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Farey sequences and resistor networks

SAMEEN AHMED KHAN

Department of Engineering, Salalah College of Technology, Post Box No. 608,
Postal Code 211, Salalah, Sultanate of Oman
E-mail: rohelakhan@yahoo.com

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Abstract. In this article, we employ the Farey sequence and Fibonacci numbers to establish strict upper and lower bounds for the order of the set of equivalent resistances for a circuit constructed from n equal resistors combined in series and in parallel. The method is applicable for networks involving bridge and non-planar circuits.

Keywords. Farey sequence; Fibonacci numbers; Lucas numbers; resistor network; bridge and non-planar circuits.

1. Introduction

The net resistance of n resistors with resistances $R_1, R_2, \ldots, R_n$ connected in series is given by

$$R_{\text{series}} = R_1 + R_2 + \cdots + R_n, \tag{1}$$

whereas the net resistance of these resistors connected in parallel is given by

$$R_{\text{parallel}} = \frac{1}{1/R_1 + 1/R_2 + \cdots + 1/R_n} \tag{2}$$

(see [8] and [4].) It is well known that the net resistance R_{series} is greater than the largest resistance among the resistances $R_1, R_2, \ldots, R_n$ and that the net resistance R_{parallel} is less than the smallest resistance among the resistances $R_1, R_2, \ldots, R_n$. The net resistance in an arbitrary circuit must therefore lie between R_{parallel} and R_{series}. Using (1) and (2), one can solve block by block any circuit configuration consisting of resistors connected in series and in parallel.

The simplest network consists of n resistors connected in series and in parallel, each of the same resistance R_0. The net resistance is proportional to R_0 which can be set (without any loss of generality) to unity. Using (2) and (1) in this order, we see that the proportionality constant is a rational number a/b in reduced form that ranges from $1/n$ to n. The problems that are associated with resistor networks include finding the net resistance between any two points in a network and the order of the set of equivalent resistances. A careful study of the resistance problem is given in [2] using diverse techniques such as Green's function, while the perturbation of a network is investigated in [3]. The problem of finding the order of the set of equivalent resistances has been studied numerically. Due to the constraint on computer memory, the problem was studied only up to $n = 23$. In this article, we provide a method that works analytically for all values of n.

Let $A(n)$ denote the set of equivalent resistances of n equal resistors put in an arbitrary combination (using series and parallel). For $n \leq 5$, we have

$$A(1) = \{1\},$$
$$A(2) = \left\{\frac{1}{2}, 2\right\},$$
$$A(3) = \left\{\frac{1}{3}, \frac{2}{3}, \frac{3}{2}, 3\right\},$$
$$A(4) = \left\{\frac{1}{4}, \frac{2}{5}, \frac{3}{5}, \frac{3}{4}, 1, \frac{4}{3}, \frac{5}{3}, \frac{5}{2}, 4\right\},$$
$$A(5) = \left\{\frac{1}{5}, \frac{2}{7}, \frac{3}{8}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{5}{8}, \frac{5}{7}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{6}, \frac{6}{5}, \frac{5}{4}, \frac{7}{5}, \frac{8}{5}, \frac{7}{4}, 2, \frac{7}{3}, \frac{8}{3}, \frac{7}{2}, 5\right\}.$$

Proceeding in this way, we see that the order of $A(n)$ grows rapidly: 1, 2, 4, 9, 22, 53, 131, 337, 869, 2213, 5691, 14517, 37017, 93731, 237465, 601093, ... It is clear that a set of higher order does not necessarily contain a set of lower order. For example, $2/3$ is in $A(3)$ but it is not in $A(4)$ or $A(5)$. The numerical results in [1] suggest that $|A(n)| \sim 2.53^n$. In Theorem 1 below, we employ the Farey sequence to establish a strict upper bound of 2.61^n on $|A(n)|$. We will use Fibonacci numbers for the order of the Farey sequence.

