\frac{\partial}{\partial x_i} + \frac{1}{2}\right)$. It is well-known that the Schwartz space of rapidly decreasing functions $\mathcal{S}(\mathbb{R}^d)$ is a core for A and the commutator of A and $-\Delta$ is computed as

$$i[-\Delta, A] = -2\Delta$$

on $\mathcal{S}(\mathbb{R}^d)$ and extends to $D(-\Delta)$. Let us set $\phi_n(x) = \phi(\frac{x-n}{r_\beta(n)})$. By assumption on ϕ, $\phi_n \in C^\infty(\mathbb{R}^d)$ and by assumption on $\mathcal{N}_{\beta,\gamma}$, the ϕ_n's have disjoint supports. The compactness of the support of μ gives uniform boundedness of ω_n in n, so $V^\omega f \in \mathcal{S}(\mathbb{R}^d)$ whenever $f \in \mathcal{S}(\mathbb{R}^d)$. Therefore if we show that $[\phi_n, A]$, computed on $\mathcal{S}(\mathbb{R}^d)$, is bounded for each $n \in \mathcal{N}_{\beta,\gamma}$, then in view of the equality

$$i[V^\omega, A] = -\sum_{n\in\mathcal{N}_{\beta,\gamma}} \omega_n \frac{1}{|n|^\alpha r_\beta(n)} x\cdot(\nabla\phi)_n(x),$$

it follows that $[V^\omega, A]$ extends to a bounded operator from $\mathcal{S}(\mathbb{R}^d)$. Since ϕ is a $C^\infty(\mathbb{R}^d)$ function of compact support, $\nabla\phi$ also has components having compact support and as a consequence $\mathrm{supp}(\nabla\phi_n)_j \subset \mathrm{supp}\,(\phi_n)$, $j = 1,\ldots,d$ for all $n \in \mathcal{N}_{\beta,\gamma}$. In addition, the supports of $\{\phi_n\}$ are mutually disjoint by the assumption of $\mathcal{N}_{\beta,\gamma}$, therefore for $x \in \mathrm{supp}(\phi_n)$, we have

$$\begin{aligned}
|i[\phi_n, A]f|(x) &\le \left|\frac{x}{|n|^\alpha r_\beta(n)}\cdot(\nabla\phi)\left(\frac{x-n}{r_\beta(n)}\right)\right||f|(x)\\
&\le \left|\frac{c}{|n|^\alpha}\frac{x-n}{|n|^\beta}\cdot(\nabla\phi)\left(\frac{x-n}{|n|^\beta}\right)\right||f|(x)\\
&\quad + \left|\frac{n}{|n|^{\alpha+\beta}}\cdot(\nabla\phi)\left(\frac{x-n}{|n|^\beta}\right)\right|_\infty |f|(x)\\
&\le c|(\nabla\phi)|_\infty|f|(x)
\end{aligned}\tag{3}$$

for each $f \in \mathcal{S}(\mathbb{R}^d)$. This inequality gives the bound

$$\begin{aligned}
\|i[V^\omega, A]f\|^2 &= \sum_{n\in\mathcal{N}_{\beta,\gamma}} \int_{\mathrm{supp}(\phi_n)} |\omega_n|^2|i[\phi_n, A]f|^2(x)\,\mathrm{d}x\\
&\le c\int_{\mathrm{supp}(\phi_n)} \sup_n |\omega_n|^2\|i[\phi_n, A]f\| \le c\|\nabla\phi\|_\infty\|f\|^2,
\end{aligned}\tag{4}$$

which gives the stated boundedness.

Therefore the commutator $[(H^\omega \pm i)^{-1}, A]$ satisfying the relation

$$[(H^\omega \pm i)^{-1}, A] = (H^\omega \pm i)^{-1}[A, H^\omega](H^\omega \pm i)^{-1}$$

also extends to a bounded operator on $\mathcal{H}$. Hence

$$\|A(H^\omega \pm i)^{-1} f\| \leq \|[A, (H^\omega \pm i)^{-1}]f\| + \|(H^\omega \pm i)^{-1} Af\|$$

implies that $(H^\omega \pm i)^{-1}$ maps $\mathcal{S}(\mathbb{R}^d)$ into $D(A)$. Thus $\mathcal{S}(\mathbb{R}^d)$ is contained in the set

$$\{f \in D(A) : (H^\omega \pm i)^{-1} f \in D(A)\}.$$

Since $\mathcal{S}(\mathbb{R}^d)$ is a core for A, so is the above set. Thus we have verified the conditions (1), (2) of the virial theorem (Theorem 2.1).

