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On non-Frattini chief factors and solvability of finite groups

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Abstract. A subgroup H of a group G is said to be a semi CAP^*-subgroup of G if there is a chief series $1 = G_0 < G_1 < \cdots < G_m = G$ of G such that for every non-Frattini chief factor G_i/G_{i-1}, H either covers G_i/G_{i-1} or avoids G_i/G_{i-1}. In this paper, some sufficient conditions for a normal subgroup of a finite group to be solvable are given based on the assumption that some maximal subgroups are semi CAP^*-subgroups.

Keywords. Frattini chief factors; solvable groups; semi cover-avoiding properties.

1. Introduction

All groups considered in this paper are finite.

Let G be a solvable group. It is well-known that every maximal subgroup M of G is a CAP-subgroup of G, that is $MH = MK$ or $M \cap H = M \cap K$ for every chief factor H/K of G. The converse of this result is also true. In fact, Guo and Shum [4] proved the following theorem.

Theorem A (Theorem 3.1 of [4]). *A group G is solvable if and only if every maximal subgroup M of G in $\mathcal{F}^{ocn}(G)$ is a CAP-subgroup of G, where $\mathcal{F}^{ocn}(G) = \{M \lessdot G|$ and there exists $P \in \mathrm{Syl}_p(G)$ with $p \neq 2$ such that $N_G(P) \leq M$, $|G : M|$ is composite and M is non-nilpotent$\}$.*

If we read the proof of Theorem A, it is easy to see that the key point of the proof is the following:

Lemma B (*Lemma* 2.8 *of* [4]). *A group G is solvable if there exists a minimal normal subgroup N of G and a solvable maximal subgroup M of G such that M avoids $N/1$, that is $M \cap N = 1$.*

It is clear that $N/1$ in Lemma B is a non-Frattini chief factor of G. So an interesting question is: How about the influence of non-Frattini chief factors on the solvability of finite groups?

In this paper, we use non-Frattini chief factors in a group to investigate the solvability of the group. In §2 we introduce a new concept: semi CAP^*-subgroups, which is related to partly non-Frattini chief factors in a group. Then, in §3, we prove some sufficient conditions for a normal subgroup of a group to be solvable provided some maximal subgroups in the group are semi CAP^*-subgroups.

2. Basic definitions and preliminary results

In this section, we give the definition of semi CAP^*-subgroups. Then we discuss some properties of semi CAP^*-subgroup.

Let K and L be normal subgroups of a group G with $L \leq K$. Then K/L is called a normal factor of G. A subgroup H of G is said to cover K/L if $HK = HL$. On the other hand, if $H \cap K = H \cap L$, then H is said to avoid K/L. If K/L is a chief factor of G and $K/L \leq \Phi(G/L)$ (respectively $K/L \nleq \Phi(G/L)$), then K/L is said to be a Frattini (respectively non-Frattini) chief factor of G.

DEFINITION 2.1

A subgroup H of a group G is said to be a semi CAP^*-subgroup of G if there is a chief series $1 = G_0 < G_1 < \cdots < G_m = G$ of G such that for every non-Frattini chief factor G_i/G_{i-1}, H either covers G_i/G_{i-1} or avoids G_i/G_{i-1}.

Recall that a subgroup H of a group G is said to be a semi CAP-subgroup of G if H covers or avoids every chief factor of some chief series of G (Definition 2.1 of [3]). It is clear that a semi CAP-subgroup of a group G must be a semi CAP^*-subgroup of G. However, the converse is not true.

Example 2.2. Let $P = \langle a\rangle \times \langle b\rangle$ be the direct product of two cyclic groups $\langle a\rangle$ and $\langle b\rangle$ of order 4 and $c \in \mathrm{Aut}(P)$ such that $a^c = a^2b^3$, $b^c = a^3b$. Then the semidirect product $K = P \rtimes \langle c\rangle$ is of order $2^4 \times 3$. We set $G = K \times \langle d\rangle$, the direct product of K and a cyclic subgroup $\langle d\rangle$ of order 2.

It is easy to see that the Frattini subgroup $\Phi(G) = \langle a^2, b^2\rangle$ of G is a minimal normal subgroup of G, the series

$$1 < \Phi(G) < P < K < G$$

is a chief series of G, $\Phi(G)/1$ is Frattini and the rest are non-Frattini. We can see that $H = \langle a^2\rangle$ avoids every non-Frattini chief factors of this series and therefore H is a semi CAP^*-subgroup of G. However, $H \cap \Phi(G) = H \neq 1 = H \cap 1$ and $H\Phi(G) = \Phi(G) \neq H$. This implies that the chief factor $\Phi(G)/1$ is neither covered nor avoided by H. Similarly, $\Phi(G)C_2/C_2$ is neither covered nor avoided by H, where $C_2 = \langle d\rangle$. On the other hand, every chief series of G must contain one of $\Phi(G)/1$ and $\Phi(G)C_2/C_2$. Thus H is not a semi CAP-subgroup of G.