Fibonacci numbers arise naturally in resistor networks. They form the sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ... The n-th Fibonacci number is defined for $n \geq 3$ by the linear recurrence relation $F_n = F_{n-1} + F_{n-2}$, using $F_1 = F_2 = 1$. A Farey sequence of order $m > 0$, denoted by $\mathcal{F}_m$, contains the most exhaustive set of fractions in the unit interval [0, 1] whose denominators are less than or equal to m. For example, the terms in the Farey sequence $\mathcal{F}_7$ are 1/7, 1/6, 1/5, 1/4, 2/7, 1/3, 2/5, 3/7, 1/2, 4/7, 3/5, 2/3, 5/7, 3/4, 4/5, 5/6, 6/7, 1 (see [5, 7, 10, 17]).

Our main result can be stated as follows:

Theorem 1. *Let $A(n)$ denote the set of equivalent resistances of n equal resistors put in an arbitrary combination, let F_{n+1} denote the $(n+1)$-th term in the Fibonacci sequence, and let $G(n)$ denote the set of Farey fractions $\mathcal{F}_{F_{n+1}}$ of order F_{n+1} in the interval $I = [1/n, 1]$ along with their reciprocals. Moreover, define $\mathcal{F}_{F_{n+1}}(I) = \mathcal{F}_{F_{n+1}} \cap I$. Then we have*

$$\frac{1}{4}(1+\sqrt{2})^n < |A(n)| < |G(n)|,$$

where

$$|G(n)| = 2|\mathcal{F}_{F_{n+1}}(I)| - 1.$$

As a corollary, we obtain the following numerical upper bound.

COROLLARY 2

Let $A(n)$ be defined as in Theorem 1. Then we have

$$|A(n)| < \left(1 - \frac{1}{n}\right)(0.318)(2.618)^n.$$

Here and in the sequel, the symbols S and P always denote series and parallel connections, respectively. Let $(1SP)_n$ denote a circuit constructed by connecting n unit resistors in series and parallel incrementally, thus resulting in a ladder network. For two resistors, we see that the resistance is $R_{1S1} = 2$; for three resistors, the resistance is $R_{1S1P1} = 2/3$; for four resistors, the resistance is $R_{1S1P1S1} = 5/3$; and so on. As it turns out, the corresponding resistances are ratios of consecutive Fibonacci numbers (see [12]). By induction, we have

$$R_{(1SP)_n} = \begin{cases} \dfrac{F_n}{F_{n+1}}, & \text{if } n \text{ is odd,} \\ \dfrac{F_{n+1}}{F_n}, & \text{if } n \text{ is even.} \end{cases}$$

The ladder network starts with two resistors in series. In principle, there can be more than two resistors in series in the beginning, thus leading to $(2SP)_n$. With two resistors in series in the beginning, we note that for three resistors the corresponding resistance is $R_{2S1} = 3$; for four resistors, the resistance is $R_{2S1P1} = 3/4$; for five resistors, the resistance is $R_{2S1P1S1} = 7/4$, and so on. Inspection shows that these fractions are ratios of consecutive Lucas numbers. Here we point out that n resistors are used to obtain the $(n-1)$-th Lucas number L_{n-1}. The same number of resistors in the ladder network gives the Fibonacci number F_{n+1}. These numbers satisfy the strict inequality $L_{n-1} < F_{n+1}$. Again by induction, we have

$$R_{(2SP)_n} = \begin{cases} \dfrac{L_{n-1}}{L_{n-2}}, & \text{if } n \text{ is odd,} \\ \dfrac{L_{n-2}}{L_{n-1}}, & \text{if } n \text{ is even.} \end{cases}$$

