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On the uniqueness of meromorphic functions that share three or two finite sets on annuli

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Abstract. The purpose of this article is to investigate the uniqueness of meromorphic functions that share three or two finite sets on annuli.

Keywords. Meromorphic function; Nevanlinna theory; the annulus.

1. Introduction and main results

In 1926, Nevanlinna [11] proved his famous five-value theorem:

For two nonconstant meromorphic functions f and g on the complex plane $\mathbb{C}$, if they have the same inverse images (ignoring multiplicities) for five distinct values, then $f(z) \equiv g(z)$.

After this very work, the uniqueness theory of meromorphic functions in $\mathbb{C}$ attracted many investigations (for references, see [13]). For the uniqueness of meromorphic functions in the unit disc, refer to [4]. For the uniqueness of meromorphic function in one angular domain, refer to [14]. However, all the above cases are in simple connected domains. Thus it is very interesting to consider the uniqueness theory of meromorphic functions in multiply connected domains.

Here we shall mainly study the uniqueness of meromorphic functions in doubly connected domains of complex plane $\mathbb{C}$. By the doubly connected mapping theorem [1] each doubly connected domain is conformally equivalent to the annulus $\{z : r < |z| < R\}$, $0 \leq r < R \leq +\infty$. We consider only two cases: $r = 0$, $R = +\infty$ simultaneously and $0 < r < R < +\infty$. In the latter case, the homothety $z \mapsto \frac{z}{\sqrt{rR}}$ reduces the given domain to the annulus $\{z : \frac{1}{R_0} < |z| < R_0\}$, where $R_0 = \sqrt{\frac{R}{r}}$. Thus, in two cases every annulus is invariant with respect to the inversion $z \mapsto \frac{1}{z}$. Hence in this paper, we consider the uniqueness of meromorphic functions in the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \leq +\infty$. We denote by S the subset of distinct elements in $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. For a function f meromorphic in $\mathbb{A}$, we define

$$E(S, f) = \bigcup_{a \in S} \{z \in \mathbb{A} : f(z) - a = 0, \quad \text{counting multiplicity}\},$$

$$\bar{E}(S, f) = \bigcup_{a \in S} \{z \in \mathbb{A} : f(z) - a = 0, \quad \text{ignoring multiplicity}\}.$$

The Nevanlinna characteristic $T_0(r, f)$ of a meromorphic function f on the annulus $\mathbb{A}$ shall be introduced in the next section.

DEFINITION 1.1 [2]

Let f be a nonconstant meromorphic function on the annulus $\mathbb{A}$. The function f is called admissible on the annulus $\mathbb{A}$ provided that

$$\limsup_{r \to \infty} \frac{T_0(r, f)}{\log r} = \infty, \quad 1 \leq r < R_0 = +\infty$$

or

$$\limsup_{r \to R_0} \frac{T_0(r, f)}{-\log(R_0 - r)} = \infty, \quad 1 \leq r < R_0 < +\infty.$$

Cao, Yi and Xu [2] proved a generalized theorem on the multiple values and uniqueness of meromorphic functions in the annulus $\mathbb{A}$, from which an analog of Nevanlinna's five-value theorem was obtained by making use of the annulus version of Nevanlinna theory (see §2). For the special case $R_0 = +\infty$, the assertion was proved by Kondratyuk and Laine [8].

Theorem 1.1 [2]. *Let f and g be two admissible meromorphic functions on the annulus $\mathbb{A}$. Let a_j $(j = 1, 2, 3, 4, 5)$ be 5 distinct complex numbers in $\bar{\mathbb{C}}$. If $\bar{E}(\{a_j\}, f) = \bar{E}(\{a_j\}, g)$ for $j = 1, 2, 3, 4, 5$, then $f(z) \equiv g(z)$.*

Recently, Cao and Yi [3] considered meromorphic functions sharing sets, and obtained two general uniqueness theorems from which uniqueness results of [2] are extended. In this paper, we continue to deal with the uniqueness problem for meromorphic functions in the annulus $\mathbb{A}$. Considering the uniqueness of two meromorphic functions in $\mathbb{A}$ sharing three finite sets, we obtain the first main theorem which is an analog of a result on $\mathbb{C}$ due to Lin and Yi [9].

Theorem 1.2. *Let f and g be two admissible meromorphic function in the annulus $\mathbb{A}$. Put $S_1 = \{0\}$, $S_2 = \{\infty\}$ and $S_3 = \{w : P(w) = 0\}$, where*

$$P(w) = aw^n - n(n-1)w^2 + 2n(n-2)bw - (n-1)(n-2)b^2,$$

$n \geq 5$ is an integer, and a and b are two nonzero complex numbers satisfying $ab^{n-2} \neq 1, 2$. If $\bar{E}(S_2, f) = \bar{E}(S_2, g)$ and $E(S_j, f) = E(S_j, g)$ for $j = 1, 3$, then $f(z) \equiv g(z)$.

We denote by $\sharp S$ the cardinality of a set S. From Theorem 1.2, we get immediately the corollary below.

COROLLARY 1.1

There exist three finite sets S_1, S_2 and S_3 with $\sharp S_1 = \sharp S_2 = 1$ and $\sharp S_3 = 5$, such that any two admissible meromorphic functions f and g must be identical if $E(S_j, f) = E(S_j, g)$ for $j = 1, 2, 3$ in the annulus $\mathbb{A}$.

Considering the case where two meromorphic functions in $\mathbb{A}$ share two finite sets, we get the second main result which is an analog of a result on $\mathbb{C}$ due to Yi [12].

Theorem 1.3. *Let f and g be two admissible meromorphic function in the annulus $\mathbb{A}$. Put $S_1 = \{\infty\}$ and $S_2 = \{w : P(w) = 0\}$, where*

$$P(w) = aw^n - n(n-1)w^2 + 2n(n-2)bw - (n-1)(n-2)b^2,$$

$n \geq 8$ is an integer, and a and b are two nonzero complex numbers satisfying $ab^{n-2} \neq 2$. If $\bar{E}(S_1, f) = \bar{E}(S_1, g)$ and $E(S_2, f) = E(S_2, g)$, then $f(z) \equiv g(z)$.

From Theorem 1.3, we get immediately the corollary below.

COROLLARY 1.2

There exist two finite sets S_1 and S_2 with $\sharp S_1 = 1$, $\sharp S_2 = 8$, such that any two admissible meromorphic functions f and g must be identical if $E(S_j, f) = E(S_j, g)$ for $j = 1, 2$ in the annulus $\mathbb{A}$.

