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Permutation groups with bounded movement having maximum orbits

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Abstract. Let G be a permutation group on a set Ω with no fixed points in Ω and let m be a positive integer. If no element of G moves any subset of Ω by more than m points (that is, $|\Gamma^g \backslash \Gamma| \leq m$ for every $\Gamma \subseteq \Omega$ and $g \in G$), and also if each G-orbit has size greater than 2, then the number t of G-orbits in Ω is at most $\frac{1}{2}(3m-1)$. Moreover, the equality holds if and only if G is an elementary abelian 3-group.

Keywords. Permutation group; bounded movement; orbit.

1. Introduction

Let G be a permutation group on a set Ω with no fixed points in Ω and let m be a positive integer. If for a subset Γ of Ω the size $|\Gamma^g \backslash \Gamma|$ is bounded, for $g \in G$, we define the movement of Γ as move(Γ) = $\max_{g\in G}|\Gamma^g \backslash \Gamma|$. If move($\Gamma$) $\leq m$ for all $\Gamma \subseteq \Omega$, then G is said to have *bounded movement* and the *movement* of G is defined as the maximum of move(Γ) over all subsets Γ, that is,

$$m := \text{move(G)} := \sup\{|\Gamma^g \backslash \Gamma| \,|\, \Gamma \subseteq \Omega, \quad g \in G\}.$$

This notion was introduced in [1,2,5]. By Theorem 1 of [5] if G has bounded movement equal to m, then Ω is finite. Moreover both the number of G-orbits in Ω and the length of each G-orbit are bounded above by linear functions of m. In particular, it was shown in Lemma 3.5 of [3] that the number of G-orbits is at most $2m-1$. In this paper we will improve this to $\frac{1}{2}(3m-1)$, when each G-orbit has size ≥ 3. If $m=1$, then $t=1$, $|\Omega|=3$ and G is Z_3 or S_3. So in the rest of this paper we suppose that m is greater than 1. We present here a classification of all groups for which the bound $\frac{1}{2}(3m-1)$ is attained. We shall say that an orbit of permutation group is nontrivial if its length is greater than 1. The main result is the following theorem.

Theorem 1.1. *Let m be a positive integer and suppose that G is a permutation group on a set Ω such that G has no fixed points in Ω, and G has bounded movement equal to* m. *If each G-orbit has size greater than* 2, *then the number t of G-orbits in Ω is at most $\frac{1}{2}(3m-1)$. Moreover, $t=\frac{1}{2}(3m-1)$, if and only if m is a power of* 3, *and G is elementary abelian of order* $3m$, *all G-orbits have length* 3, *and the pointwise stabilizers of the G-orbits are precisely $\frac{1}{2}(3m-1)$ distinct subgroups of G of index* 3.

Note that an orbit of a permutation group is non trivial if its length is greater than 1. The groups described in the next section are examples of permutation groups with bounded movement equal to m which have exactly $\frac{1}{2}(3m-1)$ nontrivial orbits.

2. Examples and preliminaries

Let $1 \neq g \in G$ and suppose that g in its disjoint cycle representations has t nontrivial cycles of lengths $l_1, \ldots, l_t$, say. We might represent g as

$$g = (a_1 a_2 \ldots a_{l_1})(b_1 b_2 \ldots b_{l_2}) \ldots (z_1 z_2 \ldots z_{l_t}).$$

Let $\Gamma(g)$ denote a subset of Ω consisting of $\lfloor l_i/2 \rfloor$ points from the i-th cycle, for each i, chosen in such a way that $\Gamma(g)^g \bigcap \Gamma(g) = \emptyset$. For example, we could choose

$$\Gamma(g) = \{a_2, a_4, \ldots, a_{k_1}, b_2, b_4, \ldots, b_{k_2}, \ldots, z_2, z_4, \ldots, z_{k_t}\},$$

where $k_i = l_i - 1$ if l_i is odd and $k_i = l_i$ if l_i is even. Note that $\Gamma(g)$ is not uniquely determined as it depends on the way each cycle is written. For any set $\Gamma(g)$ consists of *every second point of every cycle of g*. From the definition of $\Gamma(g)$ we see that

$$|\Gamma(g)^g \backslash \Gamma(g)| = |\Gamma(g)| = \sum_{i=1}^{t} \lfloor l_i/2 \rfloor.$$

The next lemma shows that this quantity is an upper bound for $|\Gamma^g \backslash \Gamma|$ for an arbitrary subset Γ of Ω.

