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On cohomology theory for topological groups

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Abstract. We construct some new cohomology theories for topological groups and Lie groups and study some of its basic properties. For example, we introduce a cohomology theory based on measurable cochains which are continuous in a neighbourhood of the identity. We show that if G and A are locally compact and second countable, then the second cohomology group based on locally continuous measurable cochains as above parametrizes the collection of locally split extensions of G by A.

Keywords. Cohomology theory; group extension; locally compact groups; Lie groups.

1. Introduction

The cohomology theory of topological groups has been studied from different perspectives by van Est, Mostow, Moore, Segal, Wigner and recently Lichtenbaum amongst others. van Est developed a cohomology theory using continuous cochains in analogy with the cochain construction of cohomology theory of finite groups. However, this definition of cohomology groups has a drawback, in that it gives long exact sequences of cohomology groups only for those short exact sequences of modules that are topologically split.

Based on a theorem of Mackey [3], guaranteeing the existence of measurable cross sections for locally compact groups, Moore developed a cohomology theory of topological groups using measurable cochains in place of continuous cochains. This cohomology theory works for the category of Polish groups G and G-modules A which are again Polish. We recall, a topological group G is said to be Polish, if its topology is induced by a complete separable metric on G. This theory satisfies the nice properties expected from a cohomology theory [13] viz., there exists long exact sequences of cohomology groups of a Polish group G corresponding to a short exact sequence of Polish G-modules, the correct interpretation of the first measurable cohomology as the space of continuous crossed homomorphism, when G and A are locally compact; an interpretation of the second cohomology $H^2_{\mathrm{m}}(G, A)$ in terms of topological extensions of G by A, etc. Here $H^*_{\mathrm{m}}(G, A)$ denotes the Moore cohomology group of a topological group G and a topological G-module A. It further agrees with the van Est continuous cohomology groups, when G is profinite and the coefficient module A is discrete.

The cohomology theory developed by Moore has had numerous applications (for some recent applications and also for further results on Moore cohomology groups, see [1]).

The motivation for us to consider the cohomology theory of topological groups, is to explore the possibility of deploying such theories to the study of the non-abelian reciprocity laws as conjectured by Langlands, just as the continuous cohomology theory of Galois groups has proved to be immensely successful in class field theory. In this context, the analogues of the Galois group like the Weil group W_k attached to a number field k (or the conjectural Langlands group whose finite dimensional representations are supposed to parametrize automorphic representations) are locally compact but not profinite in general. Indeed such a motivation led the second author to generalize a classical theorem of Tate on the vanishing of the Schur cohomology groups to the context of Weil groups [15], to show that $H^2_{\mathrm{m}}(W_k, \mathbb{C}^*)$ vanishes, where we impose the trivial module structure on $\mathbb{C}^*$.

The immediate inspiration for us is the recent work of Lichtenbaum [8], where he outlines deep conjectures explaining the special values of zeta functions of varieties in terms of Weil-étale cohomology. Here the cohomology of the generic fibre turns out to be the cohomology of the Weil group. Lichtenbaum studies the cohomology theory of topological groups from an abstract viewpoint based on the work of Grothendieck [7], where he embeds the category of G-modules in a larger abelian category with sufficiently many injectives. The cohomology groups are then the right derived functors of the functor of invariants (we refer to the paper by Flach [5] for more details and applications to the cohomology of Weil groups). Lichtenbaum imposes a Grothendieck topology on the category of G-spaces, where the covers have local sections. The required abelian category is the category of sheaves with respect to Grothendieck topology. It is interesting to study cohomology theories defined in [11,19, 20].

In this paper, we modify Moore's construction and introduce a new cohomology theory of topological groups. The main new idea is to impose a local regularity condition on the cochains in a neighbourhood of the identity (like continuity or smoothness in the context of Lie groups) but assume the cochains to be measurable everywhere. The basic observation which makes this construction viable is the following: Given a short exact sequence of Lie groups

$$1 \to G' \to G \to G'' \to 1,$$

there is a continuous section from a neighbourhood of identity in G'' to G. More generally, using the solution to Hilbert's fifth problem and the observation for Lie groups, Mostert [14] showed that every short exact sequence of finite dimensional locally compact groups

$$1 \to G' \to G \xrightarrow{\pi} G'' \to 1$$

is locally split i.e., there exists a continuous section from a neighbourhood of the identity in G'' to G.

Let $G,\ A$ be topological groups. Define the group of 0-cochains $C^0(G, A)$ to be A. For $n \geq 1$, define the group $C^n_{\mathrm{lcm}}(G, A)$ of locally continuous measurable cochains to be the space of all Borel measurable functions $f : G^n \to A$ which are continuous in a neighbourhood of the the identity in G^n. The coboundary map is given by the standard formula. Now we define our locally continuous cohomology theory $H^n_{\mathrm{lcm}}(G, A)$ as the cohomology of this cochain complex. These cohomology groups interpolate the continous cohomology and the measurable cohomology theory of Moore. There are natural maps

$$H^n_{\mathrm{cont}}(G, A) \to H^n_{\mathrm{lcm}}(G, A) \to H^n_{\mathrm{m}}(G, A),$$

where $H^n_{\mathrm{cont}}(G, A)$ denotes the continuous cohomology groups of G with values in A.

For the category of Lie groups, we replace continuity with the property of being smooth around the identity and we define the locally smooth measurable cohomology theory (denoted by $\{H^n_{\mathrm{lsm}}(G, A)\}_{n\geq 0}$) of a Lie group G that acts smoothly on A. Similarly, we can define locally holomorphic measurable cohomology theory (denoted as $\{H^n_{\mathrm{lhm}}(G, A)\}_{n\geq 0}$) in the holomorphic category, based on measurable cochains holomorphic in a neighbourhood of the identity.

We remark here that although the cohomology theories developed by Moore, Lichtenbaum and others seem sufficient for many purposes, the richness of applications of cohomology arises from the presence of different cohomology theories that can be compared to each other. The multiplicity of such theories allow the use of cohomological methods in a variety of contexts. In this regard, we expect that the principle of imposing local regularity on the cochains, will allow its use in more geometric and arithmetical contexts. For example, it is tantalizing to explore the relationship of these theories to the measurable Steinberg 2-cocycle [12], which is continuous on a dense open subset (but not at the identity!).

These locally regular cohomology theories can be related to the underlying category theoretic properties of the group and its modules. For example, suppose there is an extension

$$1 \to A \to E \to G \to 1$$

of G by A given by a measurable 2-cocycle. From the construction of this extension (as given by Moore (see page 30 of [13])), it seems difficult to relate the topology of E to that of G and A. If G and A are locally compact, it is a difficult theorem of Mackey that E is locally compact [9]. Another difficulty arises, when we work with a Lie group G and a smooth G-module A. It is not clear when an extension of G by A defined by a measurable 2-cocycle is a Lie group. Further, there does not seem to be any obvious relationship between the Moore cohomology groups and the cohomology groups of the associated Lie algebra and its module.