2. Preliminaries and some lemmas

Recently, Khrystiyanyn and Kondratyuk [6,7] proposed Nevanlinna theory for meromorphic functions on annuli, see also an important paper [8]. Let f be a meromorphic function on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 \leq r < R_0 \leq +\infty$. Denote

$$m(r, f) := \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})| \mathrm{d}\theta,$$

where $\log^+ x = \max\{\log x, 0\}$ for $x \in \mathbb{R}$. Put

$$N_1(r, f) = \int_{\frac{1}{r}}^{1} \frac{n_1(t, f)}{t}\mathrm{d}t, \quad N_2(r, f) = \int_1^r \frac{n_2(t, f)}{t}\mathrm{d}t,$$

$$\begin{aligned} m_0(r, f) &:= m(r, f) + m\left(\frac{1}{r}, f\right) - 2m(1, f), \\ N_0(r, f) &:= N_1(r, f) + N_2(r, f), \end{aligned}$$

where $n_1(t, f)$ and $n_2(t, f)$ are the counting functions of poles of the function f in $\{z : t < |z| \leq 1\}$ and $\{z : 1 < |z| \leq t\}$, respectively. Set

$$\begin{aligned} \bar{N}_0(r, \frac{1}{f-a}) &= \bar{N}_1\left(r, \frac{1}{f-a}\right) + \bar{N}_2\left(r, \frac{1}{f-a}\right) \\ &= \int_{\frac{1}{r}}^{1} \frac{\bar{n}_1(t, \frac{1}{f-a})}{t}\mathrm{d}t + \int_1^r \frac{\bar{n}_2(t, \frac{1}{f-a})}{t}\mathrm{d}t \end{aligned}$$

in which each zero of the function $f - a$ is counted only once. The Nevanlinna characteristic of f on the annulus $\mathbb{A}$ is defined by

$$T_0(r, f) = m_0(r, f) + N_0(r, f).$$

Throughout, we denote by $S(r,*)$ quantities satisfying

(i) in the case $R_0 = +\infty$,

$$S(r,*) = O(\log(rT_0(r,*)))$$

for $r \in (1,+\infty)$ except for the set $\triangle_r$ such that $\int_{\triangle_r} r^{\lambda-1}\mathrm{d}r < +\infty$;

(ii) if $R_0 < +\infty$, then

$$S(r,*) = O\left(\log\left(\frac{T_0(r,*)}{R_0-r}\right)\right)$$

for $r \in (1,R_0)$ except for the set $\triangle'_r$ such that $\int_{\triangle'_r} \frac{\mathrm{d}r}{(R_0-r)^{\lambda-1}} < +\infty$.

Thus for an admissible meromorphic function on the annulus $\mathbb{A}$, $S(r,f) = o(T_0(r,f))$ holds for all $1 \le r < R_0$ except for the set $\triangle_r$ or the set $\triangle'_r$ mentioned above, respectively.

Lemma 2.1 [6,8]. *Let f be a nonconstant meromorphic function on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 \le r < R_0 \le +\infty$. Then*

(i) $T_0(r,f) = T_0(r,\frac{1}{f})$,

(ii) $\max\{T_0(r,f_1\cdot f_2), T_0(r,\frac{f_1}{f_2}), T_0(r,f_1+f_2)\} \le T_0(r,f_1) + T_0(r,f_2) + O(1)$.

By Lemma 2.1, the first fundamental theorem on the annulus $\mathbb{A}$ is immediately obtained.

Lemma 2.2 [6,8] (*The first fundamental theorem*). *Let f be a nonconstant meromorphic function on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 \le r < R_0 \le +\infty$. Then*

$$T_0\left(r, \frac{1}{f-a}\right) = T_0(r,f) + O(1)$$

for every fixed $a \in \mathbb{C}$.

Khrystiyanyn and Kondratyuk also obtained the second fundamental theorem on the the annulus $\mathbb{A}$. We show here the reduced form due to Cao, Yi and Xu.

Lemma 2.3 [2] (*The second fundamental theorem*). *Let f be a nonconstant meromorphic function on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 \le r < R_0 \le +\infty$. Let $a_1, a_2, \dots, a_p$ be p distinct complex numbers in $\bar{\mathbb{C}}$ and $\lambda \ge 0$. Then*

$$(q-2)T_0(r,f) < \sum_{j=1}^{q} \bar{N}_0\left(r, \frac{1}{f-a_j}\right) + S(r,f).$$

Lemma 2.4 [7,8] (*The lemma of the logarithmic derivative*). *Let f be a nonconstant meromorphic function on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 \le r < R_0 \le +\infty$. Then $m_0(r, \frac{f^{(k)}}{f}) \le S(r,f)$ for every $k \in \mathbb{N}$.*

Let f be a nonconstant meromorphic function on the annulus $\mathbb{A}$, and let $a \in \bar{\mathbb{C}}$. We say that a is a Picard exceptional value of f in $\mathbb{A}$ if $f(z) - a$ has no zero in $\mathbb{A}$.

Lemma 2.5 [8] (*Picard theorem for annuli*). *Let f be an admissible meromorphic function on the annulus $\mathbb{A}$. Then f has at most two Picard exceptional values in $\mathbb{A}$.*

Lemma 2.6. *Let f be a nonconstant meromorphic function on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 \le r < R_0 \le +\infty$. Let $P(f) = a_0 f^p + a_1 f^{p-1} + \cdots + a_1 f + a_p$ ($a_0 \neq 0$) be a polynomial of f with degree p, where the coefficients a_j ($j = 0, 1, \ldots, p$) are constants, and let b_j ($j = 1, 2, \ldots, q$) be q ($q \ge p + 1$) distinct finite complex numbers. Then*

$$m_0\left(r, \frac{P(f)f'}{(f-b_1)(f-b_2)\cdots(f-b_q)}\right) = S(r, f).$$

Proof. It is easy to see that

$$\frac{P(f)}{(f-b_1)(f-b_2)\cdots(f-b_q)} = \sum_{j=1}^{q} \frac{A_j}{f-b_j},$$

where A_j are non-zero constants. Hence, by Lemma 2.4 we obtain

$$\begin{aligned} &m_0\left(r, \frac{P(f)f'}{(f-b_1)(f-b_2)\cdots(f-b_q)}\right) \\ &= m_0\left(r, \frac{A_j f'}{f-b_j}\right) \\ &\le \sum_{j=1}^{q} m_0\left(r, \frac{f'}{f-b_j}\right) + \sum_{j=1}^{q} m_0(r, A_j) + O(1) \\ &= S(r, f). \end{aligned}$$

□

Let h be meromorphic in $\mathbb{A}$. We denote by $N_0^{(1)}(r, f)$ the counting function of simple poles of h in $\mathbb{A}$, and by $\bar{N}_0^{(2)}(r, h)$ the counting function of multiple poles of h in $\mathbb{A}$, where each pole is counted only once irrespective of its multiplicity.