In the sequel, we cite various integer sequences from *The On-Line Encyclopedia of Integer Sequences* (OEIS) created and maintained by Neil Sloane [11]. For example, the sequence $A(n)$ is identified by A048211 in [18]. The first sixteen sets $A(n)$ are given in [1], while the OEIS has seven additional terms for $n = 17, 18, \ldots, 23$, which are 1519815, 3842575, 9720769, 24599577, 62283535, 157807915, 400094029. The number of equivalent resistances matches the number of configurations for $n \leq 3$. For $n = 4$, we have ten configurations that give a set of nine equivalent resistances. The two configurations $(R_0PR_0)S(R_0PR_0)$ and $(R_0SR_0)S(R_0PR_0)$ have the same equivalent resistance R_0. In this article, we focus on the set of equivalent resistances. The sequence 1, 2, 4, 10, 24, 66, 180, 522, 1532, 4624, 14136, 43930, 137908, 437502, 1399068, 4507352, … gives the growth of the number of circuit configurations corresponding to the sets $A(n)$ and occurs in different contexts such as the number of unlabeled cographs on n nodes. The OEIS enables us to see such connections among unrelated problems. Interestingly enough, the OEIS has over thousand terms of this sequence [16]. The number of configurations is much larger than the number of equivalent resistances.

2. The upper bound

The proof of the upper bound in Theorem 1 is embodied in the following two theorems.

Theorem 3 (Reciprocal theorem). *Any circuit constructed from n equal resistors in series or parallel, each resistor with resistance R_0, has an equivalent resistance $(a/b)R_0$.*

If one replaces all series connections to parallel connections and all parallel connections to series connections, then the resulting circuit will have equivalent resistance $(b/a)R_0$.

Theorem 4 (Bound of *a* and *b* theorem). *Any circuit constructed from n equal resistors in series or parallel, each resistor with resistance* R_0, *has an equivalent resistance* $(a/b)R_0$ *such that the largest possible values of a and b are bounded by the* $(n+1)$*-th term in the Fibonacci sequence,* F_{n+1}.

A proof of Theorem 3 is given by induction in [1], and Theorem 4 is discussed in detail in [6], [14], and [15]. Theorem 3 states that the resistances in $A(n)$ always occur in pairs a/b and b/a, the former being less than 1 and the latter being greater than 1. So, it suffices to count the number of circuit configurations whose equivalent resistances are less than 1. Since Theorem 4 fixes the bound on the values of a and b, the problem of estimating $|A(n)|$ thus translates to the problem of counting the number of relevant proper fractions whose denominators are bounded by $m = F_{n+1}$. As mentioned above, the Farey sequence of order $m > 0$ provides the most exhaustive set of fractions in the unit interval [0, 1] whose denominators are less than or equal to m. These fractions are restricted to the subinterval $I = [1/n, 1]$; the length of this interval is $|I| = (1 - 1/n)$. Let us recall that the resistance $1/n$ is obtained by connecting n resistors in parallel. Taking into account that all fractions in $A(n)$, except 1, have a reciprocal pair and the fact that 1 is included in the Farey sequence (and counted twice), we have

$$|G(n)| = 2|\mathcal{F}_{F_{n+1}}(I)| - 1. \tag{3}$$

By construction, the set $G(n)$ is the grand set of fractions from the Farey sequence $\mathcal{F}_{F_{n+1}}$ in the interval $[1/n, 1]$ along with their reciprocals. Its order is given by the sequence 1, 3, 7, 17, 37, 99, 243, 633, 1673, 4425, 11515, 30471, 80055, 210157, 553253, 1454817, 3821369, 10040187, ... (see [24]). Inspection shows that $G(n)$ contains all rational numbers of the form a/b such that both a and b are bounded by F_{n+1}. Since $\mathcal{F}_{F_{n+1}}$ is exhaustive, the set $G(n)$ is also exhaustive. This together with (3) lead to the required upper bound in Theorem 1,

$$|A(n)| < |G(n)| = 2|\mathcal{F}_{F_{n+1}}(I)| - 1.$$

At this point, we can derive an asymptotic formula for $G(n)$ and thus prove Corollary 2. Since $\mathcal{F}_m(I)$ grows quadratically in m (see Theorem 330 in [7]), we have the asymptotic limit

$$\mathcal{F}_m(I) \sim \left(1 - \frac{1}{n}\right) \frac{3}{\pi^2} m^2. \tag{4}$$