Lemma 2.1 (*Lemma* 2.1 *of* [4]). *Let G be a permutation group on a set Ω and suppose that $\Gamma \subseteq \Omega$. Then for each $g \in G$, $|\Gamma^g \backslash \Gamma| \leq \sum_{i=1}^{t} \lfloor l_i/2 \rfloor$, where l_i is the length of the i-th cycle of g and t is the number of nontrivial cycles of g in its disjoint cycle representation. This upper bound is attained for the $\Gamma = \Gamma(g)$ defined above.*

Now we will show that there certainly is an infinite family of 3-groups for which the maximum bound obtained in Theorem 1.1 holds.

Example 2.2. Suppose that d is a positive integer, $G := Z_3^d$, $t := \frac{1}{2}(3^d - 1)$, and let $H_1, \ldots, H_t$ be an enumeration of the subgroups of index 3 in G. Define Ω_i to be the coset space of H_i in G and $\Omega := \Omega_1 \cup \ldots \cup \Omega_t$. If $g \in G$, then g lies in $\frac{1}{2}(3^{d-1} - 1)$ of the groups H_i and therefore acts on Ω as a permutation with $\frac{1}{2}(3^d - 3)$ fixed points and 3^{d-1} disjoint 3-cycles. Taking every second point from each of these 3-cycles to form a set Γ, by Lemma 2.1 we see that $\mathrm{move}(G) = 3^{d-1}$. It follows that G has a bounded movement equal to m, and G has $\frac{1}{2}(3\mathrm{move}(G) - 1)$ nontrivial orbits in Ω.

Since $m > 1$, the classification in Theorem 1.1 follows immediately from the following theorem about subsets with movement m.

DEFINITION

Let G be a permutation group on a set Ω with orbits Ω_i, for $i \in I$. We shall say that a subset $\Gamma \subseteq \Omega$ cuts across each G-orbit if $\Gamma_i := \Gamma \cap \Omega_i \notin \{\emptyset, \Omega_i\}$, for all $i \in I$.

Now we have the following theorem which is basic.

Theorem 2.3. *Let $G \leq \mathrm{Sym}(\Omega)$ be a permutation group with t orbits for positive integer t, such that each orbit has length greater than 2. Moreover suppose that $\Gamma \subseteq \Omega$ such that* $\mathrm{move}(\Gamma) = m > 1$, *and Γ cuts across each G-orbit. Then $t \leq \frac{1}{2}(3m-1)$. Moreover, if $t = \frac{1}{2}(3m-1)$ then*:

(1) *G is an elementary abelian 3-group and every G-orbit has size 3;*
(2) *If the rank of the group G is d, then $d \geq 3$, $t = \frac{1}{2}(3^d - 1)$ and $m = 3^{d-1}$;*
(3) *The t different G-orbits are (G-equivalent to) the coset spaces of the $\frac{1}{2}(3^{d-1})$ different subgroups of index 3 in G.*

3. Proof of Theorem 2.3

Let $\Omega_1, \ldots, \Omega_t$ be t orbits of G of lengths $n_1, \ldots, n_t$. Choose $\alpha_i \in \Omega_i$ and let $T_i := G_{\alpha_i}$ so that $|G : T_i| = n_i$. For $g \in G$, set $\Gamma(g) := \{\alpha_i | \alpha_i^g \neq \alpha_i\}$ and let $\gamma(g) := |\Gamma(g)|$. Since $\Gamma(g) \cap \Gamma(g)^g = \emptyset$ it follows that $\gamma(g) \leq m$ for all $g \in G$. Let $\bar{\Omega} := \Omega_1 \cup \cdots \cup \Omega_t$, and let $\bar{G}$ and $\bar{T}_1, \ldots, \bar{T}_t$ denote the finite permutation groups on $\bar{\Omega}$ induced by G and $T_1, \ldots, T_t$ respectively. Then $n_i = |\bar{G} : \bar{T}_i|$.

For $g \in G$, let $\bar{g} \in \bar{G}$ denote the permutation of $\bar{\Omega}$ induced by g. Then as $\gamma(1_G) = 0$, we have $\sum_{\bar{g} \in \bar{G}} \gamma(g) < m|\bar{G}|$.

Now, counting the pairs $(\bar{g}, i)$ such that $\bar{g} \in \bar{G}$ and $\alpha_i^g \neq \alpha_i$ gives

$$\begin{aligned}\textstyle\sum_{\bar{g} \in \bar{G}} \gamma(g) &= \textstyle\sum_i |\{\bar{g} \in \bar{G} | \alpha_i^g \neq \alpha_i\}| = \sum_i |\{\bar{g} \in \bar{G} | g \notin T_i\}| \\ &= \textstyle\sum_i (|\bar{G}| - |\bar{T}_i|) = |\bar{G}| \sum_i (1 - \frac{1}{n_i}).\end{aligned}$$

It follows that $\sum_i (1 - \frac{1}{n_i}) < m$. Since $n_i \geq 3$ for each i, it follows that $\sum_i (1 - \frac{1}{n_i}) \geq \frac{2t}{3}$ and hence $\frac{2t}{3} < m$, that is, $t \leq \frac{1}{2}(3m-1)$.