Lemma 2.7. *Let*

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right),$$

where F and G are two nonconstant meromorphic functions in $\mathbb{A}$. If $E(\{1\}, F) = E(\{1\}, G)$ and $H(z) \not\equiv 0$, then

$$N_0^{(1)}\left(r, \frac{1}{F-1}\right) \le N_0(r, H) + S(r, F) + S(r, G).$$

Proof. By Lemma 2.4, we have $m_0(r, H) = S(r, F) + S(r, G)$. By $E(\{1\}, F) = E(\{1\}, G)$, if $z_0 \in \mathbb{A}$ is a simple zero of $F - 1$, then it must be a zero of H. Thus by Lemma 2.2 we have

$$\begin{aligned} N_0^{(1)}\left(r, \frac{1}{F-1}\right) &\leq N_0\left(r, \frac{1}{H}\right) \leq T_0(r, H) + O(1) \\ &\leq N_0(r, H) + S(r, F) + S(r, G). \end{aligned}$$

□

By simple computation, one can get the following lemma.

Lemma 2.8. *Let*

$$U = \frac{F''}{F'} - \frac{2F'}{F-1},$$

where F is a nonconstant meromorphic function in $\mathbb{A}$. If $z_0 \in \mathbb{A}$ is a simple pole of F, then U is holomorphic at z_0.

Lemma 2.9. *Let*

$$F = \frac{af^n}{n(n-1)(f-\alpha_1)(f-\alpha_2)}, \quad G = \frac{ag^n}{n(n-1)(g-\alpha_1)(g-\alpha_2)},$$

where f and g are nonconstant meromorphic functions in $\mathbb{A}$, $n \geq 4$ is an integer, and α_1 and α_2 are distinct finite complex numbers. Put

$$V = \left(\frac{F'}{F-1} - \frac{F'}{F}\right) - \left(\frac{G'}{G-1} - \frac{G'}{G}\right).$$

If $V(z) \equiv 0$ and $\bar{E}(\{\infty\}, f) = \bar{E}(\{\infty\}, g) \neq \emptyset$, then $F(z) \equiv G(z)$.

Proof. By the assumption $V \equiv 0$, we have

$$\frac{F'}{F-1} - \frac{F'}{F} \equiv \frac{G'}{G-1} - \frac{G'}{G}.$$

By integration,

$$1 - \frac{1}{F} \equiv C\left(1 - \frac{1}{G}\right),$$

where C is a nonzero constant. Since $\bar{E}(\{\infty\}, f) = \bar{E}(\{\infty\}, g) \neq \emptyset$, there exists a point $z_0 \in \mathbb{A}$ such that z_0 is a pole of both f and g. From the definitions of F and G, z_0 must be a pole of both F and G. Hence $C = 1$, and thus, $F(z) \equiv G(z)$. □

Lemma 2.10. *Let F and G be defined as in Lemma* 2.9, *and H be defined as in Lemma* 2.7. *If $E(\{0\}, F) = E(\{0\}, G)$, $E(\{1\}, F) = E(\{1\}, G)$, $\bar{E}(\{\infty\}, f) = \bar{E}(\{\infty\}, g)$ and $H(z) \not\equiv 0$, then*

$$\bar{N}_0(r, f) = \bar{N}_0(r, g) = S(r, F) + S(r, G).$$

Proof. Suppose that $\bar{E}(\{\infty\}, f) = \emptyset$, then the conclusion of this lemma holds obviously. Now we only assume that $\bar{E}(\{\infty\}, f) \neq \emptyset$. Define V as in Lemma 2.9. Since $H(z) \not\equiv 0$ and $\bar{E}(\{\infty\}, f) = \bar{E}(\{\infty\}, g) \neq \emptyset$, we can deduce from Lemma 2.9 that $V(z) \not\equiv 0$. By the definition of V we obtain

$$V = \frac{F'}{F(F-1)} - \frac{G'}{G(G-1)}. \tag{1}$$

By (1) and $\bar{E}(\{\infty\}, f) = \bar{E}(\{\infty\}, g) \neq \emptyset$, there exists a point $z_0 \in \mathbb{A}$ which is a pole of both f and g with multiplicity p and q respectively. Then z_0 must be a pole of both F and G with multiplicity $(n-2)p$ and $(n-2)q$ respectively. Thus z_0 is a zero of V with multiplicity $\geq n-3$. Hence we obtain

$$(n-3)\bar{N}_0(r, f) \leq N_0\left(r, \frac{1}{V}\right) \leq T_0(r, V) + O(1).$$

By the definition of V and Lemmas 2.1, 2.2 and 2.6, we have $m_0(r, V) = S(r, F) + S(r, G)$. Note that $E(\{0\}, F) = E(\{0\}, G)$ and $E(\{1\}, F) = E(\{1\}, G)$. Again by the definition of V we obtain $N_0(r, V) = S(r, F) + S(r, G)$. Thus we have

$$T_0(r, V) = S(r, F) + S(r, G).$$

Hence we obtain

$$(n-3)\bar{N}_0(r, f) \leq S(r, F) + S(r, G).$$

Noting $n \geq 4$, the conclusion of this lemma holds. □

Lemma 2.11 (*Page* 192 *in* [5]). *Let*

$$Q(w) = (n-1)^2(w^n - 1)(w^{n-2} - 1) - n(n-2)(w^{n-1} - 1)^2$$

be a polynomial of degree $2n-2$ $(n \geq 3)$. *Then*

$$Q(w) = (w-1)^4(w-\beta_1)(w-\beta_2)\cdots(w-\beta_{2n-6}),$$

where $\beta_j \in \mathbb{C} \setminus \{0, 1\}$ $(j = 1, 2, \ldots, 2n-6)$, *which are distinct respectively.*

Lemma 2.12. *Let* f_1 *and* f_2 *be two nonconstant meromorphic function in the annulus* $\mathbb{A}$, *and let* c_1, c_2 *and* c_3 *be three nonzero constant. If* $c_1 f_1 + c_2 f_2 \equiv c_3$, *then*

$$T_0(r, f_1) < \bar{N}_0\left(r, \frac{1}{f_1}\right) + \bar{N}_0\left(r, \frac{1}{f_2}\right) + \bar{N}_0(r, f_1) + S(r, f_1).$$