Therefore for any normalized eigenvector f^ω of H^ω, we should have

$$\langle f^\omega, i[H^\omega, A]f^\omega\rangle = 0. \tag{5}$$

However since

$$i[H^\omega, A] = 2H^\omega + ([V^\omega, A] - 2V^\omega) = 2H^\omega + B^\omega \tag{6}$$

with B^ω bounded and $\sup_\omega \|B^\omega\| = 2E_0$ finite, we see that if f^ω is the eigenvector of an eigenvalue λ^ω of H^ω satisfying $\lambda^\omega > E_0$. Then we must have

$$|\langle f^\omega, i[H^\omega, A]f^\omega\rangle| \geq |\langle f^\omega, (2H^\omega + 2B^\omega)f^\omega\rangle| \geq 2\lambda^\omega - 2E_0 > 0, \tag{7}$$

contradicting the virial relation given in equation (5). Hence there can be no eigenvalue for $H^\omega_{\beta,\gamma,\alpha}$ bigger than E_0.

To show (2) we verify the Mourre estimate in this case. Let χ_I denote the indicator function of the set I. Applying $\chi_{(E_1,\infty)}(H^\omega)$ on either side of equation (6) we see that, with $c > 0$,

$$\begin{aligned}&\chi_{(E_1,\infty)}(H^\omega) i[H^\omega, A]\chi_{(E_1,\infty)}(H^\omega)\\ &\quad > 2(E_1 - \sup_\omega \|B^\omega\|)\chi_{(E_1,\infty)}(H^\omega) > c\chi_{(E_1,\infty)}(H^\omega),\end{aligned}$$

for any $E_1 > E_0$ from the inequality (5), hence for any closed interval I in (E_0, ∞) we have

$$\chi_I(H^\omega) i[H^\omega, A]\chi_I(H^\omega) > c\chi_I(H^\omega).$$

Therefore we only need to verify that the second commutator of H^ω with respect to A is relatively bounded with respect to H^ω. Since ϕ is smooth we get that

$$i[i[H^\omega, A], A] = 4(H^\omega - V^\omega) - [[V^\omega, A], A]$$

with

$$\begin{aligned}[[V^\omega, A], A] &= \sum_{n\in\mathcal{N}_\beta} \omega_n (x\cdot\nabla)(x\cdot\nabla)\frac{1}{|n|^\alpha}(\phi)\left(\frac{x-n}{r_\beta(n)}\right)\\ &\approx \sum_{n\in\mathcal{N}_\beta} \omega_n\left(\frac{1}{|n|^{\alpha+\beta}} x\cdot(\nabla\phi) + \sum_{j,k=1}^{d} \frac{x_j x_k}{|n|^{\alpha+2\beta}}\left(\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}\phi\right)\left(\frac{x-n}{r_\beta(n)}\right)\right).\end{aligned} \tag{8}$$

The conditions on ϕ, α, β are such that the right-hand side is a bounded function of x showing that $[[V^\omega, A], A]$ extends to a bounded operator. Thus $i[i[H^\omega, A], A](H^\omega + i)^{-1}$ is bounded. These estimates show that the conditions (1)–(5) (taking $K = 0$, $S = I$ there) in Definition 3.5.5 in [8] are satisfied showing that A is a local conjugate of H^ω for each ω. Hence by Mourre's theorem (Theorem 3.5.6(ii) of [8]) there is no singular continuous spectrum for H^ω in I. These two results together show that there is no singular spectrum in any closed subinterval of (E_0, ∞), showing the theorem. □