Lemma 2.3 (*Lemma* 1.2.20 *of* [1]). *Let K/L be a chief factor of a group G. If N is a normal subgroup of G contained in L, then K/L is a Frattini chief factor of G if and only if $(K/N)/(L/N)$ is a Frattini chief factor of G/N.*

Lemma 2.4. *Let K/L be a chief factor of a group G and N a normal subgroup of G. Then*:

(1) *If N avoids K/L and $K/L \leq \Phi(G/L)$, then KN/LN is a chief factor of G and $KN/LN \leq \Phi(G/LN)$.*
(2) *If N covers K/L and $K/L \nleq \Phi(G/L)$, then $(K \cap N)/(L \cap N)$ is a chief factor of G and $(K \cap N)/(L \cap N) \nleq \Phi(G/L \cap N)$.*
(3) *If N covers K/L and $N \leq \Phi(G)$, then $K/L \leq \Phi(G/L)$ and $(K \cap N)/(L \cap N) \leq \Phi(G/L \cap N)$.*
(4) *Let N avoids K/L and $N \leq \Phi(G)$. Then $K/L \nleq \Phi(G/L)$ if and only if $KN/LN \nleq \Phi(G/LN)$.*

Proof. If N avoids K/L, then it follows from $KN/LN \cong K/L$ that KN/LN is a chief factor of G. If N covers K/L, then it follows from $(K \cap N)/(L \cap N) \cong K/L$ that $(K \cap N)/(L \cap N)$ is a chief factor of G.

(1) Since $K/L \leq \Phi(G/L)$, $(KN/L)/(LN/L) \leq \Phi((G/L)/(LN/L))$ and therefore $KN/LN \leq \Phi(G/LN)$.
(2) By hypothesis, there exists a maximal subgroup M/L of G/L such that $K/L \nleq M/L$. If $K \cap N \leq M$, then $L(K \cap N) = K \cap LN = K \cap KN = K \leq M$, a contradiction. Hence $(K \cap N)/(L \cap N) \nleq \Phi(G/L \cap N)$
(3) Since $N \leq \Phi(G)$, we can see that $(K \cap N)/(L \cap N) \leq \Phi(G)/(L \cap N) \leq \Phi(G/L \cap N)$. Also since $N \leq \Phi(G)$, we see that $NL/L \leq \Phi(G)L/L \leq \Phi(G/L)$ and therefore $KN/L = NL/L \leq \Phi(G/L)$. Hence $K/L \leq \Phi(G/L)$.
(4) By (1), we only need to prove the necessary condition. If $K/L \nleq \Phi(G/L)$, then there exists a maximal subgroup M/L of G/L such that $KM = G$. Since $N \leq \Phi(G)$, M/LN is a maximal subgroup of G/LN and $MKN = G$. This completes our proof. □

Lemma 2.5. *Let N be a normal subgroup of a group G and $N \leq H$. Then H is a semi CAP^*-subgroup of G if and only if H/N is a semi CAP^*-subgroup of G/N.*

Proof. Suppose that H is a semi CAP^*-subgroup of G. Then, there exists a chief series

$$1 = G_0 < G_1 < \cdots < G_m = G$$

of G such that H covers or avoids every non-Frattini chief factor of this series. It is easy to see that the following series:

$$1 = G_0N/N \leq G_1N/N \leq G_2N/N \leq \cdots \leq G_mN/N = G/N \qquad (*)$$

is a chief series of G/N. Suppose that $(G_iN/N)/(G_{i-1}N/N)$ is a non-Frattini chief factor of G/N. For finishing our proof, we only need to prove either $HG_iN = HG_{i-1}N$ or $H \cap G_iN = H \cap G_{i-1}N$. Since $N \trianglelefteq G$, $NG_i = NG_{i-1}$ or $N \cap G_i = N \cap G_{i-1}$. Clearly $HG_iN = HG_{i-1}N$ if $NG_i = NG_{i-1}$. So in the following we may assume that $N \cap G_i = N \cap G_{i-1}$. By Lemma 2.3, we have $G_iN/G_{i-1}N \nleq \Phi(G/G_{i-1}N)$.

Applying Lemma 2.4(1), we see $G_i/G_{i-1} \nleq \Phi(G/G_{i-1})$ and therefore H covers or avoids G_i/G_{i-1}. If H covers G_i/G_{i-1}, then it follows from $HG_i = HG_{i-1}$ that $HG_iN = HG_{i-1}N$. Hence we may assume that H avoids G_i/G_{i-1}, that is $H \cap G_i = H \cap G_{i-1}$. This implies that $H \cap G_iN = (H \cap G_i)N = H \cap G_{i-1}N$, as desired.

Conversely, if H/N is a semi CAP^*-subgroup of G/N, then there exists a chief series

$$\bar{1} = G_0/N < G_1/N < G_2/N < \cdots < G_m/N = G/N$$

of G/N such that H/N covers or avoids every non-Frattini chief factor of this series. It is easy to see that the following series

$$N = G_0 < G_1 < \cdots < G_m = G \qquad (\sharp)$$

is part of a chief series of G and H covers or avoids every non-Frattini chief factor of ($\sharp$). Moreover, H covers every non-Frattini chief factor of G contained in N. Hence H is a semi CAP^*-subgroup of G. □

Lemma 2.6. *Let N be a normal subgroup of a group G and let H be a semi CAP^*-subgroup of G. Suppose that $gcd(|H|, |N|) = 1$. Then*

(1) *HN is a semi CAP^*-subgroup of G*;
(2) *HN/N is a semi CAP^*-subgroup of G/N.*

Proof.