Proof. By Lemma 2.3 we have

$$T_0(r, f_1) < \bar{N}_0\left(r, \frac{1}{f_1}\right) + \bar{N}_0(r, f_1) + \bar{N}_0\left(r, \frac{1}{f_1 - \frac{c_3}{c_1}}\right) + S(r, f_1).$$

Note that the zeros of $f_1 - \frac{c_3}{c_1}$ are just the zeros of f_2. Hence we obtain the conclusion

$$T_0(r, f_1) < \bar{N}_0\left(r, \frac{1}{f_1}\right) + \bar{N}_0\left(r, \frac{1}{f_2}\right) + \bar{N}_0(r, f_1) + S(r, f_1).$$

□

By a similar discussion as in [10], one can obtain a stand and Valiron-Mohon'ko type result in $\mathbb{A}$ as follows:

Lemma 2.13. *Let f be a nonconstant meromorphic function in $\mathbb{A}$, $P_1(f)$ and $P_2(f)$ be two mutually prime polynomials in f with degree m and n respectively. Then*

$$T_0\left(r, \frac{P_1(f)}{P_2(f)}\right) = \max\{m, n\} T_0(r, f) + S(r, f).$$

3. Proofs of Theorems 1.2

Let

$$P(w) = aw^n - n(n-1)w^2 + 2n(n-2)bw - (n-1)(n-2)b^2,$$

where $n \geq 5$ is an integer, and a and b are two nonzero complex numbers satisfying $c := \frac{ab^{n-2}}{2} \neq 1$. Let

$$R(w) = \frac{aw^n}{n(n-1)(w-\alpha_1)(w-\alpha_2)}, \tag{2}$$

where α_1 and α_2 are two distinct roots of the equation

$$n(n-1)w^2 - 2n(n-2)bw + (n-1)(n-2)b^2 = 0.$$

Then by page 319 in [9], we get that

$$R(w) - 1 = \frac{P(w)}{n(n-1)(w-\alpha_1)(w-\alpha_2)}, \tag{3}$$

$$R(w) - c = \frac{a(w-b^3)Q_{n-3}(w)}{n(n-1)(w-\alpha_1)(w-\alpha_2)}, \tag{4}$$

where $Q_{n-3}(w)$ is a polynomial of degree $n-3$. Further, $P(w)$ has only simple zeros.

Let $F = R(f)$ and $G = R(g)$. Since $E(S_j, f) = E(S_j, g)$ for $j = 1, 3$, it is not difficult to get that $E(\{1\}, F) = E(\{1\}, G)$ and $E(\{0\}, F) = E(\{0\}, G)$. By Lemma 2.13 and (2),

$$T_0(r, f) = \frac{1}{n} T_0(r, F) + S(r, F), \quad T_0(r, g) = \frac{1}{n} T_0(r, G) + S(r, G). \tag{5}$$

Let H be as mentioned in Lemma 2.7 and suppose $H(z) \not\equiv 0$. Noting that $\bar{E}(S_2, f) = E(S_2, g)$, we deduce from Lemma 2.10 that

$$\bar{N}_0(r, f) = \bar{N}_0(r, g) = S(r, F) + S(r, G). \tag{6}$$

By (6) and the definitions of F and G, we have

$$\bar{N}_0(r, F) = \sum_{j=1}^{2} \bar{N}_0\left(\frac{1}{f-\alpha_j}\right) + S(r, F) + S(r, G), \tag{7}$$

$$\bar{N}_0(r, G) = \sum_{j=1}^{2} \bar{N}_0\left(\frac{1}{f-\alpha_j}\right) + S(r, F) + S(r, G). \tag{8}$$

From Lemmas 2.7, 2.8 and (6) we deduce that

$$\begin{aligned} N_0^{(1)}\left(r, \frac{1}{F-1}\right) \ \leq\ & \sum_{j=1}^{2} \bar{N}_0^{(2)}\left(\frac{1}{f-\alpha_j}\right) + \sum_{j=1}^{2} \bar{N}_0^{(2)}\left(\frac{1}{g-\alpha_j}\right) \\ & + \bar{N}_0^{(2)}\left(\frac{1}{F-c}\right) + \bar{N}_0^{(2)}\left(\frac{1}{G-c}\right) + N_0^{*}\left(r, \frac{1}{F'}\right) \\ & + N_0^{*}\left(r, \frac{1}{G'}\right) + S(r, F) + S(r, G), \end{aligned}$$

where $N_0^{*}(r, \frac{1}{F'})$ (or $N_0^{*}(r, \frac{1}{G'})$) means the counting function of zeros of F' (or G') but not zeros of $F(F-c)(F-1)$ (or $G(G-c)(G-1)$). Note that

$$\bar{N}_0\left(r, \frac{1}{F-1}\right) + \bar{N}_0\left(\frac{1}{G-1}\right) - N_0^{(1)}\left(r, \frac{1}{F-1}\right) \leq T_0(r, G) + S(r, G).$$

Together with Lemma 2.3, we have

$$\begin{aligned} & 2(T_0(r, F) + T_0(r, G)) \\ & \leq \sum_{j=1}^{4} \bar{N}_0\left(r\frac{1}{F-a_j}\right) + \sum_{j=1}^{4} \bar{N}_0\left(r\frac{1}{G-a_j}\right) \\ & \quad - N_0^{*}\left(r, \frac{1}{F'}\right) - N_0^{*}\left(r, \frac{1}{G'}\right) + S(r, F) + S(r, G) \\ & \leq \bar{N}_0\left(r\frac{1}{F}\right) + \bar{N}_0\left(r\frac{1}{G}\right) + \bar{N}_0\left(r\frac{1}{F-c}\right) + \bar{N}_0\left(r\frac{1}{G-c}\right) \\ & \quad + \bar{N}_0\left(r\frac{1}{f-\alpha_1}\right) + \bar{N}_0\left(r\frac{1}{g-\alpha_1}\right) \\ & \quad + \bar{N}_0\left(r\frac{1}{f-\alpha_2}\right) + \bar{N}_0\left(r\frac{1}{g-\alpha_2}\right) + \bar{N}_0^{(2)}\left(r, \frac{1}{F-c}\right) \\ & \quad + \bar{N}_0^{(2)}\left(r, \frac{1}{G-c}\right) + T_0(r, G) + S(r, F) + S(r, G), \end{aligned}$$

where $\{a_1, a_2, a_3, a_4\} = \{0, 1, c, \infty\}$. It is not difficult to get that

$$\bar{N}_0\left(r, \frac{1}{F}\right) \le \bar{N}_0\left(r, \frac{1}{f}\right) \le \frac{1}{n}T_0(r, F) + S(r, F),$$