We now describe some of our results towards establishing the legitimacy of these theories. It can be seen that these locally regular cohomology theories are cohomological, in that there exists long exact sequence of cohomology groups associated to locally split short exact sequences of modules. Further, the zeroth cohomology group is the space A^G of G-invariant elements in A. There exists a natural map

$$H^n_{\mathrm{lcm}}(G, A) \to H^n_{\mathrm{m}}(G, A).$$

When G and A are Lie groups and the G-action is smooth, the following are natural maps between cohomology groups

$$H^n_{\mathrm{lsm}}(G, A) \to H^n_{\mathrm{lcm}}(G, A) \to H^n_{\mathrm{m}}(G, A).$$

For any topological group G and continuous G-module A, the first cohomology group

$$H^1_{\mathrm{lcm}}(G, A) = \frac{\{c : G \to A \mid c(st) = c(s) + s \cdot c(t),\ c \text{ is continuous}\}}{\{c_a : G \to A \mid a \in A, c_a(s) = s \cdot a - a\}}.$$

When G, A are locally compact second countable topological groups, it follows by a theorem of Banach that measurable crossed homomophisms from G to A are continuous. Thus we have

$$H^1_{\mathrm{m}}(G, A) = H^1_{\mathrm{lcm}}(G, A) = H^1_{\mathrm{cont}}(G, A).$$

For a Lie group G and smooth G-module A (a smooth G-module A is an abelian Lie group such that the action $G \times A \to A$ is smooth) which is locally compact and second countable, the first cohomology group agrees with Moore cohomology group

$$H^1_{\mathrm{m}}(G, A) = H^1_{\mathrm{lcm}}(G, A) = H^1_{\mathrm{lsm}}(G, A)$$

which is the group of all smooth crossed homomorphisms from G to A. The advantage of working with locally regular cohomology can be seen in the holomorphic category.

PROPOSITION 1

Given a complex Lie group G and a holomorphic G-module A, $H^1_{\mathrm{lhm}}(G, A)$ is the group of all holomorphic crossed homomorphisms from G to A.

The main theorem of this paper is to show that the second cohomology group $H^2_{\mathrm{lcm}}(G, A)$ for locally compact second countable G and A, parametrizes all the locally split extensions of G by A .

Theorem 1. *If G, A are both locally compact, second countable topological groups, the second cohomology group, $H^2_{\mathrm{lcm}}(G, A)$ parametrizes all the isomorphism classes of extensions E of G by A,*

$$1 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1$$

which are locally split.

It is this theorem that confirms our expectation that these locally regular cohomology theories can be a good and potentially useful cohomology theory for topological groups. Our other attempts to construct suitable cohomology theories failed to give a suitable interpretation for the second cohomology group. It will also be interesting to compare the cohomology theory that we construct with the construction given by Segal [17], where he shows that the second cohomology group does parametrize equivalence classes of locally split extensions.

The proof of this theorem is a bit delicate. Given a locally continuous measurable 2-cocycle, we construct an abstract extension group E. We topologize E by first defining the product topology in a sufficiently small 'tubular' neighbourhood of the identity in E, and by imposing the condition that left translations are continuous. To conclude that E is a topological group, we need to show that inner conjugation by any element of E is continuous at identity. For this, we follow the idea of the proof of Banach's theorem that a measurable homomorphism of locally compact second countable groups is continuous. We find that the proof of Banach's theorem extends perfectly to prove that inner conjugation by an aribitrary element of E is continuous in a neighbourhood of the identity in E.

In the smooth category, we have the following analogue of Theorem 1.

Theorem 2. *Let G be a Lie group, A be a smooth G-module. The second cohomology groups,* $H^2_{\mathrm{lsm}}(G, A)$ *parametrizes all the locally split smooth extensions of G by A.*

Further, as a consequence of the positive solution to Hilbert's fifth problem, we have a comparison theorem as follows:

Theorem 3. *Let G be a Lie group, A be a smooth G-module. Then the natural map,*

$$H^2_{\mathrm{lsm}}(G, A) \to H^2_{\mathrm{lcm}}(G, A)$$

is an isomorphism.

The locally smooth measurable cohomology groups can be related to Lie algebra cohomology. This is easily done via the cohomology theory $H_\square(G, A)$ based on germs of smooth cochains defined in a neighbourhood of the identity developed by Świerczkowski (page 477 of [18]). We have a restriction map,

$$H^n_{\mathrm{lsm}}(G, A) \to H^n_\square(G, A),$$

given by restricting a locally smooth measurable cochain to a neighbourhood of the identity in G^n where it is smooth.

Now suppose G acts on a finite dimensional real vector space V. Let L denote the Lie algebra associated to the Lie group G and let $H(L, V)$ denote the Lie algebra cohomology. It has been proved (Theorem 2 of [18]) that

$$H_\square(G, V) \simeq H(L, V).$$

2. Basic theory

We first establish some basic properties of the locally continuous measurable cohomology groups defined in the Introduction. From the definitions, it is clear that

$$H^0_{\mathrm{lcm}}(G, A) = A^G$$

is the space of G-invariants in A. Further, these cohomology groups lie between the continuous cohomology and the measurable cohomology, i.e., there are natural maps,

$$H^n_{\mathrm{cont}}(G, A) \to H^n_{\mathrm{lcm}}(G, A) \to H^n_{\mathrm{m}}(G, A).$$

2.1 *Change of groups*

Let $A,\ A'$ be topological modules for G, G' respectively. Suppose there are continuous homomorphisms $\phi\colon G' \to G$, $\psi\colon A \to A'$ satisfying the following compatibility condition:

$$\begin{array}{ccc} G \times A & \longrightarrow & A \\ \phi\big\uparrow \quad \big\downarrow\psi & & \big\downarrow\psi \\ G' \times A' & \longrightarrow & A' \end{array},$$

$$g' \cdot \psi(a) = \psi(\phi(g') \cdot a), \quad \forall a \in A,\ g' \in G'.$$

Then there is a map of cohomology groups

$$H^n_{\rm lcm}(G, A) \to H^n_{\rm lcm}(G', A').$$

In particular, this gives functorial maps for $G = G'$,

$$H^n_{\rm lcm}(G, A) \to H^n_{\rm lcm}(G, A').$$

For $G' = H$, a subgroup of G, we have the restriction homomorphism

$$H^n_{\rm lcm}(G, A) \to H^n_{\rm lcm}(H, A).$$

2.2 *Locally split short exact sequences*

DEFINITION 1

We first recall that a short exact sequence of topological groups

$$\{1\} \to G' \xrightarrow{\imath} G \xrightarrow{\jmath} G'' \to \{1\}$$

is an algebraically exact sequence of groups with additional property that $\imath$ is a closed and $\jmath$ is open. It is said to be *locally split*, if the homomorphism $\jmath$ admits a local section, i.e., there exists an open neighbourhood U'' of the identity in G'' and a continuous map $\sigma: U'' \to G$ such that $j \circ \sigma = id_{U''}$.

The following lemma follows by an easy application of Zorn's lemma:

Lemma 1. *Consider a locally split short exact sequence of topological groups* $1 \to G' \xrightarrow{\imath} G \xrightarrow{\jmath} G'' \to 1$. *If the three groups are Polish, then there exists a measurable section* $\sigma : G'' \to G$ *which is continuous in a neighbourhood of the identity on* G''.