(1) Since H is a semi CAP^*-subgroup of G, there exists a chief series

$$1 = G_0 < G_1 < \cdots < G_m = G$$

of G such that H covers or avoids every non-Frattini chief factor of this series. Let G_i/G_{i-1} be a non-Frattini chief factor of G. If one of H and N covers G_i/G_{i-1}, then HN covers G_i/G_{i-1}. Hence we may assume that both H and N avoid G_i/G_{i-1}.

We can consider the index $|HNG_i : HNG_{i-1}|$. On the one hand, $|HNG_i : HNG_{i-1}| = |HG_i : HG_{i-1}|/|HG_i \cap N : HG_{i-1} \cap N| = |G_i : G_{i-1}|/|HG_i \cap N : HG_{i-1} \cap N|$. On the other hand, $|HNG_i : HNG_{i-1}| = |G_i : G_{i-1}|/|NG_i \cap H : NG_{i-1} \cap H|$. Therefore, $|HG_i \cap N : HG_{i-1} \cap N| = |NG_i \cap H : NG_{i-1} \cap H|$. However, $|NG_i \cap H : NG_{i-1} \cap H|$ is a π-number and $|HG_i \cap N : HG_{i-1} \cap N|$ is a π'-number, where $\pi = \pi(H)$. It follows that $HG_{i-1} \cap N = HG_i \cap N$. Moreover, $|HNG_i : HNG_{i-1}| = |G_i : G_{i-1}|/|HN \cap G_i : HN \cap G_{i-1}|$. Hence $HN \cap G_i = HN \cap G_{i-1}$ by comparing the orders. It follows that HN avoids G_i/G_{i-1}.

(2) Applying (1) and Lemma 2.5 completes the proof. □

Lemma 2.7. *Let H be a semi CAP^*-subgroup of a group G. Then H is a semi CAP-subgroup of G if one of the following holds*:

(1) *H is a maximal subgroup of G*;
(2) *H is a Hall subgroup of G.*

Proof. Since H is a semi CAP^*-subgroup of G, there exists a chief series

$$1 = G_0 < G_1 < \cdots < G_m = G \qquad (*)$$

of G such that H covers or avoids every non-Frattini chief factor of $(*)$. To finish the proof, we only need to prove that H covers or avoids every Frattini chief factor of $(*)$. Suppose that G_i/G_{i-1} is a chief factor of $(*)$ with $G_i/G_{i-1} \leq \Phi(G/G_{i-1})$.

(1) Let H be a maximal subgroup of G. If $G_{i-1} \leq H$, then $HG_{i-1} = H = HG_i$. If $G_{i-1} \nleq H$, then $HG_{i-1} = G = HG_i$. It follows that H covers G_i/G_{i-1}. Hence H is a semi CAP-subgroup of G.
(2) Let H be a Hall subgroup of G. We can see that G_i/G_{i-1} is an elementary abelian p-group for some prime p. If H is a p'-group, then it follows from $H \cap G_i = H \cap G_{i-1}$ that H avoids G_i/G_{i-1}. If p is a divisor of the order of H, then it follows from $HG_i = HG_{i-1}$ that H covers G_i/G_{i-1}, as desired. □

Let H be a normal subgroup of a group G and p a prime. We define the following families of subgroups:

$\mathcal{F}(G) = \{M | M \lessdot G\}$,
$\mathcal{F}_n(G) = \{M | M \in \mathcal{F}(G)$ and M is non-nilpotent$\}$,
$\mathcal{F}_c(G) = \{M | M \in \mathcal{F}(G)$ and $|G : M|$ is composite$\}$,
$\mathcal{F}^p(G)$=$\{M | M \in \mathcal{F}(G)$ and $N_G(P) \leq M$ for a Sylow p-subgroup P of $G\}$,
$\mathcal{F}_p(G) = \{M | M \in \mathcal{F}(G)$ and $|G : M|_p = 1\}$,
$\mathcal{F}^{op}(G) = \bigcup_{p \in \pi(G)-\{2\}} \mathcal{F}^p(G)$,
$\mathcal{F}_{pc}(G) = \mathcal{F}_p(G) \cap \mathcal{F}_c(G)$,
$\mathcal{F}^{pn}(G) = \mathcal{F}^p(G) \cap \mathcal{F}_n(G)$,
$\mathcal{F}^{pcn}(G) = \mathcal{F}^p(G) \cap \mathcal{F}_c(G) \cap \mathcal{F}_n(G)$,
$\mathcal{F}^{ocn}(G) = \mathcal{F}^{op}(G) \cap \mathcal{F}_c(G) \cap \mathcal{F}_n(G)$,
$\mathcal{F}_H(G) = \{M | M \in \mathcal{F}(G)$ and $H \nleq M\}$.