$$\begin{aligned}\bar{N}_0\left(r, \frac{1}{F-c}\right) &+ \bar{N}_0^{(2)}\left(r, \frac{1}{F-c}\right)\\ &\le 2\bar{N}_0\left(r, \frac{1}{F-b}\right) + N_0\left(r, \frac{1}{Q_{n-3}(f)}\right)\\ &\le \frac{n-1}{n}T_0(r, F) + S(r, F),\end{aligned}$$

$$\bar{N}_0\left(r, \frac{1}{f-\alpha_1}\right) + \bar{N}_0\left(r, \frac{1}{f-\alpha_2}\right) \le \frac{2}{n}T_0(r, F) + S(r, F),$$

$$\bar{N}_0\left(r, \frac{1}{G}\right) \le \bar{N}_0\left(r, \frac{1}{g}\right) \le \frac{1}{n}T_0(r, G) + S(r, G),$$

$$\begin{aligned}\bar{N}_0\left(r, \frac{1}{G-c}\right) &+ \bar{N}_0^{(2)}\left(r, \frac{1}{G-c}\right)\\ &\le 2\bar{N}_0\left(r, \frac{1}{G-b}\right) + N_0\left(r, \frac{1}{Q_{n-3}(g)}\right)\\ &\le \frac{n-1}{n}T_0(r, G) + S(r, G),\end{aligned}$$

and

$$\bar{N}_0\left(r, \frac{1}{g-\alpha_1}\right) + \bar{N}_0\left(r, \frac{1}{g-\alpha_2}\right) \le \frac{2}{n}T_0(r, G) + S(r, G).$$

Then we have

$$\left(1 - \frac{2}{n}\right)T_0(r, F) - \frac{2}{n}T_0(r, G) \le S(r, F) + S(r, G).$$

Exchanging F and G in the above discussion, we also have

$$\left(1 - \frac{2}{n}\right)T_0(r, G) - \frac{2}{n}T_0(r, F) \le S(r, F) + S(r, G).$$

Hence we obtain

$$\left(1 - \frac{4}{n}\right)(T_0(r, F) + T_0(r, G)) \le S(r, F) + S(r, G).$$

This implies $n \le 4$, which contradicts the assumption $n \ge 5$. Hence, $H(z) \equiv 0$, and thus

$$\frac{F''}{F'} - \frac{2F'}{F-1} \equiv \frac{F''}{F'} - \frac{2F'}{F-1}.$$

By integration, the above equality implies that

$$\frac{1}{G-1} \equiv \frac{A}{F-1} + B, \tag{9}$$

where $A \neq 0,\ B$ are constants. By (9) we have

$$G \equiv \frac{(B+1)F + (A-B-1)}{BF + (A-B)} \tag{10}$$

and

$$T_0(r, G) = T_0(r, F) + O(1). \tag{11}$$

We next consider three cases.

Case 1. $B \neq 0, -1$. By the assumption $\bar{E}(\{\infty\}, f) = \bar{E}(\{\infty\}, g)$ and (10) we get that ∞ is a Picard exceptional value of f and g in $\mathbb{A}$. Thus

$$\bar{N}_0(r, F) = \bar{N}_0\left(r, \frac{1}{f-\alpha_1}\right) + \bar{N}_0\left(r, \frac{1}{f-\alpha_2}\right), \tag{12}$$

$$\bar{N}_0(r, G) = \bar{N}_0\left(r, \frac{1}{g-\alpha_1}\right) + \bar{N}_0\left(r, \frac{1}{g-\alpha_2}\right). \tag{13}$$

Assume that $A - B - 1 \neq 0$. Then by Lemma 2.3, (5), (10), (11) and (12), we have

$$\begin{aligned} nT_0(r, f) &= T_0(r, F) + S(r, F) \\ &\leq \bar{N}_0(r, F) + \bar{N}_0\left(r, \frac{1}{F}\right) + \bar{N}_0\left(r, \frac{1}{F + \frac{A-B-1}{B+1}}\right) + S(r, F) \\ &\leq \bar{N}_0\left(r, \frac{1}{f-\alpha_1}\right) + \bar{N}_0\left(r, \frac{1}{f-\alpha_2}\right) \\ &\quad + \bar{N}_0\left(r, \frac{1}{F}\right) + \bar{N}_0\left(r, \frac{1}{G}\right) + S(r, F) \\ &\leq \bar{N}_0\left(r, \frac{1}{f-\alpha_1}\right) + \bar{N}_0\left(r, \frac{1}{f-\alpha_2}\right) \\ &\quad + \bar{N}_0\left(r, \frac{1}{f}\right) + \bar{N}_0\left(r, \frac{1}{g}\right) + S(r, f) \\ &\leq 4T_0(r, f) + S(r, f), \end{aligned}$$

which implies $n \leq 4$, a contradiction. Hence, $A - B - 1 = 0$. Then we rewrite (10) as

$$G \equiv \frac{(B+1)F}{BF + 1}. \tag{14}$$

By (13) and (14) we have

$$\bar{N}_0\left(r \frac{1}{F + \frac{1}{B}}\right) = \bar{N}_0(r, G) = \bar{N}_0\left(r, \frac{1}{g-\alpha_1}\right) + \bar{N}_0\left(r, \frac{1}{g-\alpha_2}\right). \tag{15}$$

Assume that $c = \frac{ab^{n-2}}{2} \neq \frac{-1}{B}$. By (4), (5), Lemma 2.2 and the definition of F, we obtain

$$\bar{N}_0\left(r, \frac{1}{F-c}\right) \leq \frac{n-2}{n} T_0(r, F) + S(r, F) \leq (n-2)T_0(r, f) + S(r, f). \tag{16}$$

By Lemmas 2.1, 2.2, 2.3, (5), (11), (12), (15) and (16), we get that

$$\begin{aligned}
2nT_0(r,f) &= 2T_0(r,F)+S(r,F)\\
&\le \bar{N}_0(r,F)+\bar{N}_0\left(r,\frac{1}{F}\right)\\
&\quad +\bar{N}_0\left(r,\frac{1}{F+\frac{1}{B}}\right)+\bar{N}_0\left(r,\frac{1}{F-c}\right)+S(r,F)\\
&\le \bar{N}_0\left(r,\frac{1}{f-\alpha_1}\right)+\bar{N}_0\left(r,\frac{1}{f-\alpha_2}\right)\\
&\quad +\bar{N}_0\left(r,\frac{1}{g-\alpha_1}\right)+\bar{N}_0\left(r,\frac{1}{g-\alpha_2}\right)\\
&\quad +\bar{N}_0\left(r,\frac{1}{f}\right)+(n-2)T_0(r,f)+S(r,f)\\
&\le (n+3)T_0(r,f)+S(r,f).
\end{aligned}$$

This contradicts the assumption $n \ge 5$.