Now, the closed form expression for the n-th Fibonacci number F_n is given by $F_n = [\phi^n/\sqrt{5}]$, where $\phi = 1.6180339887\ldots$ is the golden ratio (see [12]). Here $[x]$ is used to denote the maximal integer less than or equal to x. Using (3) and (4) with the substitution $m = F_{n+1}$, we obtain

$$\begin{aligned} |G(n)| &\sim \left(1 - \frac{1}{n}\right) \frac{6}{\pi^2} \left(\frac{\phi^{n+1}}{\sqrt{5}}\right)^2 \\ &= \left(1 - \frac{1}{n}\right) (0.318)(2.618)^n. \end{aligned}$$

This is consistent with the asymptotic limit $A(n) \sim 2.53^n$ obtained from numerical computations up to $n = 16$ in [1] and up to $n = 23$ in [18]. Here we remark that the asymptotic limit $G(n) \sim 2.618^n$ strictly fixes an upper bound on $|A(n)|$. When using $G(n)$ in place of $A(n)$, there is a certain amount of overcounting as the Farey sequence contains some terms that are absent in the actual circuit configuration.

We next show that the Farey sequence method is applicable for circuits with n or fewer equal resistors. To this end, let $C(n)$ denote the set of equivalent resistances from one or more n equal resistors put in an arbitrary combination. The order of the first 16 sets $C(n)$ are 1, 3, 7, 15, 35, 77, 179, 429, 1039, 2525, 6235, 15463, 38513, 96231, 241519, 607339 (see [19]).

We may write $C(n)$ as

$$C(n) = \bigcup_{i=1}^{n} A(i).$$

Here we note that each $A(i)$ is a subset of $G(i)$ which is constructed from $\mathcal{F}_{F_{i+1}}$ and that a Farey sequence of a given order contains all Farey sequences of lower orders. Thus, we have

$$|A(n)| < |C(n)| < |G(n)|.$$

Finally, bridge circuits can be analysed by first converting them to the usual series-parallel equivalents with transformed resistances (see [4]). The transformed resistances satisfy Theorem 4. In fact, the bounds turn out to be less than F_{n+1}. Although individual bridge circuits do not necessarily satisfy Theorem 3, the set of bridge circuits satisfy Theorem 3 for $n \leq 8$. When the bridge circuits in the set $A(n)$ are modified to equivalent circuits in $B(n)$, their orders are the terms in the sequence 1, 2, 4, 9, 23, 57, 151, 409, ... (see [20]). Moreover, circuits of equal resistors forming geometries such as polygons and polyhedral structures are studied in [9] and [13]. From these investigations, it is evident that Theorem 4 is satisfied by these circuits. Hence, any larger set of circuit configurations involving such geometries will be bounded by $|G(n)|$. Because of this, the Farey sequence method is applicable. Hence, all equivalent resistances of configurations containing bridge circuits belong to $G(n)$. Note that the set $A(n)$ is a proper subset of the set $B(n)$ which additionally contains bridge circuits. These two sets satisfy the inequalities

$$|A(n)| < |B(n)| < |G(n)|.$$

For the sets $A(n)$ and $C(n)$, by virtue of Theorem 3 we conclude that there are equal number of configurations on either side of 1 with the reciprocal relation for each pair. In the absence of Theorem 3, the same cannot be said of set $B(n)$. Still larger sets $D(n)$ are obtained by including the bridge circuits in the set $C(n)$. The order of the first few sets of $D(n)$ are 1, 3, 7, 15, 35, 79, 193, 489, ... (see [21]). All four sets $A(n)$, $C(n)$, $D(n)$ and $G(n)$ satisfy the inequalities

$$|A(n)| < |C(n)| < |D(n)| < |G(n)|.$$

3. The lower bound

In this section we establish the lower bound in Theorem 1. The order of the set $A(n+1)$ can be estimated from the order of the set $A(n)$. Treating the elements of $A(n)$ as single blocks the $(n+1)$-th resistor can be connected either in series or in parallel, and so the resulting sets are called either series or parallel. We denote these sets either by $1SA(n)$ or $1PA(n)$. We can also add the $(n+1)$-th resistor somewhere within the $A(n)$ blocks and call the resulting set the cross set, which we denote by $1 \otimes A(n)$.