COROLLARY 1

Consider a locally split short exact sequence of topological G-modules, $0 \to A' \to A \to A'' \to 0$. *Assume that the group G is Polish and the G-modules* A, A', A'' *are Polish*, *then there is a short exact sequence of cochain complexes*,

$$0 \to C^*_{\rm lcm}(G, A') \xrightarrow{\tilde{\imath}} C^*_{\rm lcm}(G, A) \xrightarrow{\tilde{\jmath}} C^*_{\rm lcm}(G, A'') \to 0.$$

Hence, *there is a long exact sequence of locally continuous measurable cohomology groups*,

$$0 \to H^0_{\rm lcm}(G, A') \to H^0_{\rm lcm}(G, A) \to H^0_{\rm lcm}(G, A'') \xrightarrow{\delta} H^1_{\rm lcm}(G, A') \to \cdots .$$

Proof. Since the construction of the connecting homomorphism provided us with the initial idea to our construction of cohomology theories, we briefly indicate the construction of the connecting homomorphism

$$\delta : H^n_{\rm lcm}(G, A'') \to H^{n+1}_{\rm lcm}(G, A').$$

We choose a locally continous measurable section $\sigma : A'' \to A$ as given by Lemma 2.2. The connecting homomorphism is defined as

$$\delta(s) = d(\sigma \circ s), \quad s \in Z^n_{\mathrm{lcm}}(G, A'').$$

This gives a well-defined cocycle with values in A', and the cohomology class defined by this cocycle is independent of the choice of the section σ. □

2.2.1 *Finite dimensional groups*

We recall the definition of a finite dimensional topological space and topological group:

DEFINITION 2

A topological space X has finite topological dimension k, if every covering $\mathcal{U}$ of X has a refinement $\mathcal{U}'$ in which every point of X occurs in at most $k+1$ sets in $\mathcal{U}'$, and k is the smallest such integer. Finite dimensional topological groups are topological groups that have finite dimension as a topological space.

The following theorem due to Mostert (page 647 of [14]) provides examples of locally split short exact sequences:

Theorem. *Let G be a finite dimensional locally compact group and H be a closed normal subgroup of G. Then G/H admits a local cross-section.*

This theorem is obvious when G is a Lie group, and it follows from the fact that that any finite dimensional locally compact group is an inverse limit of Lie groups. For detailed proof, refer to [14].

2.2.2 *The first cohomology group*

PROPOSITION 2

Suppose G is a topological group and and A is a topological G-module. Then

$$Z^1_{\mathrm{lcm}}(G, A) = Z^1_{\mathrm{cont}}(G, A)$$

is the space of continuous crossed homomorphisms from G to A.

Proof. A locally continuous measurable 1-cocycle is a measurable function $c : G \to A$ satisfying the cocycle condition

$$c(s_1 s_2) = s_1 \cdot c(s_2) + c(s_1), \qquad \forall s_1, s_2 \in G.$$

Further, there exists an open set $U \subset G$ containing the identity such that $c|_U$ is continuous. For any $x \in G$ arbitrary, the map $c|_{xU}$ satisfies the following formula.

$$c(xs) = x \cdot c(s) + c(x), \qquad \text{for all } s \in U.$$

Since the group action is continuous and the map of translation by $c(x)$ is continuous on A, we see that c is continuous on xU. □

Remark 1. This holds in greater generality in the context of the measurable cohomology groups constructed by Moore. Using Banach's theorem that any measurable homomorphism between two polish groups is continuous, it can be seen that if G and A are locally compact and G acts continuously on A, then the first measurable cohomology group is the group of all continuous crossed homomorphisms from G to A.

2.3 *Extensions, other constructions*

We first recall Moore's construction of an extension of G by A corresponding to a measurable 2-cocycle. Suppose that $G,\ A$ are Polish topological groups and A is a topological G-module. Denote by $I(A)$ the group of measurable maps from G to A.

Since A is Polish, there exists a metric ρ on A whose underlying topology is the same as the original topology on A. Further, we can assume that ρ is bounded. We take a finite measure $\mathrm{d}\nu$ on G, which is equivalent to the Haar measure on G [13]. Define a metric on $I(A)$ as follows:

$$\bar{\rho}(f_1, f_2) = \int_G \rho(f_1(x), f_2(x))\mathrm{d}\nu(x).$$

This makes $I(A)$ a Polish group. We define a G-action on $I(A)$ by

$$(s \cdot f)(t) = sf(s^{-1}t), \quad \forall\, s,\ t \in G,\ f \in U(G, A).$$

Via this action $I(A)$ becomes a topological G-module and A embeds as a G-submodule of $I(A)$ as submodule of the constant maps. It can be seen that the higher measurable cohomology groups of $I(A)$ are trivial [13]. We have a short exact sequence,

$$0 \to A \to I(A) \to U(A) \to 0.$$

Moore showed that the second measurable cohomology group parametrizes the collection of extensions.

PROPOSITION 3

Suppose a Polish group G acts continuously on an abelian Polish group A. Then $H^2_{\mathrm{m}}(G, A)$ parametrizes the isomorphism classes of topological extensions of G by A,

$$1 \to A \to E \to G \to 1.$$

Proof. Given a topological extension E of G by A, the construction of the 2-cocycle corresponding to the extension is done (as before) using the existence of the measurable cross-section guaranteed by the theorem of Dixmier-Mackey.

For our purpose, we briefly recall the proof of the converse. From the short exact sequence, and the cohomological triviality of $I(A)$, we obtain an isomorphism

$$H^2_{\mathrm{m}}(G, A) \simeq H^1_{\mathrm{m}}(G, U(A)).$$

Corresponding to a 2-cocycle $b \in Z^2_{\mathrm{m}}(G, A)$, we obtain a measurable crossed homomorphism, i.e., a continuous homomorphism $T\ :\ G \to:\ U(A) \rtimes G$. Let $\mathcal{E}$ be the image of $T(G)$. We have a short exact sequence

$$0 \to A \to I(A) \rtimes G \to U(A) \rtimes G \to 0.$$

The required extension group E is obtained as the inverse image of $\mathcal{E}$ in $I(A) \rtimes G$. We equip the group E with subspace topology of $I(A) \rtimes G$. It can be verified that E is a closed subgroup of $I(A) \rtimes G$, and hence it is a Polish group. □

Remark 2. From this construction, it does not seem possible to directly relate the topology of E to that of G and A; for example, if G and A are locally compact, will E be locally compact? This question was already answered in the affirmative by Mackey [9], but the proof is neither easy nor direct.

A similar problem arises when we work with Lie groups and we want to relate the manifold structure on E to that of G and A. This provides us another motivation (apart from the work of Lichtenbaum) for the construction of a cohomology theory based on measurable cochains which are continuous (or more generally are regular in a suitable sense) in a neighbourhood of the identity.

Remark 3. It is possible to construct other cohomology theories imposing different conditions on the nature of the cochains. For example, the proof of Lemma 2.2 can be modified to extend a continuous section to a dense, open subset of G''. We can then construct a cohomology theory based on continuous cochains defined on dense open subsets of products of G (or even measurable cochains which are continuous on dense open subsets of products of G). However, such a cohomology theory does not have restriction maps to subgroups in general. Further, it seems difficult to relate the second cohomology group (based on continuous cochains defined on dense open subsets of products of G) to extensions of G.

Another construction can be based on set theoretic cochains which are continuous in a neighbourhood of the identity of G^n. But here again, the second cohomology group does not seem to correspond to extensions of G having local sections.