Proof of Theorem 1.5. When $\alpha > 0$, the potential is relatively compact with respect to $-\Delta$, so Weyl's theorem implies the statement on the essential spectrum. On the other hand, since $H^\omega_{\beta,\gamma,\alpha}$ is an unbounded self-adjoint operator its spectrum cannot be bounded, hence the statement for $\alpha = 0$. □

Proof of Theorem 1.6. The proof of Theorem 1.6(ii) is almost as in the proof of Theorem 3.5 of [12] with a minor modification. Equation (17) of [12] should be replaced by

$$\begin{aligned} P_{\mathbf{n}}(H^\omega_\Lambda - E - i\epsilon)^{-1} P_{\mathbf{m}} &= P_{\mathbf{n}}(H_0 - E - i\epsilon)^{-1} P_{\mathbf{m}} \\ &\quad - \sum_{\mathbf{k} \in I_\Lambda} P_{\mathbf{n}}(H_0 - E - i\epsilon)^{-1} \omega_{\mathbf{k}} P_{\mathbf{k}} (H^\omega_\Lambda - E - i\epsilon)^{-1} P_{\mathbf{m}}. \end{aligned}$$

Then the estimate in the inequality (21) of [12] should be redone as

$$\begin{aligned} &\mathbb{E}\{\| P_{\mathbf{n}}(H^\omega_\Lambda - z)^{-1} P_{\mathbf{m}}\|^s\} \\ &\quad \le \| P_{\mathbf{n}}(H_0 - z)^{-1} P_{\mathbf{m}}\|^s \\ &\qquad + K_s \sum_{\mathbf{k} \in I_\Lambda} \| P_{\mathbf{n}}(H_0 - z)^{-1} P_{\mathbf{k}}\|^s \ \mathbb{E}\{\| P_k(H^\omega_\Lambda - z)^{-1} P_{\mathbf{m}}\|^s\} \\ &\quad \le \| P_{\mathbf{n}}(H_0 - z)^{-1} P_{\mathbf{m}}\|^s \\ &\qquad + K_s \sum_{\mathbf{k} \in I} \| P_{\mathbf{n}}(H_0 - z)^{-1} P_{\mathbf{k}}\|^s \ \mathbb{E}\{\| P_k(H^\omega_\Lambda - z)^{-1} P_{\mathbf{m}}\|^s\}. \end{aligned}$$

Now the proof goes through exactly as that of Theorem 3.5 of [12].

To show that the a.c. spectrum in Theorem 1.6(i) contains $[0, \infty)$ we prove that wave operators for the pair $(H^\omega_1, -\Delta)$ exist almost everywhere.

The existence of wave operators follows if we show that for a dense set of $f \in L^2(\mathbb{R}^d)$, the limits

$$\lim \mathrm{e}^{iH^\omega_1 t} \mathrm{e}^{i\Delta t} f$$

exist strongly as t goes to ∞. Thus by Cook's method

$$\lim_{s,t\to\infty} \|\mathrm{e}^{iH^\omega_1 t} \mathrm{e}^{i\Delta t} - \mathrm{e}^{iH^\omega_1 s} \mathrm{e}^{i\Delta s} f\| \le \lim_{s,t\to\infty} \int_s^t \mathrm{d}w \|(H^\omega_1 + \Delta) \mathrm{e}^{i\Delta w} f\| = 0$$

for a dense set of f. This follows if the integral

$$\int_1^\infty \mathrm{d}t \ \|V^\omega \mathrm{e}^{i\Delta t} f\| < \infty, \tag{9}$$

for a dense set of f.

Let the union of the coordinate axes in $\mathbb{R}^d$ be denoted by A_0, thus $A_0 = \{x \in \mathbb{R}^d : x_i = 0 \text{ for some } i = 1, \ldots, d\}$. We pick the dense set to be

$$\mathcal{D} = \{f \in L^2(\mathbb{R}^d) : \text{supp } \hat{f} \subset \mathbb{R}^d \backslash A_0 \text{ and supp } \hat{f} \text{ compact}\}.$$

We therefore consider the integrand and get the estimate for each ω,

$$\begin{aligned} \|V^\omega \mathrm{e}^{i\Delta t} f\| &= \left\| \sum_{\mathbf{n} \in I_\Lambda} \omega_n \langle \Phi_{\mathbf{n}}, \mathrm{e}^{i\Delta t} f \rangle \Phi_{\mathbf{n}} \right\| \\ &\leq C \sum_{\mathbf{n} \in I_\Lambda} |\langle \Phi_{\mathbf{n}}, \mathrm{e}^{i\Delta t} f \rangle| \end{aligned} \tag{10}$$

since ω_n are bounded and $\{\Phi_{\mathbf{n}}\}$ is an orthonormal set. We will show that the sum in inequality (10) converges.