Consequently G has at most $\frac{1}{2}(3m-1)$ orbits in Ω. Suppose that G has exactly t orbits say, $\Omega_1, \Omega_2, \ldots, \Omega_t$, where $t = \frac{1}{2}(3m-1)$. Suppose further that $\Gamma \subseteq \Omega$ has move $(\Gamma) = m$ that cuts across each of the G-orbits Ω_i. For each i, set $\Gamma_i = \Gamma \cap \Omega_i$. Note that $0 < |\Gamma_i| < n_i$. Then we have the following claim.

Claim 3.1. If Theorem 2.3 holds for the special case in which $|\Gamma_i| = 1$ for each $i = 1, \ldots (3m-1)/2$, then it holds in general.

Proof. Let Theorem 2.3 hold for the special case in which $|\Gamma_i| = 1$. For each $i = 1, \ldots, \frac{1}{2}(3m-1)$, define $\sum_i := \{\Gamma_i^g | g \in G\}$, and note that $|\sum_i| \geq 2$ since Γ cuts across Ω_i. Set $\Sigma := \cup_{i \geq 1} \sum_i$. Then G induces a natural action on Σ for which the G-orbits are $\Sigma_1, \ldots, \Sigma_t$. Let G^{Σ} denote the permutation group induced by G on Σ, and let K denote the kernel of this action.

We claim that the t-element subset $\Gamma_\Sigma = \{\Gamma_1, \ldots, \Gamma_t\} \subseteq \Sigma$ has movement equal to m relative to G^Σ, and that Γ_Σ cuts across each G^Σ-orbit Σ_i. For each $g \in G$, $|\Gamma^g - \Gamma| \leq m$ and hence $|\Gamma_\Sigma^g - \Gamma_\Sigma| \leq m$. Thus move $(\Gamma_\Sigma) \leq m$. Also, since $|\Sigma_i| \geq 2$ and $\Gamma_\Sigma \cap \Sigma_i$ consists of the single element Γ_i of Σ_i, the set Γ_Σ cuts across each of the $\frac{1}{2}(3m-1)$ orbits Σ_i. However, it follows that the number of G^Σ-orbits is at most $\frac{1}{2}(3\ \text{move}(\Gamma_\Sigma) - 1)$, and hence move $(\Gamma_\Sigma) = m$.

Thus the hypotheses of Theorem 2.3 hold for the subset $\Gamma_\Sigma \subseteq \Sigma$ relative to G^Σ, and Γ_Σ meets each G^Σ-orbit in exactly one point. By our assumption it follows that $t = \frac{1}{2}(3^d - 1) = \frac{1}{2}(3m - 1)$ for some $d > 1$, and that $G^\Sigma = Z_3^d$ and each $|\Sigma_i| = 3$. Further, the subgroups H_i of G fixing Γ_i setwise range over the $\frac{1}{2}(3^d - 1)$ distinct subgroups which have index 3 in G and which contain K. In particular, for each i, H_i is normal in G and hence the H_i-orbits in Ω_i are blocks of imprimitivity for G, and their number is at most $|G : H| = 3$. Since H_i fixes Γ_i setwise it follows that Γ_i is an H_i-orbit and $n_i = 3|\Gamma_i|$.

Let $g \in G \backslash K$. Then as in Example 2.2, in its action on Σ, g moves exactly m of the Γ_i. Since the Γ_i are blocks of imprimitivity for G, each Γ_i^g is either Γ_i or $\Gamma_i \cap {\Gamma_i}^g = \phi$. It follows that $|\Gamma^g \backslash \Gamma|$ is equal to the sum of the sizes of the m subsets Γ_i moved by g. However, since move $(\Gamma) = m$, each of these m subsets Γ_i must have size 1. Since for each i we may choose an element g which moves Γ_i, we deduce that each of the Γ_i has size 1, and that K is the identity subgroup. It follows that Theorem 2.3 holds for G. Thus the claim is proved.

From now on we may and shall assume that each $|\Gamma_i| = 1$. Let $\Gamma_i = \{\omega_i\}$. Further we may assume that $n_1 \leq n_2 \leq \cdots \leq n_t$. For $g \in G$, let $c(g)$ denote the number of integers i such that $\omega_i^g = \omega_i$. Note that since move $(\Gamma) = m$, we have

$$c(g) > t - m = \frac{1}{2}(3m-1) - m = (m-1)/2 \text{ and also } c(1_G) = t > (m-1). \qquad (*)$$

Lemma 3.2. *With the above notion we have* $n_1 = 3$.