3. Main results

In this section, we study the solvability of a normal subgroup H of a group G when some subgroups are assumed to be semi CAP^*-subgroups of G.

Theorem 3.1. *Let H be a normal subgroup of a group G and let p be the largest prime dividing the order of G. If every maximal subgroup M of G in $\mathcal{F}_{pc}(G) \cap \mathcal{F}_H(G)$ is a semi CAP^*-subgroup of G, then H is solvable.*

Proof. If $\mathcal{F}_{pc}(G) \cap \mathcal{F}_H(G) = \emptyset$, then we claim that H is solvable. In fact, if $\mathcal{F}_{pc}(G) = \emptyset$, by Theorem 8 of [8], G is solvable and so is H. If $\mathcal{F}_{pc}(G) \neq \emptyset$, then H is contained in every maximal subgroup of G in $\mathcal{F}_{pc}(G)$. Applying Theorem 8 of [8] again, H is solvable. Now we may assume that $\mathcal{F}_{pc}(G) \cap \mathcal{F}_H(G) \neq \emptyset$.

Let N be a minimal normal subgroup of G, and let M/N be a maximal subgroup of G/N with $M/N \in \mathcal{F}_{pc}(G/N) \cap \mathcal{F}_H(G/N)$. Then $M \in \mathcal{F}_{pc}(G) \cap \mathcal{F}_H(G)$. Furthermore, M/N is a semi CAP^*-subgroup of G/N by Lemma 2.5. It is clear that G/N satisfies the hypothesis of the theorem for the normal subgroup HN/N and so HN/N is solvable by induction. If $N \nleq H$, then $H \cong HN/N$ is solvable, as desired. Hence we may assume that $N \leq H$. Now a routine argument shows that G is solvable if G has two different minimal normal subgroup of G. Hence we may assume that N is the unique minimal normal subgroup of G.

Suppose that N is non-solvable. Let q be the largest prime dividing the order of N and Q a Sylow q-subgroup of N. Then $G = N_G(Q)N$ by the Frattini argument. So there exists a maximal subgroup M of G which contains $N_G(Q)$, but $N \nleq M$. By hypothesis, $p \geq q$. If $p > q$, it is clear that $|G : M|_p = |N : M \cap N|_p$=1. If $p = q$, then $N_G(Q)$ contains a Sylow p-subgroup of G. Thus we conclude that $|G : M|_p = 1$ in these two cases. If $|G : M| = r$ for some prime r, then, since $M_G = 1$, we have that G is isomorphic to a subgroup of the symmetric group S_r of degree r. This implies that $|G|\big|r!$ and so $|N|\big|r!$, in contradiction to q being the largest prime in $\pi(N)$. Hence we conclude that $M \in \mathcal{F}_{pc}(G) \cap \mathcal{F}_H(G)$. By the hypothesis, M is a semi CAP^*-subgroup of G and $N \nleq \Phi(G)$, by the fact that $MN = M$ or $M \cap N = 1$. But these two situations are clearly impossible as $N_G(Q)$ is contained in M and $N \nleq M$, a contradiction. This shows that N is solvable and therefore H is solvable. □

In Theorem 3.1, the group G is not necessary solvable.

Example 3.2. Let K, H be the alternating groups of degrees 5 and 4, respectively and let $G = K \times H$. Suppose that $M = K \times C_3$, where C_3 is a cyclic group of order 3 of H. Then M is a maximal subgroup of G. It is clear that $H \nleq M$ and $|G : M| = 4$. Thus $M \in \mathcal{F}_{pc}(G) \cap \mathcal{F}_H(G)$ and we can also see that $\mathcal{F}_{pc}(G) \cap \mathcal{F}_H(G) = \{M^g | g \in G\}$. Furthermore, it is easy to see that the following series

$$1 < K < K \times K_4 < G$$

is a chief series of G and that M^g avoids $(K_4 \times K)/K$ and covers the other non-Frattini chief factors of G, where K_4 is the Klein four group contained in H. That is, M is a semi CAP^*-subgroup of G. However, G is not solvable.

Noticing that a group G is nilpotent if and only if every maximal subgroup of G is normal by Chapter A, Theorem 8.3 of [2]. We have the following result.

Theorem 3.3. *Let H be a normal subgroup of a group G. If every maximal subgroup M of G in $\mathcal{F}_{G'}(G) \cap \mathcal{F}_H(G)$ is a semi CAP^*-subgroup of G, then H is solvable.*