Assume that $c=\frac{-1}{B}$. Then we get from (14) that

$$F\equiv \frac{cG}{G-(1-c)}. \tag{17}$$

From (12) and (17), we get

$$\bar{N}_0\left(r\frac{1}{G-(1-c)}\right)=\bar{N}_0(r,F)=\bar{N}_0\left(r,\frac{1}{f-\alpha_1}\right)+\bar{N}_0\left(r,\frac{1}{f-\alpha_2}\right). \tag{18}$$

By (4), (5), Lemma 2.2 and the definition of G, we obtain

$$\bar{N}_0\left(r,\frac{1}{G-c}\right)\le \frac{n-2}{n}T_0(r,G)+S(r,G)\le (n-2)T_0(r,g)+S(r,g). \tag{19}$$

By the assumption $2c=ab^{n-2}\ne 1,2$, we have $1-c\ne c$. By Lemmas 2.1, 2.2, 2.3, (5), (11), (13), (18) and (19), we get that

$$\begin{aligned}
2nT_0(r,g) &= 2T_0(r,G)+S(r,G)\\
&\le \bar{N}_0(r,G)+\bar{N}_0\left(r,\frac{1}{G}\right)\\
&\quad +\bar{N}_0\left(r,\frac{1}{G-(1-c)}\right)+\bar{N}_0\left(r,\frac{1}{G-c}\right)+S(r,G)\\
&\le \bar{N}_0\left(r,\frac{1}{g-\alpha_1}\right)+\bar{N}_0\left(r,\frac{1}{g-\alpha_2}\right)\\
&\quad +\bar{N}_0\left(r,\frac{1}{g-\alpha_1}\right)+\bar{N}_0\left(r,\frac{1}{g-\alpha_2}\right)\\
&\quad +\bar{N}_0\left(r,\frac{1}{g}\right)+(n-2)T_0(r,g)+S(r,g)\\
&\le (n+3)T_0(r,g)+S(r,g).
\end{aligned}$$

This contradicts the assumption $n \geq 5$.

Case 2. $B = -1$. We rewrite (10) as

$$G \equiv \frac{A}{(A+1) - F}. \tag{20}$$

Assume that $A + 1 \neq 0$. Note that $A \neq 0$ and $E(\{0\}, F) = E(\{0\}, G)$. Then by (20) we get that 0 and $\frac{A}{A+1}$ are Picard exceptional values of G in $\mathbb{A}$. By (5), Lemma 2.3 and the definition of G, we have

$$\begin{aligned} nT_0(r, g) &= T_0(r, G) + S(r, G) \\ &\leq \bar{N}_0(r, G) + \bar{N}_0\left(r, \frac{1}{G}\right) + \bar{N}_0\left(r, \frac{1}{G - \frac{A}{A+1}}\right) + S(r, G) \\ &\leq \bar{N}_0(r, g) + \bar{N}_0\left(r, \frac{1}{g - \alpha_1}\right) + \bar{N}_0\left(r, \frac{1}{g - \alpha_2}\right) + S(r, g) \\ &\leq 3T_0(r, g) + S(r, g). \end{aligned}$$

This contradicts the assumption $n \geq 5$.

Assume that $A + 1 = 0$. Then $F(z)G(z) \equiv 1$. Thus

$$\frac{f^n g^n}{(f - \alpha_1)(f - \alpha_2)(g - \alpha_1)(g - \alpha_2)} \equiv \frac{n^2(n-1)^2}{a^2}. \tag{21}$$

Note that $E(\{0\}, f) = E(\{0\}, g)$ and $\bar{E}(\{\infty\}, f) = \bar{E}(\{\infty\}, g)$. We get from (21) that $0, \infty, \alpha_1$ and α_2 are Picard exceptional values of f. By Lemma 2.5 we have a contradiction.

Case 3. $B = 0$. We rewrite (10) as

$$G \equiv \frac{F + (A-1)}{A}. \tag{22}$$

Assume that $A \neq 1$. Note that $E(\{0\}, F) = E(\{0\}, G)$. Then by (22) we get that 0 and $1 - A$ are Picard exceptional values of F in $\mathbb{A}$. By (5), Lemma 2.3 and the definition of F, we have

$$\begin{aligned} nT_0(r, f) &= T_0(r, F) + S(r, F) \\ &\leq \bar{N}_0(r, F) + \bar{N}_0\left(r, \frac{1}{F}\right) + \bar{N}_0\left(r, \frac{1}{F - (1-A)}\right) + S(r, F) \\ &\leq \bar{N}_0(r, f) + \bar{N}_0\left(r, \frac{1}{f - \alpha_1}\right) + \bar{N}_0\left(r, \frac{1}{f - \alpha_2}\right) + S(r, f) \\ &\leq 3T_0(r, f) + S(r, f). \end{aligned}$$

This contradicts the assumption $n \geq 5$.

Assume that $A = 1$. Thus $F(z) \equiv G(z)$. Together with $\bar{E}(\{\infty\}, f) = \bar{E}(\{\infty\}, g)$, we have $E(\{\infty\}, f) = E(\{\infty\}, g)$. Further, together with (2), we get

$$\begin{aligned} &n(n-1)f^2g^2(f^{n-2} - g^{n-2}) - 2bn(n-2)fg(f^{n-1} - g^{n-1}) \\ &\quad + b^2(n-1)(n-2)(f^n - g^n) \equiv 0. \end{aligned} \tag{23}$$

Set $h = \frac{f}{g}$. Noting that $E(\{\infty\}, f) = E(\{\infty\}, g)$ and $E(\{0\}, f) = E(\{0\}, g)$, we get that h is holomorphic in $\mathbb{A}$. Substituting $f = hg$ into (23),

$$\begin{aligned} &n(n-1)h^2g^2(h^{n-2}-1) - 2bn(n-2)hg(h^{n-1}-1) \\ &\quad + b^2(n-1)(n-2)(h^n-1) \equiv 0, \end{aligned}$$

and thus

$$\begin{aligned} &n^2(n-1)^2h^2g^2(h^{n-2}-1)^2 - 2bn^2(n-1)(n-2)hg(h^{n-1}-1)(h^{n-2}-1) \\ &\quad \equiv -b^2n(n-1)^2(n-2)(h^n-1)(h^{n-2}-1). \end{aligned}$$