The sets $1SA(n)$ and $1PA(n)$ each has exactly $|A(n)|$ configurations and $|A(n)|$ equivalent resistances. They are disjoint and contribute $2|A(n)|$ elements to the set $A(n+1)$. Additionally, they are the source of 2^n configurations. The cross set is not as straightforward, as it is generated by placing the $(n+1)$-th resistor anywhere within the blocks of $A(n)$. The cross set is the source of all extra configurations which do not necessarily result in new equivalent resistances. Moreover, it has at least $|A(n-1)|$ elements, since the set $A(n)$ has $|A(n-1)|$ connections corresponding to those in the set $1 \otimes A(n-1)$. This argument works for $n \geq 6$ and leads to the inequality

$$|A(n+1)| > 2|A(n)| + |A(n-1)|,$$

which, in turn, leads to the required lower bound in Theorem 1,

$$\frac{1}{4}(1+\sqrt{2})^n < |A(n)|.$$

Appendix

In this section we derive several properties of the set $A(n)$. First, the *scaling property* is the statement that if $a/b \in A(m)$, then one can construct the resistances $k(a/b)$ and $(1/k)(a/b)$ using k such blocks in series and in parallel, respectively, with km number of unit resistors. Thus, we have $kA(m) \in A(km)$ and $(1/k)A(m) \in A(km)$.

From (1), we see that a block of i resistors in series has an equivalent resistance i. If i such blocks are combined in parallel, using (2) we get back the unit resistance. From this, we conclude that $1 \in A(i^2)$. The same result can be obtained by taking i blocks in series, each containing i unit resistors connected in parallel. Once the unit resistor has been obtained, using i^2 resistors (or much less as we shall soon see), we can use it to construct other equivalent resistances. Every set $A(m)$ is made from m unit resistors. The same set can be replicated by using m unit resistors constructed with i^2 resistors. Hence $A(m) \subset A(i^2 m)$. Whenever 1 belongs to some set $A(i)$, we label it as 1_i to indicate that it has been constructed from i basic unit resistors R_0.

The *translation property* is the statement that $1 \in A(i)$ implies $1 \in A(i+3)$. This can be seen by taking either of the combination of 1_i with three basic unit resistors $(1S1)P(1S1_i)$ so that $R_{2P2} = 1$ or with $(1P1)S(1P1_i)$, so that $R_{(1/2)S(1/2)} = 1$. So whenever $1 \in A(i)$, it follows that $1 \in A(i+3)$. We shall use the translation property to prove the following theorem.

Theorem A1. *We have* $1 \in A(n)$ *for* $n \neq 2$, $n \neq 3$, *and* $n \neq 5$.

From an exhaustive search (or otherwise), we know that 1 belongs to $A(6)$, $A(7)$ and $A(8)$. Using the translational property, we see that 1 also belongs to $A(9)$, $A(10)$ and $A(11)$; and to $A(12)$, $A(13)$ and $A(14)$; and so on. Thus we conclude that 1 belongs to $A(n)$ for $n \geq 6$. As for the lower $A(i)$, 1 belongs to $A(1)$ and $A(4)$; but 1 does not belong to $A(2)$, $A(3)$ and $A(5)$. Hence the theorem is proved.

Next, recall that all elements in $A(n)$ have a reciprocal pair a/b and b/a, with 1 being its own partner. The presence of 1 implies that $|A(n)|$ is always odd, with the exception of $|A(2)| = 2$, $|A(3)| = 4$ and $|A(5)| = 22$. Recall also that each Farey sequence $\mathcal{F}_m$ contains an odd number of elements, with the exception of $m = 1$, where $|\mathcal{F}_1| = 2$.