Proof. If $\mathcal{F}_{G'}(G) \cap \mathcal{F}_H(G)$=$\emptyset$, then we claim that H is solvable. In fact, if $\mathcal{F}_{G'}(G)$=$\emptyset$, then, for any maximal subgroup M of G, $M \geq G'$, it is clear that $G' \leq \Phi(G)$. So G is nilpotent by Theorem 5.2.16 of [9]. If $\mathcal{F}_{G'}(G) \neq \emptyset$, then H is contained in every maximal subgroup M of G in $\mathcal{F}_{G'}(G)$. Suppose that $\mathcal{L}^d(G) = \bigcap\{M | M \in \mathcal{F}_{G'}(G)\}$. It is clear that $\mathcal{L}^d(G)$ is a characteristic subgroup of G and the following inclusion $\Phi(G) \leq \mathcal{L}^d(G)$ always holds. Let $x \in \mathcal{L}^d(G) \backslash \Phi(G)$. Then $g^{-1}x^{-1}gx = [g, x] \in G' \cap \mathcal{L}^d(G) \leq \Phi(G)$ for any $g \in G$. Therefore $x\mathcal{L}^d(G) \in Z(G/\Phi(G))$. This means that $\mathcal{L}^d(G)/\Phi(G) \leq Z(G/\Phi(G))$. Applying Theorem 5.2.15 of [9], H is solvable. Hence we may assume that $\mathcal{F}_{G'}(G) \cap \mathcal{F}_H(G) \neq \emptyset$.

Let N be a minimal normal subgroup of G. If $M/N \in \mathcal{F}_{G'}(G/N) \cap \mathcal{F}_H(G/N)$, then $M \ngeq G'$ and $G = HM$. It follows from Lemma 2.5 that M/N is a semi CAP^*-subgroup of G/N. Thus, G/N satisfies the hypothesis of our theorem for the normal subgroup HN/N. By induction, HN/N is solvable and we may assume that N is contained in H and N is the unique minimal normal subgroup of G. If $N \leq \Phi(G)$, then H is solvable. We may assume that G is a primitive group. Let p be a prime dividing the order of N and P a Sylow p-subgroup of N. Then $G = N_G(P)N$ by the Frattini argument. If N is not a p-group, then there exists a maximal subgroup M of G such that $M \geq N_G(P)$ and so $G = MN$. This implies that $M \in \mathcal{F}_{G'}(G) \cap \mathcal{F}_H(G)$. By hypothesis, M covers or avoids $N/1$. However, these two cases are impossible. It follows that H is solvable. Thus, our proof is complete. □

In Theorem 3.3, the group G is not necessary solvable.

Example 3.4. Let $G = A_5 \times S_3$, a direct product of the alternating group of degree 5 and the symmetric group of degree 3. Then $G' = A_5 \times C_3$, where C_3 is a cyclic group of order 3 of S_3. Let $H = S_3$ and $M = A_5 \times C_2$, where C_2 is a cyclic group of order 2 of S_3. Thus $M \in \mathcal{F}_{G'}(G) \cap \mathcal{F}_H(G)$ and we can also see that $\mathcal{F}_{G'}(G) \cap \mathcal{F}_H(G) = \{M^g | g \in G\}$. Furthermore, it is easy to see that the following series

$$1 < A_5 < A_5 \times C_3 < G$$

is a chief series of G and that M^g avoids $(A_5 \times C_3)/A_5$ and covers the other non-Frattini chief factors of G. That is, M is a semi CAP^*-subgroup of G. However, G is not solvable.

Theorem 3.5. *Let H be a normal subgroup of a group G and let p be the largest prime dividing the order of G. If every maximal subgroup M of G in $\mathcal{F}^{pcn}(G) \cap \mathcal{F}_H(G)$ is a semi CAP^*-subgroup of G, then H is p-solvable.*

Proof. If $\mathcal{F}^{pcn}(G) \cap \mathcal{F}_H(G) = \emptyset$, then we can see that H is p-solvable by Lemma 2.4 of [4]. Now, we may assume that $\mathcal{F}^{pcn}(G) \cap \mathcal{F}_H(G) \neq \emptyset$. Let $P \in \mathrm{Syl}_p(G)$. If P is normal in G, then G is certainly p-solvable and so is H. So we may assume that $N_G(P) < G$.

Let N be a minimal normal subgroup of G. It is clear that G/N satisfies the hypothesis of the theorem for the normal subgroup HN/N and so HN/N is p-solvable by induction. By a routine argument, we can assume that N is contained in H and N is the unique minimal normal subgroup of G.

Suppose that N is not p-solvable. It follows that $N/1$ is a non-Frattini chief factor of G. Then p is a divisor of the order of N. We know that $N \cap P \in \mathrm{Syl}_p(N)$ and $P \cap N$ is not a normal subgroup of N. By Frattini argument, we have that $G = N_G(P \cap N)N$. So there exists a maximal subgroup M of G which contains $N_G(P \cap N)$ and $M \not\geq N$. It is clear that $N_G(P) \leq M$. If $|G : M| = q$ is a prime, then by Sylow's theorem, we have $q = 1 + kp$ and $q \big| |N|$. This contradicts p being the largest prime which divides the order of N. Hence $|G : M|$ must be a composite number. If M is nilpotent, then the Sylow 2-subgroup M_2 of M is not identity by Theorem 10.4.2 of [9]. Let $M_{2'}$ be a Hall $2'$-subgroup of M. By Theorem 1 of [10], $M_{2'}$ is normal in G and therefore $P \trianglelefteq G$ since P is a characteristic subgroup of $M_{2'}$. It follows that $P \cap N \trianglelefteq G$, a contradiction. Thus, $M \in \mathcal{F}^{pcn}(G) \cap \mathcal{F}_H(G)$. By the hypothesis, M is a semi CAP^*-subgroup of G and so $MN = M$ or $M \cap N = 1$. However, these two situations are impossible. This shows that N is p-solvable and therefore H is p-solvable. The proof of the theorem is now complete. □

In Theorem 3.5, the group G need not be p-solvable as the following example shows.