We first note that undertaking Fourier transforms we have

$$\begin{aligned} \langle \Phi_{\mathbf{n}}, \mathrm{e}^{i\Delta t} f \rangle &= \int \widehat{\overline{\Phi_{\mathbf{n}}}}(\xi) \mathrm{e}^{-i\xi^2 t} \hat{f}(\xi)\, \mathrm{d}\xi \\ &= 2^{-\frac{\mathrm{d}n_1}{2}} \int \overline{\widehat{\Psi_{c(\mathbf{n})}(\xi)}}\, \mathrm{e}^{-i2^{-n_1} n_2 \cdot \xi - i\xi^2 t} \hat{f}(\xi)\, \mathrm{d}\xi. \end{aligned} \tag{11}$$

We recall from equations (A1), (A2) that the function $\widehat{\Psi_{c(\mathbf{n})}}$ has at least one factor ψ_j (which is supported in the set $\{|\xi_j| \in [2\pi/3, 8\pi/3]\}$) so that for at least one coordinate ξ_j of ξ we have the condition $|2^{-n_1}\xi_j| \in [2\pi/3, 8\pi/3]$. In addition, by the choice of f we have $\hat{f}(\xi) = 0$ if $|\xi_j| \notin [c, d]$ for some $0 < c < d < \infty$ for all $j = 1, \ldots, d$. These two conditions together imply that the integral is zero unless there is an $R < \infty$ such that $-R < n_1 < R$, where R depends on c, d. Thus the sum over n_1 is reduced to a finite sum in equation (10).

The idea is now to get arbitrary decay in t from the integral with respect to ξ_1 and get decay in each of the variables n_{2j} in exchange for some growth in t from each of the other variables $\xi_2, \ldots, \xi_d$. These estimates together give decay of the integral in both t and $|n|$.

By assumption on Λ, writing $n_2 = (n_{21}, \ldots, n_{2d})$, n_{2k} is finite for some $k = 1, \ldots d$, without loss of generality. Let $|n_{21}| < K < \infty$. We set $a_1(\xi_1) = -i\frac{\partial}{\partial \xi_1}(2^{n_1} n_{21}\xi_1 - \xi_1^2 t)$. Then we have

$$|a_1(\xi_1)| = \left| \frac{\partial}{\partial \xi_1}(2^{n_1} n_{21}\xi_1 - \xi_1^2 t) \right| \geq 2|t| \left| |\xi_1| - \frac{2^{n_1 - 1} n_{21}}{t} \right| \geq 2|t|c/2 = c|t|, \tag{12}$$

if $2^{R-1}K/t < c/2$, whenever $\xi \in \text{supp } \hat{f}$. Under the Hypothesis 4.1, $\widehat{\Psi_{\mathbf{n}}}$ has $2d+2$ partial derivatives in each of the ξ_j's, so we can do repeated integration by parts with respect to the variable ξ_1 in the above integral equation (11) to get, for every $\ell \in \{1, 2, \ldots, 2d+2\}$,

$$\langle \Phi_{\mathbf{n}}, \mathrm{e}^{i\Delta t} f \rangle = (-1)^\ell \int \mathrm{e}^{-i2^{-n_1} n_2 \cdot \xi - i\xi^2 t} \left(\frac{\partial}{\partial \xi_1} \frac{1}{a_1(\xi_1)} \right)^\ell B_{\mathbf{n}}(\xi)\, \mathrm{d}\xi, \tag{13}$$

where we took $2^{-\frac{\mathrm{d}n_1}{2}} \overline{\widehat{\Psi_{c(\mathbf{n})}}}(2^{-n_1}\xi)\hat{f}(\xi) = B(\mathbf{n}, \xi)$.