COROLLARY A2

We have $1/2 \in A(n)$ *for* $n \neq 1$, $n \neq 3$, $n \neq 4$, *and* $n \neq 6$.

The parallel combination of one basic unit resistor, with 1_i for $i = 4$ and $i \geq 6$ results in an equivalent resistance of 1/2. This is because $R_{1P1_i} = (1 \times 1_i)/(1 + 1_i) = 1/2$. Consequently, $1/2 \in A(i+1)$ for $i = 4$ and $i \geq 6$. Corollary A2 is proved for $n = 5$ and $n \geq 7$. Resorting to an exhaustive search, we note that $1/2 \in A(2)$. The four exceptional sets are $A(1)$, $A(3)$, $A(4)$ and $A(6)$, which do not contain 1/2. Here we note that the Farey sequence $\mathcal{F}_m$ contains 1/2, with the exception of $\mathcal{F}_1$.

Theorem A3 (Modular theorem). *We have*

(i) $A(m) \subset A(m+3)$,
(ii) $A(m) \subset A(m+i)$ *for* $i \geq 5$.

Every set $A(m)$ is constructed from m basic unit resistors R_0. If we replace any one of these basic unit resistors with 1_i for $i = 4$ and $i \geq 6$, we will reproduce the complete set $A(m)$ using $(m+i-1)$ resistors. Consequently, $A(m) \subset A(m+i-1)$ for $i = 4$ and $i \geq 6$. Thus, Theorem A3 is proved. Specifically, every set $A(m)$ is completely contained in all subsequent and larger sets $A(m+3)$ along with infinite and complete sequence of sets $A(m+5)$, $A(m+6)$, $A(m+7)$, ... However, it is very surprising to note that the infinite range theorem is silent about the three important sets: the nearest neighbour $A(m+1)$, next-nearest neighbour $A(m+2)$, and the near-neighbour $A(m+4)$. From the modular relation $A(m) \subset A(m+i)$ for $i \geq 5$, we conclude that $A(n-5) \subset A(n) \cap A(n+1)$ for $n \geq 6$. This is the closest that we can get to understand the overlap between $A(n)$ and its nearest neighbor $A(n+1)$. An immediate consequence of Theorem A3 is that the set $C(n)$ can be rewritten as follows:

$$C(n) = \bigcup_{i=1}^{n} A(i) = \bigcup_{i=n-2}^{n} A(i) = A(n-2) \cup A(n-1) \cup A(n),$$

which tells us that it is enough to consider only the last three sets $A(n-2)$, $A(n-1)$ and $A(n)$ in the union. So it is not surprising that the ratio $C(n)/A(n)$ is close to 1.

We now turn our attention to the *decomposition* of $A(n)$. When deriving the lower bound in Theorem 1, we observed that $A(n)$ is the union of the three sets formed by different ways of adding the n-th resistor. The decomposition

$$A(n) = 1PA(n-1) \cup 1SA(n-1) \cup 1 \otimes A(n-1)$$

enables us to understand some properties of $A(n)$.

All elements of the parallel set are strictly less than 1; this is because $R_{1P(a/b)} = a/(a+b) < 1$. Likewise, all elements of the series set are strictly greater than 1; this is because $R_{1S(a/b)} = (a+b)/b > 1$. So $1PA(n-1) \cap 1SA(n-1) = \emptyset$, and the element 1 necessarily belongs to the cross set alone.