Example 3.6. Let $H = C_2 \times C_2 \times C_2 \times C_2$ be an elementary abelian group of order 2^4. Then there is a subgroup $M = A_5$ in the automorphism group of H, where A_5 is the alternating group of degree 5. Let $G = (C_2 \times C_2 \times C_2 \times C_2) \rtimes A_5$ be the corresponding semidirect product. We can deduce that $\mathcal{F}^{pcn}(G)\cap\mathcal{F}_H(G) = \{M^g | g \in G\}$. Furthermore, there exists a chief series

$$1 < H < G$$

of G such that M^g covers or avoids non-Frattini chief factors G/H and $H/1$. Thus, M^g is a semi CAP^*-subgroup of G. That is, G satisfies the hypothesis of Theorem 3.5 for normal subgroup H. However, G is not 5-solvable.

If we remove the hypothesis that p is the largest prime, then we have the following theorem.

Theorem 3.7. *Let H be a normal subgroup of a group G and let p be an odd prime dividing the order of G. If every maximal subgroup M of G in $\mathcal{F}^{pn}(G)\cap\mathcal{F}_H(G)$ is a semi CAP^*-subgroup of G, then H is p-solvable.*

Proof. If $\mathcal{F}^{pn}(G) \cap \mathcal{F}_H(G) = \emptyset$, then we claim that H is p-solvable. In fact, if $\mathcal{F}^{pn}(G) = \emptyset$, then, for any Sylow p-subgroup P of G, $N_G(P) = G$ or there exists a maximal subgroup M of G such that M is nilpotent and $N_G(P) \leq M$. If $N_G(P) = G$, then G is p-solvable and so H is as well. If $N_G(P) \leq M$ and M is nilpotent, then, by a result of Thompson (Theorem 10.4.2 of [9]), the subgroup M is of even order. Now, let $M_{2'}$ be a Hall $2'$-subgroup of M. Then, by a result of Rose (Theorem 1 of [10]), $M_{2'} \trianglelefteq G$. Since P is a characteristic subgroup of $M_{2'}$, P is normal in G and therefore G is p-solvable, as desired. If $\mathcal{F}^{pn}(G) \neq \emptyset$, then H is contained in every maximal subgroup M of G in $\mathcal{F}^{pn}(G)$. Suppose that $\mathcal{S}^{pn}(G) = \bigcap\{M | M \in \mathcal{F}^{pn}(G)\}$. It is clear that $\mathcal{S}^{pn}(G)$ is a characteristic subgroup of G. Let $Q \in \mathrm{Syl}_p(\mathcal{S}^{pn}(G))$. Then $G = N_G(Q)\mathcal{S}^{pn}(G)$ by the Frattini argument. If $N_G(Q) \neq G$, then there exists a maximal subgroup M of G such that $M \geq N_G(Q)$. If M is nilpotent, then G is p-solvable by using the above argument. Suppose that M is non-nilpotent. This implies that $M \in \mathcal{F}^{pn}(G)$. It follows that $G = N_G(Q)\mathcal{S}^{pn}(G) \leq M$, a contradiction. Thus, we can see that $\mathcal{S}^{pn}(G)$ is p-solvable and so is H. Hence we may assume that $\mathcal{F}^{pn}(G) \cap \mathcal{F}_H(G) \neq \emptyset$.

Let N be a minimal normal subgroup of G. Then, by Lemma 2.5, it is easy to see that G/N satisfies the hypothesis of the theorem for the normal subgroup HN/N. Now, by induction, we know that HN/N is p-solvable. By a routine argument, we can assume that N is contained in H and N is the unique minimal normal subgroup of G. If N is a p'-group, then H is p-solvable. Suppose that N is not a p'-group and $N/1$ is a non-Frattini chief factor of G. Now, we will prove that N is a p-group and therefore H is p-solvable.

Let M be a maximal subgroup of G in $\mathcal{F}^{pn}(G) \cap \mathcal{F}_H(G)$. It follows that $N \leq M$ or $N \cap M = 1$. But the latter situation is clearly impossible as every Sylow p-subgroup of M is a Sylow p-subgroup of G. Hence $N \leq M$ for every maximal subgroup of G in $\mathcal{F}^{pn}(G) \cap \mathcal{F}_H(G)$.