From this equality and Lemma 2.11 and we can deduce that

$$(n(n-1)h(h^{n-2}-1)g - n(n-2)b(h^{n-1}-1))^2 \equiv -b^2n(n-2)Q(h), \tag{24}$$

where $Q(h) = (h-1)^4(h-\beta_1)(h-\beta_2)\cdots(h-\beta_{2n-6})$, $\beta_j \in \mathbb{C}\backslash\{0, 1\}$ $(j = 1, 2, \ldots, 2n-6)$, which are pairwise distinct. If h is not a constant, then by (24) we get that the multiplicity of every zero of $h-\beta_j$ $(j = 1, 2, \ldots, 2n-6)$ is at least 2. By Lemma 2.3 we can get that

$$\begin{aligned} (2n-7)T_0(r, h) &< \sum_{j=1}^{2n-6} \bar{N}_0\left(r, \frac{1}{h-\beta_j}\right) + \bar{N}_0(r, h) + S(r, h) \\ &\le \frac{1}{2}\sum_{j=1}^{2n-6} N_0\left(r, \frac{1}{h-\beta_j}\right) + S(r, h) \\ &\le \frac{2n-6}{2}T_0(r, h) + S(r, h). \end{aligned}$$

This contradicts the assumption $n \ge 5$. Therefore, h is a constant. Thus $h^n - 1 = 0$, $h^{n-1} - 1 = 0$ and $h^{n-2} - 1 = 0$. This implies $h = 1$, and so $f(z) \equiv g(z)$.

4. Proof of Theorem 1.3

Let $F = R(f)$ and $G = R(g)$. By a similar argument as in the proof of Theorem 1.2, we also have $E(\{1\}, F) = E(\{1\}, G)$ and (2)–(5).

Let H be mentioned in Lemma 2.7 and suppose $H(z) \not\equiv 0$. By Lemma 2 we have

$$N_0^{(1)}\left(r, \frac{1}{F-1}\right) \le N_0(r, H) + S(r, F) + S(r, G). \tag{25}$$

By the definitions of F and G, we get

$$\begin{aligned} F' &= \frac{(n-2)af^{n-1}(f-b)^2f'}{n(n-1)(f-\alpha_1)^2(f-\alpha_2)^2}, \\ G' &= \frac{(n-2)ag^{n-1}(g-b)^2g'}{n(n-1)(g-\alpha_1)^2(g-\alpha_2)^2}. \end{aligned} \tag{26}$$

It is obvious that any simple zero of $f-\alpha_1,\ f-\alpha_2$ in $\mathbb{A}$ is a simple pole of F in $\mathbb{A}$, that any multiple zero of $f-\alpha_1,\ f-\alpha_2$ in $\mathbb{A}$ is a zero of f' in $\mathbb{A}$, any simple zero of $g-\alpha_1,\ g-\alpha_2$ in $\mathbb{A}$ is a simple pole of G in $\mathbb{A}$, and that any multiple zero of $g-\alpha_1$, $g-\alpha_2$ in $\mathbb{A}$ is a zero of g' in $\mathbb{A}$. Noting $\bar{E}(S_2, f) = E(S_2, g)$, we deduce from Lemma 2.8, $E(\{1\}, F) = E(\{1\}, G)$, (26) and the definitions of F, G and H that

$$\begin{aligned} N_0(r, H) \ \le\ & \bar{N}_0\left(r, \frac{1}{f}\right) + \bar{N}_0\left(r, \frac{1}{f-b}\right) + \bar{N}_0\left(r, \frac{1}{g}\right) \\ & + \bar{N}_0\left(r, \frac{1}{g-b}\right) + N_0^*\left(r, \frac{1}{f'}\right) + N_0^*\left(r, \frac{1}{g'}\right), \end{aligned} \tag{27}$$

where $N_0^*(r, \frac{1}{f'})$ (or $N_0^*(r, \frac{1}{g'})$) means the counting function of zeros of f' (or g') but not zeros of $f(f-b)$ (or $g(g-b)$) and $F-1$ (or $G-1$). Then by Lemma 2, (25), (27), (5), $E(\{1\}, F) = E(\{1\}, G)$ and $\bar{E}(S_2, f) = \bar{E}(S_2, g)$, we can deduce that

$$\begin{aligned} & (n+1)(T_0(r, f) + T_0(r, g)) \\ \le\ & \bar{N}_0\left(r, \frac{1}{F-1}\right) + \bar{N}_0\left(r, \frac{1}{f}\right) + \bar{N}_0\left(r, \frac{1}{f-b}\right) + \bar{N}_0(r, f) \\ & + \bar{N}_0\left(r, \frac{1}{G-1}\right) + \bar{N}_0\left(r, \frac{1}{g}\right) + \bar{N}_0\left(r, \frac{1}{g-b}\right) + \bar{N}_0(r, g) \\ & - N_0^*\left(r, \frac{1}{f'}\right) - N_0^*\left(r, \frac{1}{g'}\right) + S(r, f) + S(r, g) \\ \le\ & 2\bar{N}_0\left(r, \frac{1}{f}\right) + 2\bar{N}_0\left(r, \frac{1}{f-b}\right) + 3\bar{N}_0(r, f) + 2\bar{N}_0\left(r, \frac{1}{g}\right) \\ & + 2\bar{N}_0\left(r, \frac{1}{g-b}\right) + \bar{N}_0\left(r, \frac{1}{F-1}\right) + \bar{N}_0\left(r, \frac{1}{G-1}\right) \\ & - \bar{N}_0^{(1)}\left(r, \frac{1}{F-1}\right) + S(r, f) + S(r, g) \\ \le\ & 4T_0(r, f) + 4T_0(r, g) + \frac{1}{2}N_0\left(r, \frac{1}{F-1}\right) + \frac{1}{2}N_0\left(r, \frac{1}{G-1}\right) \\ & + 3\bar{N}_0(r, f) + S(r, f) + S(r, g), \end{aligned}$$

and thus

$$(n-6)(T_0(r, f) + T_0(r, g)) \le 6\bar{N}_0(r, f) + S(r, f) + S(r, g). \tag{28}$$