Remark 4. It is possible to introduce reduced locally continuous cohomology groups just as in [13] by considering two locally continuous measurable cochains to be equivalent if they differ on a set of measure zero on G. We do not study these cohomology groups out here, nor the comparison of the cohomology groups constructed out here with the other theories already in existence.

3. Locally split extensions and $H^2_{\mathrm{lcm}}(G, A)$

Let $G,\ A$ be locally compact second countable topological groups, and assume that A is a continuous G-module. Our aim here is to prove Theorem 1, giving a bijective correspondence between the second cohomology group $H^2_{\mathrm{lcm}}(G, A)$ and equivalence classes of *locally split* extensions E of G by A, i.e., those extensions for which there exists a continuous section for the map $\pi : E \to G$ in some neighbourhood of the identity.

Consider a locally split extension E of G by A. We now associate a unique cohomology class in $H^2_{\mathrm{lcm}}(G, A)$. By Lemma 1, choose a measurable section $\sigma : G \to E$ which is continuous in a neighbourhood of the identity in G. Define $f_\sigma : G \times G \to A$ as $f_\sigma(s_1, s_2) = \sigma(s_1)\sigma(s_2)\sigma(s_1 s_2)^{-1}$. It can be verified that f_σ satisfies the 2-cocycle condition, is continuous in a neighbourhood of the identity in $G \times G$ and defines a class in $H^2_{\mathrm{lcm}}(G, A)$ independent of the choice of the section σ.

Conversely, given a measurable 2-cocycle $F\colon G \times G \to A$ which is continuous on a neighbourhood $U_F \times U_F \subset G \times G$ of the identity. Since it is an abstract 2-cocycle, we get an abstract extension

$$1 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1.$$

In order to topologize E, we define a base $\mathcal{B}$ for the neighbourhoods of the identity in E that consists of sets of the form $U_A \times U_G$, where U_A and U_G are open neighbourhoods of the identity in A and G respectively, such that $F|_{U_G \times U_G}$ is continuous. It is easy to see that $\mathcal{B}$ is a filter base in the terminology of Bourbaki (Chapter 1, Section 6.3 of [2]). Let us call any subset of E, containing some member of $\mathcal{B}$, as a neighbourhood of the identity in E.

We topologize E by considering the left translates $x\mathcal{B}$ as a base for the open neighbourhoods of $x \in E$. With this topology left multiplication by any element $x \in E$ is a continuous map from E to E. It is easy to observe the following proposition listing some basic properties of the topological space E.

PROPOSITION 4

(i) *The homomorphisms ι, π are continuous and π is an open map.*
(ii) *There exists an open neighbourhood $U_F \subset G$ of the identity in G, and a continuous section $\sigma\colon U_F \to E$.*
(iii) *The inclusion $\iota\colon A \to E$ is a homeomorphism onto its image and $\iota(A)$ is a closed subset of E.*
(iv) *E is a locally compact, second countable, Hausdorff space.*
(v) *The Borel algebra on E is generated by members of the filters $\cup_{x\in E} x\mathcal{B}$. Moreover, the measure structure on E is product of measure structures on G and A.*
(vi) *The group law and the inverse map on E are Borel measurable functions. Hence, the map $\iota_x\colon E \to E$ of inner conjugation by any $x \in E$ is Borel measurable.*

Proof. Since E is second countable, its Borel measurable sets are generated by small open sets, namely the members of $\cup_{x\in E} x\mathcal{B}$. We observe the formulae for group law and the inverse map:

$$(a_1, s_1)(a_2, s_2) = (a_1 + s_1 \cdot a_2 + F(s_1, s_2),\ s_1 s_2),$$

$$(a, g)^{-1} = (s^{-1} \cdot (-a) + s^{-1}(-F(s, s^{-1})),\ e_G).$$

The cocycle, $F : G \times G \to A$ is measurable, and G, A are topological groups. Therefore, the group law and the inverse map are measurable on the product measure space $\mathcal{M}_A \times \mathcal{M}_G$ which is the Borel measure space $\mathcal{M}_E$ on E. □

Since the cocycle F is continuous in a neighbourhood of the identity, it can be verified that the multiplication map $E \times E \to E$ (resp. the inverse map $E \to E$) are continuous in a neighbourhood of the identity. From Proposition 1 of Bourbaki (Chapter 3, Section 1.2, page 221 of [2]), for E to be a topological group with $\mathcal{B}$ as a base for the neighbourhoods at identity, it is necessary and sufficient that inner conjugation by any element $a \in E$ is continuous at identity: for $a \in \tilde{E}$ and any $V \in \tilde{\mathcal{B}}$, there exists $V' \in \tilde{\mathcal{B}}$ such that $V' \subset a \cdot V \cdot a^{-1}$. We single this out as a theorem.

Theorem 4. *Let E be an extension of the group G by A corresponding to the 2-cocycle $F : G \times G \to A$ and is provided with the neighbourhood topology $\mathcal{B}$ defined above. Then for any $x \in E$, the map of inner conjugation $\iota_x : E \to E$ is continuous at identity.*

The proof of this theorem is modelled on the proof of Banach's theorem that measurable homomorphims of second countable locally compact topological groups are continuous. Heuristically, this can be considered as saying that the topology of a locally compact group can be recovered from the underlying measure theory. Our proof of the above theorem makes this heuristic precise.

In our situation, E can be equipped with a measure structure since the cocycle F is measurable. We topologize E with a neighbourhood base filter $\mathcal{B}$, imposing the condition that left translation is continuous. We show that there exists a left invariant measure on E. This will allow us to define convolution of measurable functions. We then use the fact that the multiplication and the inverse maps are continuous in a neighbourhood of the identity e, together with a global argument to prove the above theorem.

3.1 *Construction of left invariant measure on E*

For the construction of the left invariant integral on E, we follow the standard method of constructing the Haar measure, for example as given in Chapter 3, Section 7 of [4]. In our setting, E is not a topological group, but only a group equipped with a topology which is locally compact. However the proof goes through, and the only change that is required is given by Lemma 3.1, analogous to the uniform continuity lemma given by Chapter 2, Proposition 1.9 of [4].

Let $C_c(E)$ denote the space of real valued continuous functions with compact support on E, and $C_c^+(E) \subset C_c(E)$ the subspace of functions taking nonnegative real values. We denote by f, g, h the functions in $C_c(E)$. For a function f on E and $u \in E$, let $f_u(x) = f(ux),\ x \in E$ denote the left translation of f by u.

Lemma 2. *Let E be as in Proposition 4, and let $f,\ g \in C_c^+(E)$. Let $g \neq 0$ with nonnegative values. Then there exist finitely many positive real numbers $c_1, c_2, \ldots, c_r$ and elements $u_1, u_2, \ldots, u_r \in E$ such that*

$$f \leq c_1 g_{u_1} + c_2 g_{u_2} + \cdots + c_r g_{u_r}, \tag{1}$$

where $g_{u_i} : E \to \mathbb{R}$ is defined as $g_{u_i}(s) = g(u_i s)$, for all $s \in E$.