Now let P_1 be a Sylow p-subgroup of N, and P a Sylow p-subgroup of G such that $P_1 = P \cap N$. It is clear that $N_G(P) \leq N_G(P_1)$. If N is not a p-group, then $N_G(P_1) < G$ and therefore there is a maximal subgroup L of G such that $N_G(P_1) \leq L$. By using the same arguments as above, we can show that L is not nilpotent and therefore $L \in \mathcal{F}^{pn}(G) \cap \mathcal{F}_H(G)$. Thus, $N \leq L$ and therefore $NN_G(P_1) \leq M$. On the other hand, by using Frattini argument, we can show that $G = NN_G(P_1)$. It follows that $G \leq L$, a contradiction. This shows that N is a p-group and therefore H is a p-solvable group. □

Remark 3.8. Example 3.2 also illustrates that the group G is not necessarily p-solvable in Theorem 3.7. In fact, let P be a Sylow 3-subgroup of G. Then $N_G(P) = C_3 \times S_3$. It is clear that $N_G(P) \leq M$ and M is non-nilpotent. Thus $M \in \mathcal{F}^{pn}(G) \cap \mathcal{F}_H(G)$ and we can also see that $\mathcal{F}^{pc}(G) \cap \mathcal{F}_H(G)$=$\{M^g | g \in G\}$. Furthermore, M is a semi CAP^*-subgroup of G. However, G is not 3-solvable.

Notice that a group G is solvable if and only if G is 2-solvable. We have the following result.

Theorem 3.9. *Let H be a normal subgroup of a group G. If every maximal subgroup M of G in $\mathcal{F}^2(G) \cap \mathcal{F}_H(G)$ is a semi CAP^*-subgroup of G, then H is solvable.*

Proof. If $\mathcal{F}^2(G) \cap \mathcal{F}_H(G) = \emptyset$, then we claim that H is solvable. In fact, if $\mathcal{F}^2(G) = \emptyset$, then for any Sylow 2-subgroup P of G, $N_G(P) = G$. It follows that G is 2-solvable and therefore G is solvable since groups with odd order are solvable. If $\mathcal{F}^2(G) \neq \emptyset$, then H is contained in every maximal subgroup M of G in $\mathcal{F}^2(G)$. Suppose that $\mathcal{Z}(G) = \bigcap\{M | M \in \mathcal{F}^2(G)\}$. It is clear that $\mathcal{Z}(G)$ is a characteristic subgroup of G. Let $P_1 \in \mathrm{Syl}_2(\mathcal{Z}(G))$. By Sylow's theorem, there exists $P \in \mathrm{Syl}_2(G)$ such that $P_1 = P \cap \mathcal{Z}(G)$. If $P_1 \ntrianglelefteq G$, then there exists $M \lessdot G$ such that $N_G(P) \leq N_G(P_1) \leq M$ and so $\mathcal{Z}(G) \leq M \in \mathcal{F}^2(G)$. By the Frattini argument, $G = \mathcal{Z}(G)N_G(P_1) \leq M$, a contradiction. Therefore $P_1 \trianglelefteq G$ and so H is solvable. Hence we may assume that $\mathcal{F}^2(G) \cap \mathcal{F}_H(G) \neq \emptyset$.

Let N be a minimal normal subgroup of G. Then, by Lemma 2.5, it is easy to see that G/N satisfies the hypothesis of the theorem for the normal subgroup HN/N. Now, by induction, we know that HN/N is solvable. By a routine argument, we can assume that N is contained in H and N is the unique minimal normal subgroup of G. If N is a group of odd order, then H is solvable. Suppose that N is a group of even order and $N/1$ is a non-Frattini chief factor of G.

Let M be a maximal subgroup of G in $\mathcal{F}^2(G) \cap \mathcal{F}_H(G)$. It follows that $N \leq M$ or $N \cap M = 1$. But the latter situation is clearly impossible as every Sylow 2-subgroup of M is a Sylow 2-subgroup of G. Hence $N \leq M$ for every maximal subgroup of G in $\mathcal{F}^2(G) \cap \mathcal{F}_H(G)$. By using arguments similar to the proof of Theorem 3.7, we can prove that N is a 2-group, and therefore H is a solvable group. □

Remark 3.10. Example 3.4 also illustrates that the group G is not necessarily solvable in Theorem 3.9. In fact, we can deduce that $N_G(P) = A_4 \times C_2$, where P is a Sylow 2-subgroup of G. Thus $M \in \mathcal{F}^2(G) \cap \mathcal{F}_H(G)$ and we can also see that $\mathcal{F}^2(G) \cap \mathcal{F}_H(G) = \{M^g | g \in G\}$. Furthermore, M is a semi CAP^*-subgroup of G. However, G is not solvable.

Now, we use another family of maximal subgroups in a group to characterize the solvability of some normal subgroup of the group.

Theorem 3.11. *Let H be a normal subgroup of a group G. If every maximal subgroup M of G in $\mathcal{F}^{ocn}(G) \cap \mathcal{F}_H(G)$ is a semi CAP^*-subgroup of G, then H is solvable.*

Proof. If $\mathcal{F}^{ocn}(G) \cap \mathcal{F}_H(G) = \emptyset$, then we can see that H is solvable by Lemma 2.6 of [4]. Now, we may assume that $\mathcal{F}^{ocn}(G) \cap \mathcal{F}_H(G) \neq \emptyset$.

Let N be a minimal normal subgroup of G. It is clear that G/N satisfies the hypothesis of the theorem for the normal subgroup HN/N and so HN/N is solvable by induction. By a routine argument, we can assume that N is contained in H and N is the unique minimal normal subgroup of G.