We now take

$$B(t,\ell,\mathbf{n},\xi)=\mathrm{e}^{-i\xi^2 t}\left(\frac{\partial}{\partial\xi_1}\frac{1}{a_1(\xi_1)}\right)^{\ell}B_{\mathbf{n}}(\xi) \tag{14}$$

and do integration by parts twice with respect to each of the variables $\xi_2,\ldots,\xi_d$ to get

$$\langle\Phi_{\mathbf{n}},\mathrm{e}^{i\Delta t}f\rangle=\left(\prod_{j=2}^{d}\frac{-1}{2^{-2n_1}n_{2j}^2}\right)(-1)^{\ell}\int \mathrm{e}^{-i2^{-n_1}n_2\cdot\xi}\prod_{j=2}^{d}\frac{\partial^2}{\partial\xi_j^2}B(t,\ell,\mathbf{n},\xi)\,\mathrm{d}\xi. \tag{15}$$

It is now a tedious but not difficult calculation to see, using inequalities/equations (12–15), that if we take $\ell=2d$, then

$$|\langle\Phi_{\mathbf{n}},\mathrm{e}^{i\Delta t}f\rangle|\le C\prod_{j=2}^{d}\frac{1}{1+|n_{2j}|^2}|t|^{2d-2}|t|^{-2d}\int W(n_1,\xi)\,\mathrm{d}\xi,$$

where the factor $|t|^{2d-2}$ is the maximum power of $|t|$ possible by taking derivatives of the factor $\mathrm{e}^{-i\xi^2 t}$ with respect to the variables $\xi_2,\ldots,\xi_d$, while the factor $|t|^{-\ell}$ comes from the factor $1/a_1(t,\xi)$ occurring ℓ times and we clubbed all the rest of the integrand in W. Using the fact that $|n_1|<R$ and that W has compact support in ξ and so is integrable, the above inequality implies that

$$\int_1^{\infty}\sum_{\mathbf{n}\in\Lambda}|\langle\Phi_{\mathbf{n}},\mathrm{e}^{i\Delta t}f\rangle|\,\mathrm{d}t<C\int_1^{\infty}t^{-2}\,\mathrm{d}t\sum_{|n_1|<\infty}\sum_{n_{22},\ldots,n_{2d}\in\mathbb{Z}}\frac{1}{1+|n_{2j}|^2}<\infty.$$

This estimate together with the inequality (10) proves the required inequality (9). □

3. Examples

There are lots of examples of sets $\mathcal{N}_{\beta,\gamma}$ mentioned before equation (1).

Examples 3.1. We shall give an example in $d=2$ of the potentials that have neither 'decay' nor supported on a 'sparse' set. This example is motivated by the paper of Rodnianski and Schlag [23].

Consider a fixed $R>0$ and consider the squares $B_k=\{x\in\mathbb{R}^2:|x_i|\le 2^kR, i=1,2\}$, $k\in\mathbb{Z}^+$, which are centered at the origin and have side length $2^{k+1}R$. Then $\cup_k B_k=\mathbb{R}^2$ and we consider the annulus $A_k=B_{k+1}\backslash B_k$. The area of B_k is $(2^{k+1}R)^2$ and so the area of the annulus is Area(A_k) =Area$(B_{k+1})-$Area$(B_k)=3(2^{k+1}R)^2$. Clearly we can cover A_k with 12 squares of side length 2^kR each, with the centres of these squares falling on the lines $|x_1|=32^{k-1}R$ or $|x_2|=32^{k-1}R$. We take the squares S_y of side length 2^kR centred at the points y in the set

$$\begin{aligned}C_k &= \left\{x:x_2=\pm\frac{3}{2}\times 2^kR,\,x_1=\pm 2^{k-1}R,\pm 3\times 2^{k-1}R\right\}\\ &\bigcup\left\{x:x_1=\pm\frac{3}{2}\times 2^kR,\,x_2=\pm 2^{k-1}R,\pm 3\times 2^{k-1}R\right\}\end{aligned} \tag{16}$$