The proof follows from the compactness of the support of f. This allows us to define the following:

DEFINITION 3

Suppose $f, g \in C_c^+(E)$ are as above, we define the approximate integral of f relative to g as

$$(f; g) = \inf \left\{ \sum_{i=1}^{r} c_i \right\},$$

where the tuple $(c_1, c_2, \ldots, c_r\}$ runs over all the finite sequences of non-negative numbers for which there exist group elements $u_1, u_2, \ldots, u_r$ satisfying the proposition above. By linearity, we define $(f; g)$ for any $f \in C_c(E)$.

DEFINITION 4

Fix a compactly supported function $g : E \to \mathbb{R}^+$. If $f, \phi \in C_c^+(E)$ and $\phi \neq 0$, define $I_\phi(f) = (g; \phi)^{-1}(f; \phi)$.

It can be seen that the approximate integral $I_\phi(f)$ satisfies the following properties. The arguments are similar and follows from analogous properties satisfied by $(f; g)$ (see Chapter 3, Lemma 7.4 and page 202 of [4]).

Lemma 3. *Let $f, f_1, f_2, \in C_c^+(E)$. Then*

(i) *If $f \neq 0$, then $(g; f)^{-1} \leq I_\phi(f) \leq (f; g)$;*
(ii) $I_\phi(f_x) = I_\phi(f)$ for all $x \in G$;
(iii) $I_\phi(f_1 + f_2) \leq I_\phi(f_1) + I_\phi(f_2)$;
(iv) $I_\phi(cf) = cI_\phi(f)$ *for all* $c \in \mathbb{R}_{\geq 0}$.

We now need to show that, if ϕ has a small compact support, I_ϕ is 'nearly additive'. For this purpose, we require a lemma on uniform continuity, the analogue of Corollary 1.10, page 167 of [4], whose proof we give since we do not yet have an uniform structure on E.

Lemma 4. *Let f be a real valued continuous function on E and $\epsilon > 0$. Suppose C is a compact subset of E. Then, there is a neighbourhood V of e such that*

$$|f(x) - f(y)| < \epsilon \quad \text{whenever } x,\ y \in C,\ x^{-1}y \in V.$$

Proof. Suppose the lemma is not true. Then there exists $\epsilon > 0$, a sequence V_i of neighbourhoods of the identity e in E with $\cap V_i = \{e\}$, elements $x_i,\ y_i \in C$ with $x_i^{-1}y_i \in V_i$, such that

$$|f(x_i) - f(y_i)| \geq \epsilon.$$

We can assume that $V_{i+1}V_{i+1} \subset V_i$. Since $x_i,\ y_i$ are in a compact set C, we can assume by passing to a subsequence, that the sequence x_i (resp. y_i) converges to x_0 (resp. y_0). Since f is continuous,

$$|f(x_0) - f(y_0)| \geq \epsilon.$$

Since, $\{x_i\}$ converges to x_0, choose $N_k \geq k+1$ such that $x_i \in x_0 V_{k+1}$ for $i \geq N_k \geq k+1$. Now,

$$y_i \in x_i V_i \subset (x_0 V_{k+1}) \cdot V_i.$$

Since $i \geq k+1$,

$$(x_0 V_{k+1}) \cdot V_i \subset (x_0 V_{k+1}) \cdot V_{k+1} \subset x_0 V_k.$$

Therefore, $y_i \in x_0 V_k$ whenever$i \geq N_k$. Hence the sequence $\{y_i\}$ converges to x_0. Since f is continuous, this gives a contradiction. □

We now state the condition that I_ϕ is nearly additive, when ϕ has sufficiently small compact support.

Lemma 5. *Given $f_1, f_2 \in C_c^+(E)$ and $\epsilon > 0$, we can find a neighbourhood V of e such that, if $\phi \in C_c^+(E)$ is non-zero and* $\text{supp}(\phi) \subset V$, *then*

$$|I_\phi(f_1) + I_\phi(f_2) - I_\phi(f_1 + f_2)| \leq \epsilon. \tag{2}$$

The proof is similar to the proof of the Lemma in Chapter 3, 7.7 of [4], only we use Lemma 4 in place of the lemma on uniform continuity Chapter 3, 7.1 of [4] available for a locally compact topological group, and we skip the details.

From Lemma 3 we also have

$$(g; f)^{-1} \leq I_\phi(f) \leq (f; g)$$

whenever $g \neq 0$ and f, ϕ, g are compactly supported real valued functions on E. This combined with an application of Tychnoff's theorem and well known arguments give the existence of the left invariant integral.

PROPOSITION 5

There exists a non-zero left invariant integral on E.

Remark 5. This integral defines a left invariant measure on E. The integral is positive, i.e., $I(f) > 0$, whenever $0 \neq f \in C_c^+(E)$. Let μ be the Borel measure on E, corresponding to the left invariant integral I on E. If $x \in G$, we see that

$$\mu(W) = \mu(xW)$$

where W is a Borel subset of E whose closure is compact. Further, if K is any compact subset of E then $\mu(K) < \infty$.

3.2 *Global argument*

We now derive a consequence of the existence of a non-trivial left invariant integral on E. We start with a general observation with Lindelöf property. We recall that a topological space X is said to satisfy Lindelöf property, if every open cover X admits a countable subcover.

Lemma 6. *Let G_1, G_2 be two groups, and $f : G_1 \to G_2$ be a group homomorphism. Suppose that G_1, G_2 are topological spaces, and G_2 satisfies Lindelöf property. Assume further that there exist non-zero left invariant measures μ_1 (resp. μ_2), on the Borel subalgebra of G_1 (resp. of G_2).*

Let f be measurable and $W \subset G_2$ be an arbitrary open subset. Then

- *The measure $\mu_2(W) > 0$.*
- *If f is surjective, the measure of the preimage $\mu_1(f^{-1}(W)) > 0$.*

Proof. Since W is open in G_2 and G_2 is Lindelöf, there exist countably many left translates $\{t_i W\}_{i \in \mathbb{N}}$ which cover G_2. Since the measure is left invariant and non-zero, it follows that the measure of W is positive.

Since f is surjective, there exist elements $s_i \in G_1$, such that $f(s_i) = t_i$. Since f is measurable, the inverse image $f^{-1}(W)$ is a measurable subset of G_1. Since $\{t_i W\}_{i \in \mathbb{N}}$ cover G_2, the collection $\{s_i f^{-1}(W)\}_{i \in \mathbb{N}}$ covers G_1. Since the measure μ_1 is non-zero on G_1, it follows that $\mu_1(f^{-1}(W)) > 0$. □

We apply this global argument when $E = G_1 = G_2$ with $f = \iota_x$ inner conjugation by an element $x \in E$.

COROLLARY 2

Let E be as above and μ denote the left invariant measure constructed in the foregoing subsection. Let W be an open subset of E and x an element of E. Then

$$\mu(i_x^{-1}(W)) > 0.$$

3.3 *Convolution*

The proof of Banach's theorem for locally compact groups proceeds by first showing that convolution of measurable functions satisfying suitable properties is continuous. In our context, we can carry out such an argument for measurable functions supported in a sufficiently small neighbourhood of the identity in E. However, here we establish directly a statement that suffices for proving Theorem 4. The proof makes more use of symmetric subsets, has the advantage of simplifying the required arguments in our context by reducing the requirement of uniform continuity to Lemma 4. The key proposition is the following:

PROPOSITION 6

Let M be a measurable, symmetric (i.e. $M = M^{-1}$) subset of E. Suppose that $M \subset \pi^{-1}(U_2)$, for U_2 a symmetric relatively compact open neighbourhood of the identity in G such that the product of the closures $\overline{U}_2\overline{U}_2 \subset U_F$. Assume that the identity $e \in M$ and measure $\mu(M)$ is positive and finite. Then the set

$$MM = \{xy \colon x \in M, y \in M\}$$

contains an open neighbourhood of the identity in E.