Proof. Let X denote the number of pairs (g, i) such that $g \in G$, $1 \leq i \leq t$ and $\omega_i^g = \omega_i$. Then $X = \sum_{g \in G} c(g)$, and by our observations, $X > |G|.\frac{1}{2}(m - 1)$, because by the relation of (*) we have

$$X = \sum_{g \in G} c(g) = \sum_{1 \neq g \in G} c(g) + c(1_G) > (|G| - 1)\tfrac{1}{2}(m - 1) + (m - 1)$$

$$= |G|\tfrac{1}{2}(m - 1) + \tfrac{1}{2}(m - 1) > |G|\tfrac{1}{2}(m - 1).$$

On the other hand, for each i, the number of elements of G which fix ω_i is $|G_{\omega_i}| = \frac{|G|}{n_i}$, and hence $X = |G| \sum_{i=1}^{t} n_i^{-1}$. If all $n_i \geq 4$, then $X \leq |G|\frac{t}{4} \leq |G|(m - 1)/2$ (since $m \geq 2$) which is a contradiction. Hence $n_i = 3$.

A similar argument to this enables us to show that all $n_i = 3$, for each $(2 \leq i \leq t)$ and hence G is an elementary abelian 3-group.

Lemma 3.3. *Let* $G = Z_3^r$ *for some* $r \geq 2$. *Moreover for each* $n_i = 3$, *the stabilizers* G_{ω_i} $(2 \leq i \leq t)$ *are pairwise distinct subgroups of index* 3 *in* G, *and for each* $g \neq 1, c(g) = (m - 1)/2$, *such that* $g \in G$, $2 \leq i \leq t$, *and* $\omega_i^g = \omega_i$.

Proof. By Lemma 3.2, $n_1 = 3$. Thus $H := G_{\omega_1}$ is a subgroup of index 3. This time we compute the number Y of pairs (g, i) such that $g \in G\backslash H, 2 \le i \le t$, and $\omega_i^g = \omega_i$. For each such g, $\omega_1^g \ne \omega_1$ and hence there are $c(g)$ of these pairs with first entry g. Thus $Y = \sum_{g\in G\backslash H} c(g) \ge |G\backslash H|(m-1)/2 = (|G|-|H|)(m-1)/2 = (2|H|/3)(m-1)/2 = |G| \cdot (m-1)/3$.

On the other hand, for each $i \ge 2$, the number of elements of $G\backslash H$, which fix ω_i is $|G_{\omega_i}\backslash H|$. If $G_{\omega_i} = H$, $|G_{\omega_i}\backslash H| = 0$ while if $G_{\omega_i} \ne H$, then $|G_{\omega_i}\backslash H| = |G_{\omega_i}\backslash(H \cap G_{\omega_i})| = |G_{\omega_i}| - |G_{\omega_i} \cap H| = |G|(1 - 1/3) = 2/3|G|$. Hence

$$\begin{aligned} Y &= \textstyle\sum_{i=2}^{t} |G\backslash H| = \frac{2}{3}|G| \sum_{i=2}^{t} \frac{1}{n_i} \le \frac{2}{9}|G|(t-1) \\ &= \frac{2}{9}|G|(\frac{1}{2}(3m-1) - 1) = \frac{|G|}{3}(m-1). \end{aligned}$$

So we have

$$|G|\frac{m-1}{3} \le Y \le |G|\frac{m-1}{3}.$$

It follows that equality holds in both of the displayed approximations for Y. This means, in particular, that each $n_i = 3$. Further, for each $i \ge 2$, $G_{\omega_i} \ne H$ and so $r \ge 2$. Arguing in the same way with H replaced by G_{ω_i}, for some $i \ge 2$, we see that $G_{\omega_i} \ne G_{\omega_j}$ if $j \ne i$, and also if $g \in G_{\omega_i}$ then $c(g) = (m-1)/2$. Thus the stabilizers $G_{\omega_i} (1 \le i \le t)$ are pairwise distinct, and if $g \ne 1$, then $c(g) = (m-1)/2$.

Finally we determine m.

Lemma 3.4. $m = 3^{r-1}$.

Proof. We use the information in Lemma 3.3 to determine precisely the quantity $X = \sum_{g\in G} c(g) = t + (|G| - 1).(m-1)/2 = \frac{1}{2}(3m-1) + (3^r - 1)(m-1)/2$.

On the other hand, from the proof of Lemma 3.2,

$$X = |G|\sum_{i=1}^{t} n_i^{-1} = \frac{|G|}{3}t = 3^{r-1} \,.\, \frac{1}{2}(3m-1).$$

Thus $(3^{r-1} - 1)(\frac{1}{2}(3m-1)) = (3^r - 1)(m-1)/2$, and this implies that $m = 3^{r-1}$.

The proof of Theorem 2.3 now follows from Lemmas 3.2–3.4.

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