If $N \leq \Phi(G)$, then H is solvable. Hence we may assume that $N/1$ is a non-Frattini chief factor of G.

Suppose that N is not solvable. We let p be the largest prime dividing the order of N and S a Sylow p-subgroup of N such that $S \leq P$, where P is a Sylow p-subgroup of G. Then there exists a maximal subgroup M of G such that $N_G(P) \leq N_G(S) \leq N_G(Z(J(S))) \leq M$, where $J(S)$ is the Thompson subgroup of S. Using the Frattini argument, we obtain that $G = NN_G(S) = NM$. If $|G : M| = q$ is a prime, then, by Sylow's theorem, we have $q = 1 + kp$ and $q \big| |N|$. This contradicts p being the largest prime number which divides the order of N. Hence $|G : M|$ must be a composite number. If M is nilpotent, then so is $N_G(Z(J(S)))$ and therefore $N_N(Z(J(S)))$ is nilpotent. Since we may assume that $p > 2$, by the Glauberman–Thompson theorem (Theorem 8.3.1 of [6]), we see that N is p-nilpotent. However, since N is a minimal normal subgroup of G and $p \big| |N|$, we see that N is a p-group, a contradiction. Thus, M must be a non-nilpotent group and so $M \in \mathcal{F}^{ocn}(G) \cap \mathcal{F}_H(G)$. By hypothesis, M covers or avoids $N/1$. However, these two situations are impossible. This completes the proof. □

By using the 2-maximal subgroups, we obtain the following theorem.

Theorem 3.12. *If there is a 2-maximal subgroup H of a group G such that H is a solvable semi CAP^*-subgroup of G, then G is solvable.*

Proof. If G is simple, then $H = 1$. Thus G has a maximal subgroup with prime order and therefore G is solvable (IV. 7.4 Satz of [7]). Hence we may assume that G is not simple. If $H_G \neq 1$, then it is easy to see that G/H_G satisfies the hypothesis of the theorem. An inductive argument shows that G/H_G is solvable and so is G. Suppose that $H_G = 1$. Since H is a semi CAP^*-subgroup of G, there is a chief series

$$1 = G_0 < G_1 = N < \cdots < G_m = G \qquad (*)$$

of G such that H covers or avoids every non-Frattini chief factor of this series. It is clear that $N \nleq H$. If $N \leq \Phi(G)$, then HN is a maximal subgroup of G and HN is solvable. Furthermore, for any non-Frattini chief factor G_i/G_{i-1} of $(*)$, H covers or avoids G_i/G_{i-1} by hypothesis. If H covers G_i/G_{i-1}, then it follows from $HG_i = HG_{i-1}$ that $HNG_i = HNG_{i-1}$. If H avoids G_i/G_{i-1}, then $HN \cap G_i = HN \cap G_{i-1}$ by $H \cap G_i = H \cap G_{i-1}$. Hence HN is a semi CAP^*-subgroup of G. In view of Corollary 2.11 of [5] and Lemma 2.7, G is solvable. Suppose that N is a non-Frattini chief factor of G. Then $HN = H$ or $H \cap N = 1$. It follows from $H_G = 1$ that $H \cap N = 1$.

We claim that $HN < G$. Otherwise, $HN = G$. By hypothesis, there is a maximal subgroup M of G such that H is a maximal subgroup of M. It is clear that $M = M \cap G = H(M \cap N)$. Noticing that $M \cap N$ is normal in M and $(M \cap N) \cap H \leq N \cap H = 1$, we see that $M \cap N$ is a minimal normal subgroup of M. Applying Lemma B, M is solvable and so $M \cap N$ is an elementary p-group for some prime p. Let $P = M \cap N$, then we can see that $M \leq N_G(P)$. If $N_G(P) = G$, then $N = P$ by the minimality of N. This implies that $G = HN \leq M$, a contradiction. Hence $N_G(P) = M$. It follows that $N_N(P) = P = C_N(P)$. By the Burnside theorem, we see that N is p-nilpotent. However, because N is a minimal normal subgroup of G, N is a p-group and $N = P \leq M = N_G(P) < G$, which contradicts $HN = G$.

We claim that HN is a maximal subgroup of G. In fact, since H is a 2-maximal subgroup of G, there is a maximal subgroup M of G such that H is a maximal subgroup of M. If $N \nleq M$, then $G = MN$ and $HN < MN$. Let K be a subgroup of G with $HN \leq K \leq MN = G$. Then $K = N(K \cap M)$ and $H \leq K \cap M \leq M$. Noticing that H is a maximal subgroup of M, we have that $K \cap M = H$ or M. It follows that $K = HN$ or $K = MN = G$. Hence HN is a maximal subgroup of G. If $N \leq M$, then $HN \leq M$. Since H is a maximal subgroup of M, we see that $HN = M$ is also a maximal subgroup of G. This implies that N is a minimal normal subgroup of HN since H is a 2-maximal subgroup of G. By Lemma B, HN is solvable. By the above argument, we can see that HN is a semi CAP^*-subgroup of G. Applying Corollary 2.11 of [5] and Lemma 2.7, G is solvable. Thus, the proof is complete. □

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