Granting this proposition, we now prove Theorem 4.

Proof of Theorem 4. We need to show that for any sufficiently small neighbourhood V of the identity in E the set $\iota_x^{-1}(V)$ contains an open neighbourhood of the identity $e \in E$. Let W be a symmetric open neighbourhood of e in E satisfying the following:

(i) W is symmetric (i.e., $W = W^{-1}$);
(ii) $W \subset \pi^{-1}(xU_2x^{-1})$;
(iii) $WW \subset V$.

Let $M' = \iota_x^{-1}(W)$. By Corollary 2, $\mu(M') > 0$. Since $W \subset \pi^{-1}(xU_2x^{-1})$, we have $\iota_x^{-1}(W) \subset \pi^{-1}(U_2)$. Intersecting wih a symmetric compact set K containing the identity

$e \in E$, we can assume that $M = \iota_x^{-1}(W) \cap K$ has finite, positive measure, and is contained inside $\pi^{-1}(U_2)$. By Proposition 6, we see that the product set MM contains an open neighbourhood V'_x of e. Now

$$\iota_x^{-1}(V) \supseteq \iota_x^{-1}(WW) \supset MM \supset V'_x.$$

This proves Theorem 4. □

We now proceed to the proof of Proposition 6. For $x \in E$, define the function

$$u(x) = \mu(M \cap xM).$$

The proof of the proposition reduces to the following lemma.

Lemma 7. *Under the hypothesis of Proposition* 6, *u is a continuous function.*

Assuming Lemma 7, we now prove Proposition 6.

Proof of Proposition 6. If $u(x) \neq 0$, then $M \cap xM \neq \emptyset$. Hence $x \in MM^{-1} = MM$ as M is assumed to be symmetric. Further, $u(e) = \mu(M) > 0$. Since u is continuous, this proves Proposition 6. □

We now proceed to the proof of Lemma 7.

Proof of Lemma 7. Let χ_M denote the characteristic function of M. Then $u(x)$ can be defined by the following integral:

$$u(x) = \mu(M \cap xM) = \int_{M \cap xM} \mathrm{d}\mu(y) = \int_M \chi_M(x^{-1}y)\mathrm{d}\mu(y)$$
$$= \int_E \chi_M(y)\chi_M(x^{-1}y)\mathrm{d}\mu(y).$$

We observe that support of u is contained inside $MM \subseteq U_2U_2$. Since $M \subset \pi^{-1}(U_2)$, by Lusin's theorem [16] choose a function $f \in C_C(E)$ with support contained inside $\pi^{-1}(\overline{U}_2)$ such that

$$\int_E |\chi_M(y) - f(y)|\, \mathrm{d}\mu(y) < \epsilon_1,$$

for some sufficiently small $\epsilon_1 > 0$.

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence converging to x_0 in $\pi^{-1}(U_2U_2)$. To show the continuity of u restricted to $\pi^{-1}(U_2U_2)$, it is enough to show that the sequence $\{u(x_n)\}$ converges to $u(x_0)$. We have

$$|u(x_n) - u(x_0)| = \left| \int_E \chi_M(y)(\chi_M(x_n^{-1}y) - \chi_M(x_0^{-1}y))\mathrm{d}\mu(y) \right|.$$

Since M is symmetric, we also have $\chi_M(y^{-1}) = \chi_M(y)$. Therefore,

$$|u(x_n) - u(x_0)| \leq \int_E \chi_M(y)|\chi_M(y^{-1}x_n) - \chi_M(y^{-1}x_0)|\mathrm{d}\mu(y).$$

We have

$$|u(x_n) - u(x_0)| \le \int_E \chi_M(y)|\chi_M(y^{-1}x_n) - \tilde{f}(y^{-1}x_n)|\mathrm{d}\mu(y)$$
$$+ \int_E \chi_M(y)|\tilde{f}(y^{-1}x_n) - \tilde{f}(y^{-1}x_0)|\mathrm{d}\mu(y)$$
$$+ \int_E \chi_M(y)|\tilde{f}(y^{-1}x_0) - \chi_M(y^{-1}x_0)|\mathrm{d}\mu(y),$$

where $\tilde{f}(z) = f(z^{-1})$ for $z \in E$. Since the integral is left invariant, by replacing y by $x_n y$ (resp. by $x_0 y$) in the first (resp. third) term on the right, we see that

$$\int_E \chi_M(y)|\chi_M(y^{-1}x_n) - \tilde{f}(y^{-1}x_n)|\mathrm{d}\mu(y)$$
$$= \int_E \chi_M(x_n y)|\chi_M(y^{-1}) - \tilde{f}(y^{-1})|\mathrm{d}\mu(y)$$
$$\le \int_E |\chi_M(y^{-1}) - \tilde{f}(y^{-1})|\mathrm{d}\mu(y). = \int_E |\chi_M(y) - f(y)|\mathrm{d}\mu(y).$$
$$< \epsilon_1.$$

Here we have used the definition of $\tilde{f}$ and the fact that M is symmetric. Similarly, we obtain

$$\int_E \chi_M(y)|\tilde{f}(y^{-1}x_0) - \chi_M(y^{-1}x_0)|\mathrm{d}\mu(y) < \epsilon_1.$$

Now we estimate the middle term. Since the inverse map is continuous in $\pi^{-1}(U_F)$ and support of f is contained inside $\pi^{-1}(U_F)$, the function $\tilde{f}(y) = f(y^{-1})$ is continuous. Given $\epsilon_2 > 0$, by Lemma 4, there exists a symmetric neighbourhood W_2 contained inside $\pi^{-1}(U_F)$ (here again we are using the fact that the inverse map is continuous on $\pi^{-1}(U_F)$) such that

$$|\tilde{f}(z_1) - \tilde{f}(z_2)| < \epsilon_2, \quad \text{for } z_1^{-1}z_2 \in W_2.$$

Since x_n converges to x_0, there exists a natural number N such that for $n \ge N$,

$$x_n \in x_0 W_2, \quad \text{i.e. } x_0^{-1}x_n \in W_2.$$

Since W_2 is symmetric, this condition can be rewritten as

$$x_n^{-1}x_0 \in W_2.$$

Hence,

$$(y^{-1}x_n)^{-1}(y^{-1}x_0) = x_n^{-1}x_0 \in W_2.$$

By applying Lemma 4 to the continuous function $\tilde{f}$, we obtain

$$|\tilde{f}(y^{-1}x_n) - \tilde{f}(y^{-1}x_0)| \leq \epsilon_2, \quad \text{for } n \geq N.$$

Hence for $n \geq N$, the middle term can be estimated as

$$\int_E \chi_M(y)|\tilde{f}(y^{-1}x_n) - \tilde{f}(y^{-1}x_0)|\mathrm{d}\mu(y) < \epsilon_2 \int_E \chi_M(y) < \epsilon_2\mu(M).$$

Combining the above estimates, we obtain

$$|u(x_n) - u(x_0)| < 2\epsilon_1 + \epsilon_2\mu(M), \quad \text{for all } n \geq N.$$

This establishes continuity of u and hence Lemma 7 is proved. □

3.4 *Comparison with other cohomology theories*

Suppose G is a locally compact group acting on a locally compact group A. Then we have a natural map

$$H^2_{\mathrm{lcm}}(G, A) \to H^2_{\mathrm{m}}(G, A).$$

As a corollary to Theorem 4, we show that the above map is injective:

COROLLARY 3

Let G, A be locally compact, second countable groups. Then the natural map

$$H^2_{\mathrm{lcm}}(G, A) \to H^2_{\mathrm{m}}(G, A)$$

is injective.