The series and the parallel sets each have exactly $|A(n-1)|$ configurations and the same number of equivalent resistances. Let c/d and d/c be any reciprocal pair (ensured by Theorem 3) in $A(n-1)$. Then $R_{1P(c/d)} = c/(c+d)$ and $R_{1P(d/c)} = d/(c+d)$ belong to the set $1PA(n-1)$. Similarly, $R_{1S(c/d)} = (c+d)/d$ and $R_{1S(d/c)} = (c+d)/c$ belong to the set $1SA(n-1)$. This shows that all the reciprocal partners of the set $1PA(n-1)$ always belong to the set $1SA(n-1)$, and vice versa. Consequently, all elements in the cross set $1 \otimes A(n-1)$ have their reciprocal partners in $1 \otimes A(n-1)$, with 1 being its own partner. These two disjoint sets contribute $2A(n-1)$ elements to the set $A(n)$ and are the source of 2^n configurations. The order of the cross set $1 \otimes A(n)$ is $A(n+1) - 2A(n)$ and is given by the terms in the sequence 0, 0, 0, 1, 4, 9, 25, 75, 195, 475, 1265, 3135, … (see [22]). It is the cross set which takes the count beyond 2^n to 2.53^n numerically and maximally to 2.61^n, strictly fixed by the Farey sequence method. For $n \geq 7$, all the three basic sets have odd number of elements since $A(n)$ is odd for $n \geq 6$. For $n > 6$, the cross set has at least $|A(n-2)|$ elements since $A(n-1)$ has $|A(n-2)|$ connections corresponding to the set $1 \otimes A(n-2)$. This leads to the recurrence relation which gives the lower bound for $|A(n)|$.

The cross set is expected to be dense near 1 with few of its elements below a half (recall that $1/2 \in 1PA(n)$ for $n \geq 6$ and $1/2$ is not a member of the cross set). This is reflected by the fact that cross sets up to $1 \otimes A(7)$ do not have a single element below a half. The successive cross sets have 1, 6, 9, 24, 58, 124, 312, …, elements below a half respectively (see [23]); a small percentage compared to the size of the cross sets 195, 475, 1265, 3125, … (see [22]).

It is straightforward to carry over the set theoretic relations to the bridge circuits sets, since $A(n) \subset B(n)$ (see [20]). Unlike the sets $A(i)$, the sets $B(i)$ have the additional feature $1 \in B(5)$. So the various statements must be modified accordingly. In particular, we have

(i) $1 \in B(n)$ for $n \neq 2$ and $n \neq 3$,
(ii) $1/2 \in B(n)$ for $n \neq 1$, $n \neq 3$ and $n \neq 4$,
(iii) $B(m) \subset B(mi)$ for $i = 1$ and $i \geq 4$,
(iv) $B(m) \subset B(m+i)$ for $i \geq 3$,
(v) $B(n-3) \subset B(n) \cap B(n+1)$ for $n \geq 4$.

Finally, the *complementary property* is the statement that every set $A(n)$ with $n \geq 3$ has some complementary pair such that their sum is equal to 1. As an example, in $A(3)$ we have the pair (1/3, 2/3); in $A(4)$ we have two pairs (1/4, 3/4) and (2/5, 3/5); and so on.

By virtue of Corollary A2, we see that $1/2$ can be treated as its own complementary partner. We shall soon conclude that each element of the set $1PA(n-1)$ has a complementary partner in $1PA(n-1)$. By Theorem 3, the elements c/d and d/c occur as reciprocal pairs in $A(n-1)$. So in $1PA(n-1)$ we have

$$\left(1P\frac{c}{d}\right) + \left(1P\frac{d}{c}\right) = \frac{c}{c+d} + \frac{d}{d+c} = 1.$$

Consequently, all elements of $1PA(n-1)$ have a complementary partner in $1PA(n-1)$. For $n \geq 7$, the number of such pairs in $1PA(n-1)$ is $(A(n-1)-1)/2$. This is because $A(n-1)$ is odd for $n \geq 7$ and $1/2 \in A(n)$ for $n \geq 7$. It is obvious that the set $1SA(n-1)$ does not have complementary pairs. Here we recall that elements in the Farey sequence are complementary with respect 1; the median point $1/2$ is the only exception and may be treated as its own partner.

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