Let V be defined as in Lemma 2.9. If $V(z) \not\equiv 0$, then we can deduce that

$$\frac{1}{F} - \frac{A}{G} \equiv 1 - A,$$

where $A \neq 0$ is a constant. Together with (5), we have $T_0(r, f) = T_0(r, g) + S(r, f)$. Set $f_1 = \frac{1}{F}$ and $f_2 = \frac{-A}{G}$. Then we get $f_1 + f_2 \equiv 1 - A$. Suppose that $A \neq 1$. Then by Lemma 2.12 and (5) we have

$$\begin{aligned} nT_0(r, f) \ \le\ & \sum_{j=1}^{3} \bar{N}_0\left(\frac{1}{f-a_j}\right) + \sum_{j=1}^{3} \bar{N}_0\left(\frac{1}{g-a_j}\right) + \bar{N}_0\left(\frac{1}{f}\right) + S(r, f) \\ \le\ & 7\,T_0(r, f) + S(r, f), \end{aligned}$$

where $\{a_1, a_2, a_3\} = \{\infty, \alpha_1, \alpha_2\}$. This contradicts the assumption $n \geq 8$. Hence $A = 1$, and thus $F(z) \equiv G(z)$. This implies $H(z) \equiv 0$, a contradiction. Therefore, $V(z) \not\equiv 0$. By the definition of V and Lemmas 2.1, 2.2 or 2.6, we have (1) and

$$m_0(r, V) = S(r, F) + S(r, G).$$

From (1), we get that any pole of F and G is not a pole of V. Noting that $E(\{1\}, F) = E(\{1\}, G)$, by the definition of V we get that any zero of $F - 1$ and $G - 1$ is not a pole of V. Thus by (5), we have

$$\begin{aligned} N_0(r, V) &\leq \bar{N}_0\left(\frac{1}{F}\right) + \bar{N}_0\left(\frac{1}{G}\right) = \bar{N}_0\left(\frac{1}{f}\right) + \bar{N}_0\left(\frac{1}{g}\right) \\ &\leq T_0(r, f) + T_0(r, g) + O(1). \end{aligned}$$

Hence we get

$$T_0(r, V) \leq T_0(r, f) + T_0(r, g) + S(r, f) + S(r, g).$$

If $z_1 \in \mathbb{A}$ is a pole of both f and g with multiplicity p and q respectively, then by the definitions of F and G, we get that z_1 must be a pole of both F and G with multiplicity $(n-2)p$ and $(n-2)q$ respectively. Thus z_1 is a zero of V with multiplicity $\geq n-3$. Hence we obtain

$$\begin{aligned} (n-3)\bar{N}_0(r, f) &\leq N_0\left(r, \frac{1}{V}\right) \leq T_0(r, V) + O(1) \\ &\leq T_0(r, f) + T_0(r, g) + S(r, f) + S(r, g). \end{aligned} \tag{29}$$

By (28) and (29), we have

$$\left(n - 6 - \frac{6}{n-3}\right)(T_0(r, f) + T_0(r, g)) \leq S(r, f) + S(r, g).$$

This contradicts the assumption $n \geq 8$. Therefore, $H(z) \equiv 0$, and thus (9)–(11) hold. We next consider three cases similarly as in the proof of Theorem 1.2.

Case 1. $B \neq 0, -1$. By the same argument as in the proof of Theorem 1.2, we obtain a contradiction.

Case 2. $B = -1$. Then (20) holds.

Assume that $A + 1 \neq 0$. Then $\bar{N}_0(r, \frac{1}{F-(A+1)}) = \bar{N}_0(r, G)$. By Lemma 2.3 and (5), (11) and the definitions of F and G, we have

$$\begin{aligned} T_0(r, F) &\leq \bar{N}_0\left(r, \frac{1}{F}\right) + \bar{N}_0\left(r, \frac{1}{F-(A+1)}\right) + \bar{N}_0(r, F) + S(r, F) \\ &= \bar{N}_0\left(r, \frac{1}{F}\right) + \bar{N}_0(r, F) + \bar{N}_0(r, G) + S(r, F) \\ &\leq \frac{7}{n} T_0(r, F) + S(r, F). \end{aligned}$$

This contradicts the assumption $n \geq 8$.

Assume that $A+1=0$. Then $F(z)G(z)\equiv 1$, and thus (21) holds. Noting that $\bar{E}(\{\infty\}, f)=\bar{E}(\{\infty\}, g)$, by (21) we get that ∞ is a Picard exceptional value of f and g in $\mathbb{A}$, and that the multiplicity of $f-\alpha_j$ $(j=1,2)$ is at least n. By Lemma 2.3 we have

$$\begin{aligned} T_0(r,f) &\le \bar{N}_0(r,f)+\bar{N}_0\left(r,\frac{1}{f-\alpha_1}\right)+\bar{N}_0\left(r,\frac{1}{f-\alpha_2}\right)+S(r,f)\\ &\le \frac{1}{n}N_0\left(r,\frac{1}{f-\alpha_1}\right)+N_0\left(r,\frac{1}{f-\alpha_2}\right)+S(r,f)\\ &\le \frac{2}{n}T_0(r,f)+S(r,f). \end{aligned}$$

This contradicts the assumption $n\ge 8$.

Case 3. $B=0$. Then (22) holds.

Assume that $A\ne 1$. Then $\bar{N}_0(r,\frac{1}{F+(A-1)})=\bar{N}_0(r,\frac{1}{G})$. By Lemma 2.3, (5), (11) and the definitions of F and G, we have

$$\begin{aligned} T_0(r,F) &\le \bar{N}_0\left(r,\frac{1}{F}\right)+\bar{N}_0\left(r,\frac{1}{F+(A-1)}\right)+\bar{N}_0(r,F)+S(r,F)\\ &= \bar{N}_0\left(r,\frac{1}{F}\right)+\bar{N}_0(r,F)+\bar{N}_0\left(r,\frac{1}{G}\right)+S(r,F)\\ &\le \frac{5}{n}T_0(r,F)+S(r,F). \end{aligned}$$

This contradicts the assumption $n\ge 8$.

Assume that $A=1$. Thus $F(z)\equiv G(z)$. Set $h=\frac{f}{g}$. Then (23) and (24) hold. If h is not a constant, then by (24) we get that the multiplicity of every zero of $h-\beta_j$ $(j=1,2,\ldots,2n-6)$ is at least 2. By Lemma 2.3 we can get that

$$\begin{aligned} (2n-8)T_0(r,h) &< \sum_{j=1}^{2n-6}\bar{N}_0\left(r,\frac{1}{h-\beta_j}\right)+S(r,h)\\ &\le \frac{1}{2}\sum_{j=1}^{2n-6}N_0\left(r,\frac{1}{h-\beta_j}\right)+S(r,h)\\ &\le (n-3)T_0(r,h)+S(r,h). \end{aligned}$$

This contradicts the assumption $n\ge 8$. Therefore, h is a constant. Thus $h^n-1=0$, $h^{n-1}-1=0$ and $h^{n-2}-1=0$. This implies $h=1$, and so $f(z)\equiv g(z)$.

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