and take the respective discs of radius $2^{k-1}R$ with the same centres and inscribed in the squares. We can then take a bump function ϕ supported in the unit disk, nowhere vanishing in the open disk but vanishing on its boundary. We take $r_1(n) = 2^{k-1}R$ for $n \in C_k$. Since the points of C_k have absolute value $\sqrt{10}2^{k-1}R$ or $\sqrt{18}2^{k-1}R$, we find that the condition

$$\frac{1}{3\sqrt{2}}|n| \leq r_1(n) = 2^{k-1}R \leq \frac{1}{\sqrt{10}}|n|, \quad n \in C_k$$

is valid. Then the functions $\phi(\frac{x-y}{2^{k-1}R})$ with $y \in C_k$ give a collection of functions such that

$$\sup_{k\in\mathbb{Z}^+}\sup_{x\in\mathbb{R}^2}\left|(x\cdot\nabla\phi)\left(\frac{x-y}{2^{k-1}R}\right)\right| < \infty \quad \text{and} \quad \sup_{k\in\mathbb{Z}^+}\sup_{x\in\mathbb{R}^2}\left|(x\cdot\nabla\phi)^2\left(\frac{x-y}{2^{k-1}R}\right)\right| < \infty.$$

Further we note that by construction, for each k we have

$$\text{Area}(A_k) = \cup_{y\in C_k}\text{Area}(S_y) = 12(2^k R)^2.$$

The area of the discs inscribed in each S_y, $y \in C_k$ is $\pi(2^{k-1}R)^2$, so the total area of these discs contained in A_k is $12\pi(2^{k-1}R)^2$. Thus in each of the annuli A_k the area of the discs is $\frac{\pi}{4}$ Area(A_k). Adding up we find that the union of the discs we constructed with centers at all points in C_k is a fraction $\frac{\pi}{4}$, so it also forms the same fraction of the area of the squares B_k . This shows that the union of the supports of the functions $\phi(\frac{x-y}{2^{k-1}R})$, $y \in C_k$ has positive density in $\mathbb{R}^2$ (the density of $\frac{\pi}{4}$).

Remark 3.2.

1. It is clear from the construction above example that if we took a product of bump functions $\prod_{j=1}^{d} f_j$ each supported on $[-1, 1]$, then we can get ϕ_y's to have full support in the annulus A_k and then the resulting potential

$$V^{\omega}(x) = \sum_{n\in\cup_{k=1}^{\infty}C_k}\omega_n\phi_n(x)$$

is a random potential which is non-vanishing on a set of full measure on $\mathbb{R}^d$ for which Theorem 1.1 will be valid. Of course there are many more possibilities.
2. The above example can be extended to any $\mathbb{R}^d$ with spheres replacing discs, but the centers chosen to fall in between cubes Λ_k of side lengths $2\gamma^k R$, $\gamma > 1$ centred at the origin. The spheres can be chosen to lie in the region $\Lambda_{k+1}\backslash\Lambda_k$ with centres chosen so they pack a positive density (which is independent of k but depends on the dimension d) of the volume of this region. Such sets give rise to independent random potentials (which are supported on these sets) that are neither 'decaying' nor have 'sparse' supports. Nevertheless there is no localization at large energies for them.

Appendix

We reproduce verbatim the construction of the projections $P_{\mathbf{n}}$, used in equation (2), using the Lemare–Meyer wavelets from [12], for easy reference.

In order to construct the projections $P_{\mathbf{n}}$, we first recall the definition of wavelets in higher dimensions.

A *wavelet* in one dimension is a function ψ with the property that the collection of translated and diadically dilated functions $\{\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k) \mid j, k \in \mathbb{Z}\}$, forms an orthonormal basis for $L^2(\mathbb{R})$. Associated with the wavelet ψ is the *scaling function* ϕ. The scaling function ϕ is used to construct the wavelet ψ through a procedure called *multiresolution analysis* (cf. [6,26]).