Proof. Suppose a 2-cohomology class c in $H^2_{\mathrm{lcm}}(G, A)$ is trivial in $H^2_{\mathrm{m}}(G, A)$. Construct the corresponding extension E of c. Then $c = 0$ in $H^2_{\mathrm{m}}(G, A)$, implies that there exists a measurable section $\sigma : G \to E$ which is a group homomorphism. By Theorem 3, we know that E is locally compact. It can be seen that E is also second countable. Hence by Banach's theorem, σ is a continuous group homorphism, and this implies that the extension is a semi-direct product $E \simeq A \rtimes G$. □

COROLLARY 4

Suppose that either of the following conditions hold:

(i) *G is a profinite group and A is a discrete G-module.*
(ii) *G is a Lie group and A is a finite dimensional vector space.*

Then the natural maps

$$H^2_{\mathrm{cont}}(G, A) \to H^2_{\mathrm{lcm}}(G, A) \to H^2_{\mathrm{m}}(G, A),$$

are isomorphisms.

Proof. By [13], in the above cases there is an isomorphism

$$H^2_{\text{cont}}(G, A) \to H^2_{\text{m}}(G, A).$$

This implies that the map $H^2_{\text{lcm}}(G, A) \to H^2_{\text{m}}(G, A)$ is surjective, and hence the corollary follows from the previous one. □

4. Cohomology theory for Lie groups

In this section, we work in the smooth category in the context of Lie groups G, A with smooth actions $G \times A \to A$. Here we can define an analogous cohomology theory $H^n_{\text{lsm}}(G, A)$ where we impose the condition that the Borel measurable cochains are smooth in a neighbourhood of the identity of G. It is easy to see that $H^0_{\text{lsm}}(G, A) = A^G$ and the first cohomology group $H^1_{\text{lsm}}(G, A)$ is the group of all smooth crossed homomorphisms from G to A. Further, given a short exact sequence of G-modules

$$0 \to A \xrightarrow{\iota} A \xrightarrow{\jmath} A \to 0,$$

there is a long exact sequence of cohomology groups

$$\cdots \to H^i_{\text{lsm}}(G, A') \to H^i_{\text{lsm}}(G, A) \to H^i_{\text{lsm}}(G, A'') \xrightarrow{\delta} H^{i+1}_{\text{lsm}}(G, A) \to \cdots.$$

Our aim here is to give a direct proof that the second locally smooth measurable cohomology group $H^2_{\text{lsm}}(G, A)$ parametrizes the collection of locally split extensions of G by A.

Theorem 5. *Let G be a Lie group and A be a smooth G-module. Then the second cohomology group $H^2_{\text{lsm}}(G, A)$ parametrizes equivalence classes of extensions E of G by A,*

$$1 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 0,$$

where E is a Lie group and π admits a measurable cross section $\sigma : G \to E$ such that σ is smooth around the identity in G.

Proof. Given an extension E of G by A with a locally smooth measurable cross section $\sigma : G \to E$, we assign to it the 2-cohomology class of $F_\sigma : G \times G \to A \in Z^2_{\text{lcm}}(G, A)$ that takes $(s_1, s_2) \in G \times G$ to $F_\sigma(s_1)F_\sigma(s_2)F_\sigma(s_1 s_2)^{-1}$.

For proving the converse, we take an arbitrary cohomology class $\bar{F} \in H^2_{\text{lsm}}(G, A)$. Choose a representative F and construct an abstract group extension E

$$1 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1.$$

Suppose G^0 is the connected component of the identity in G, we claim that the subgroup $E^0 := \pi^{-1}(G^0) \prec E$ is a Lie group. Since G^0 is a normal subgroup of G and π is surjective, the subgroup E^0 is normal in E. We shall first verify that E^0 is a Lie group (we remark here that we can carry out a similar argument in the continuous case to show directly that E^0 is a topological group, instead of using Theorem 4). We then use the measurability condition to show that the extension E is a Lie group.

Since the cocycle F is smooth in a neighbourhood of the identity, we assume that U_G is sufficiently small so that the following holds:

- The product map

$$(x, y, z) \mapsto xyz$$

  is smooth from $U_G \times U_G \times U_G$ to U_F. This can be ensured by assuming that the following functions are smooth on $U_G \times U_G \times U_G$:

$$(s_1, s_2, s_3) \mapsto F(s_1 s_2, s_3) \quad \text{and} \quad (s_1, s_2, s_3) \mapsto F(s_1, s_2 s_3). \tag{3}$$

- The map $s \mapsto s^{-1}$ is smooth from $\pi^{-1}(U_G)$ to $\pi^{-1}(U_G)$ (here we have assumed U_G is symmetric).

We define an atlas on E^0 by imposing that left translations are diffeomorphisms and imposing the product of smooth structure on $\pi^{-1}(U_G) \simeq A \times U_G$, i.e., the atlas consists of $(xU, \phi \circ L_{x^{-1}})$, where $x \in E^0$ and U is an open subset of $\pi^{-1}(U_G)$. Here (U, ϕ) is a part of the atlas for the product smooth structure on $\pi^{-1}(U_G)$.

We first claim that this gives us an atlas: suppose there exists elements $x,\ y \in E^0$ and open sets $U,\ V$ contained inside $\pi^{-1}(U_G)$ such that $xU \cap yV \neq \emptyset$. By taking the union of U and V, we can assume that $U = V$. We have the charts

$$xU \xrightarrow{L_{x^{-1}}} U \xrightarrow{\phi} W,$$

$$yU \xrightarrow{L_{y^{-1}}} U \xrightarrow{\phi} W,$$

where W is an open subset in some Euclidean space. Let $V = U \cap x^{-1}yU$ be the image of $L_{x^{-1}}(xU \cap yU)$. We need to show that the map

$$\phi \circ L_{y^{-1}} \circ L_x \circ \phi^{-1} : \phi(V) \to W \text{ is smooth.}$$

For this it is enough to show that

$$L_{y^{-1}x} : V \to U \text{ is smooth.}$$

The hypothesis implies that there exists elements $z,\ z'$ in U such that $xz = yz'$, i.e., $y^{-1}x = z'z^{-1}$. This implies that

$$y^{-1}x \in \pi^{-1}(U_G) \times \pi^{-1}(U_G).$$

Hence the required smoothness follows from the assumption that the triple product is smooth from $\pi^{-1}(U_G) \times \pi^{-1}(U_G) \times \pi^{-1}(U_G)$ to $\pi^{-1}(U_F)$. This concludes the proof that E^0 with the above atlas is a smooth manifold. We remark that the manifold structure is such that left translations are diffeomorphisms.