To define a wavelet in higher dimensions (as in Proposition 5.2 of [26]), we first start with a collection $\{\phi_1, \ldots, \phi_d, \psi_1, \ldots, \psi_d\}$ of $2d$ functions on $\mathbb{R}$ of which the ϕ_j are scaling functions and the ψ_j are the associated wavelets constructed from the ϕ_j. We note that we may take all the $\phi_j \equiv \phi$, $\psi_j \equiv \psi$, $j = 1, \ldots, d$., although this is not necessary. Let us define an index set $F = \{0, 1\}^d \setminus (0, 0, \ldots, 0)$. For each $c \in F$, we define a function on $\mathbb{R}^d$ by

$$\Psi_c(x) = \prod_{j=1}^{d} (\delta_{c_j,0}\phi_j + \delta_{c_j,1}\psi_j)(x_j), \quad c \in F. \tag{A1}$$

Here, the $\delta_{c_j,k}$, for $c = (c_1, \ldots, c_d) \in F$ and $k = 0, 1$ is the Kronecker delta. In the product, the function $\phi_j(x_j)$ is present if the index c_j is zero, and $\psi_j(x_j)$ is present otherwise. Note that there is at least one factor ψ_j in Ψ_c for any $c \in F$. We consider the set of dyadic dilations and $\mathbb{Z}^d$-lattice translations of these functions. We denote by I the countable index set $I = F \times \mathbb{Z} \times \mathbb{Z}^d$. An element $\mathbf{n} \in I$ is a triple $\mathbf{n} = (c(\mathbf{n}), n_1, n_2)$. The collection of dilated and translated functions

$$\Phi_{\mathbf{n}}(x) = 2^{n_1 d/2}\Psi_{c(\mathbf{n})}(2^{n_1}x - n_2), \;\; c \in F, \;\; n_1 \in \mathbb{Z} \;\, n_2 \in \mathbb{Z}^d, \;\; x \in \mathbb{R}^d, \tag{A2}$$

is called a *multi-variable wavelet* if the collection forms an orthonormal basis for $L^2(\mathbb{R}^d)$.

In the following we shall, notationally, always refer to the collection of functions $\{\Psi_c \mid c \in F\}$ simply as Ψ, and any property stated for Ψ is by definition to be take to be valid for each member of this collection. Thus a statement that the property P is valid for $\hat{\Psi}$ means that P is valid for each of the Fourier transforms $\widehat{\Psi_c}$, for each $c \in F$, and so on.

We assume the following conditions on the multi-variable wavelet and the distribution of the random variables $\{\omega_n \mid \mathbf{n} \in I\}$.

Hypothesis 4.1. Let Ψ be a multi-variable wavelet formed out of the scaling functions ϕ_i, $i = 1, \ldots, d$ and the wavelets ψ_i, $i = 1, \ldots, d$ such that

1. the functions $\widehat{\phi_j} \in \mathcal{C}^{2d+2}(\mathbb{R})$, $\widehat{\psi_j} \in \mathcal{C}_0^{2d+2}(\mathbb{R})$, $\;j = 1, \ldots, d$;
2. the functions $\widehat{\phi}_j^{(\alpha)}$, for $|\alpha| \le 2d + 2$, decay rapidly;
3. the functions are normalized, $\int |\Psi|^2 \, \mathrm{d}x = 1$.

Hypothesis 4.2. Let $I = F \times \mathbb{Z} \times \mathbb{Z}^d$, and let $\{\omega_n \mid \mathbf{n} \in I\}$ be independent and identically distributed random variables with their common probability distribution μ being absolutely continuous and of compact support in $\mathbb{R}$.

Remark. Any one-dimensional Lemarié–Meyer wavelet ψ, and its related scaling function ϕ, satisfy Hypothesis 4.1. Typically, a Meyer wavelet can be constructed to be in the Schwartz class, $\psi \in \mathcal{S}(\mathbb{R})$, and its Fourier transform $\hat{\psi}$ is compactly supported in the set $[-8\pi/3, -2\pi/3] \cup [2\pi/3, 8\pi/3]$. The corresponding scaling function can also be chosen to satisfy $\phi \in \mathcal{S}(\mathbb{R})$, so that $\hat{\phi}$ has compact support in $[-4\pi/3, 4\pi/3]$, cf. [19,26]. A large number of additional examples are constructed in the paper of Auscher *et al.* [4].

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