We now have to show that E^0 is a Lie group. For this we first observe that inner conjugation by any element $x \in E^0$ is smooth at identity. Since G^0 is a connected Lie group, the neighbourhood U_G generates G as a group. It follows that the group E^0 is generated by $\pi^{-1}(U_G)$. Hence any element $x \in E^0$ can be written as

$$x = x_1 \ldots x_r, \text{ where each } x_i \in \pi^{-1}(U_G).$$

By our choice of U_G, inner conjugation by any $x_i \in \pi^{-1}(U_G)$ is smooth at identity. Since the inner conjugation by x is a composite of inner conjugations by the elements x_i, it follows that inner conjugation by any element of $x \in E^0$ is smooth at identity.

We now show that the multiplication map $E^0 \times E^0 \rightarrow E^0$ is smooth. Suppose $x, y \in E^0$. Let U be a sufficiently small neighbourhood of the identity in E such that the conjugation map $z \mapsto y^{-1}zy$ is smooth where $z \in U$. Now the multiplication map $xU \times yU$ can be written as

$$(xz)(yz') = (xy)(y^{-1}zy)z'z, \quad z' \in U.$$

We can assume that $U, \ y^{-1}Uy \subset \pi^{-1}(U_G)$. Since left multiplication by xy is smooth, and multiplication is smooth on $\pi^{-1}(U_G) \times \pi^{-1}(U_G)$, we conclude that multiplication is a smooth map from $E^0 \times E^0$ to E^0.

Similarly, to show that the inverse map is smooth on E^0, say around $x \in E^0$, we take U to be a sufficiently small neighbourhood of the identity in E^0 such that $z \mapsto xz^{-1}x^{-1}$, $z \in U$ is smooth on U. (We use the fact that the inverse map is smooth on $\pi^{-1}(U_G)$ and assume that $U \subset \pi^{-1}(U_G)$.) Now

$$(xz)^{-1} = x^{-1}(xz^{-1}x^{-1}), \quad z \in U.$$

As left translations are smooth, it follows that the inverse map is smooth on E^0. This concludes the proof that E^0 is a Lie group.

Now we want to conclude that E is a Lie group. For this, we first show that E is a topological group. Since the cocycle is measurable, we see that inner conjugation i_x by any element $x \in E$ is a measurable automorphism of E^0. By Banach's theorem, it follows that i_x is continuous on E^0 (in particular, it follows that E is a topological group).

Since ι_x is continuous on E^0, the graph of ι_x is closed in $E^0 \times E^0$. Therefore, the graph of ι_x is a closed subgroup of the Lie group $E^0 \times E^0$. Therefore,the graph of ι_x is a Lie group of $E^0 \times E^0$. Therefore, that i_x is a smooth diffeomorphism of E^0. To conclude, we now argue as above that E is a Lie group. Therefore we get the following short exact sequences of topological groups:

$$1 \rightarrow A \xrightarrow{\iota} E \xrightarrow{\pi} G \rightarrow 1.$$

Since E and G are Lie groups with a continuous group homomorphism $\pi : E \rightarrow G$, we see that graph of π is a closed subgroup of $E \times G$ which is a Lie group. Therefore graph of π is a Lie subgroup. This implies that π is smooth. By the implicit function theorem, π admits a smooth cross section in a neighbourhood of the identity. We use arguments similar to those used in proving Lemma 1 and extend this to a locally smooth measurable cross section σ from G to E. This concludes the proof of Theorem 5. □

Remark 6. We remark again that the above arguments do not require the fact that E or E^0 is a topological group. The above arguments, carried out in the continuous category, will directly yield that E^0 is a topological group. We have only used the fact that any neighbourhood of the identity in G generates G as a group and that cocyles are locally regular (locally regular means locally continuous or locally smooth depending on the setting).

4.1 *A comparison theorem*

In this section, as a corollary of positive solution to Hilbert's fifth problem, we show the following:

Theorem 6. *Let G be a Lie group and A be a smooth G-module. Then the natural map,*

$$H^2_{\mathrm{lsm}}(G, A) \to H^2_{\mathrm{lcm}}(G, A),$$

is an isomorphism.

Proof. Let $F : G \times G \to A$ be a locally continuous measurable 2-cocycle on G with values in A. We shall show that F is cohomologous to a locally smooth measurable 2-cocycle b. By Theorem 4, we obtain a locally split (topological) extension E of G by A:

$$1 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1.$$

Denote by E^c the connected component of E containing the identity. Since the extension is locally split and G and A are Lie groups, it follows that E^c is locally Euclidean. Hence by positive solution to Hilbert's fifth problem [6,10,21], we conclude that E^c is a Lie group.

Now the map $\pi|_{E^c} \to G$ is a continuous homomorphism. Hence the graph of $\pi|_{E^c}$ is a closed subgroup of the Lie group $E^c \times G$. Therefore, it is a Lie subgroup and this shows that the projection map $\pi|_{E^c}$ is smooth. Applying the implicit function theorem, we can find a smooth cross section of π in a neighbourhood of the identity on G to E^c. By arguments similar to Lemma 1, we extend this to a measurable section σ from G to E.

The section σ gives rise to a 2-cocycle $b_\sigma : G \times G \to A$ in $Z^2_{\mathrm{lsm}}(G, A)$ given by the formula $b_\sigma(s_1, s_2) = \sigma(s_1)\sigma(s_2)\sigma(s_1 s_2)^{-1}$. One can observe that b_σ is cohomologous to F in $Z^2_{\mathrm{lcm}}(G, A)$. This yields a surjective map

$$H^2_{\mathrm{lsm}}(G, A) \to H^2_{\mathrm{lcm}}(G, A).$$

Next, we claim this map to be injective. Suppose a class $\underline{b} \in H^2_{\mathrm{lsm}}(G, A)$ is trivial in $H^2_{\mathrm{lcm}}(G, A)$. Corresponding to $\underline{b} \in H^2_{\mathrm{lsm}}(G, A)$, by Theorem 5 we obtain a Lie group E which is an extension of G by A. Since $\underline{b} = 0$ in $H^2_{\mathrm{lcm}}(G, A)$, there exists a locally continuous measurable section $\sigma : G \to E$ which is a group homomorphism. Since it is continuous at identity, it is continuous everywhere. Hence we obtain a continuous isomorphism between the Lie groups E and $A \rtimes G$. By an application of the closed graph theorem, this isomorphism is smooth. Therefore, the cohomology class $\underline{b}$ is trivial in $H^2_{\mathrm{lsm}}(G, A)$. Hence it follows that

$$H^2_{\mathrm{lsm}}(G, A) \to H^2_{\mathrm{lcm}}(G, A)$$

is an isomorphism. □

Remark 7. We can introduce an analogous cohomology theory in the holomorphic context based on measurable cochains which are holomorphic in a neighbourhood of the identity. In this context, we observe that the first cohomology group $H^1_{\mathrm{lhm}}(G, A)$ is the space of all holomorphic crossed homomorphisms from G to A (see Proposition 1). Since the closed graph theorem is not applicable in the holomorphic context, we cannot obtain this result from the smooth version by an application of arguments as above. It also remains to understand the higher cohomology groups in the holomorphic context.

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