Proc. Indian Acad. Sci. (Math. Sci.) Vol. 122, No. 1, February 2012, pp. 139–152.

Existence of positive solutions for the system of higher order two-point boundary value problems

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MS received 28 March 2010; revised 9 December 2010

Abstract. In this paper, we establish the existence of at least one and two positive solutions for the system of higher order boundary value problems by using the Krasnosel'skii fixed point theorem.

Keywords. System of boundary value problem; positive solutions; cone.

1. Introduction

Boundary value problems (BVPs) play a major role in many fields of engineering design and manufacturing. Major established industries such as the automobile, aerospace, chemical, pharmaceutical, petroleum, electronics and communications, as well as emerging technologies such as nanotechnology and biotechnology rely on the BVPs to simulate complex phenomena at different scales for design and manufactures of high-technology products. In these applied settings, positive solutions are meaningful [1,9,13,21]. Due to their important role in both theory and applications, the BVPs have generated a great deal of interest in recent years.

It should be pointed out that Eloe and Henderson [6] discussed the boundary value problem

$$u^{(n)}(t) + a(t)f(u(t)) = 0, \quad 0 < t < 1,$$

$$u^{(i)}(0) = u^{(n-2)}(1) = 0, \quad 0 \le i \le n-2.$$

By using Krasnoselskii's fixed point theorem, the existence of solutions are obtained in the case when, either f is superlinear, or f is sublinear. Yang and Sun [22] considered the boundary value problem of the system of equations

$$-u'' = f(t, v), \quad -v'' = g(t, u), \quad u(0) = u(1) = 0, \quad v(0) = v(1) = 0.$$

By appealing to the degree theory, the existence of solutions are established. Note that, there is only one differential equation in [6] and the BVP in [22] contains simple boundary conditions.

In this paper, we shall consider the nonlinear system of differential equations,

$$\begin{cases} y_1^{(m)} + f_1(t, y_1, y_2) = 0, & t \in [a, b] \\ y_2^{(n)} + f_2(t, y_1, y_2) = 0, & t \in [a, b] \end{cases} \tag{1.1}$$

satisfying the two-point boundary conditions

$$\begin{cases} y_1^{(i)}(a) = 0, & 0 \le i \le m-2, \\ y_1^{(p)}(b) = 0, & (1 \le p \le m-1, \text{ but fixed}) \\ y_2^{(j)}(a) = 0, & 0 \le j \le n-2, \\ y_2^{(q)}(b) = 0, & (1 \le q \le n-1, \text{ but fixed}) \end{cases} \tag{1.2}$$

where $m, n \geq 2$, and $f_i : [a, b] \times \mathbb{R}^2 \to \mathbb{R}$ for $i = 1, 2$ are continuous.

By using the cone theory techniques, we establish some sufficient conditions for the existence results of positive solutions to the BVP (1.1) and (1.2). The rest of the paper is organized as follows. In Section 2, we prove some inequalities for Green's function, which are needed later. In Section 3, we obtain the existence of a solution for the BVP (1.1) and (1.2), due to Schauder fixed point theorem. Criteria for the existence of one and two positive solutions of the BVP (1.1) and (1.2) are established in Section 4, using the Guo–Krasnosel'skii fixed point theorem.

2. The Green's function and bounds

In this section, we construct the Green's function for the homogeneous boundary value problem corresponding to the BVP (1.1) and (1.2) by using Cauchy function concept. And then we prove some inequalities on bounds of the Green's function which are needed later.

To obtain a solution $y(t) = (y_1(t), y_2(t))$ of the BVP (1.1) and (1.2), we let $G_l(t, s)$, $(l \geq 2)$ be the Green's function for the boundary value problem,

$$-y^{(l)} = 0, \quad t \in [a, b] \tag{2.1}$$

$$y^{(i)}(a) = 0, \quad 0 \le i \le l-2, \tag{2.2}$$

$$y^{(j_1)}(b) = 0, \quad 1 \le j_1 \le l-1, \text{ but fixed.} \tag{2.3}$$

Using the Cauchy function concept $G_l(t, s)$ is given by

$$G_l(t,s) = \begin{cases} \dfrac{(t-a)^{l-1}(b-s)^{l-j_1-1}}{(l-1)!(b-a)^{l-j_1-1}}, & t \le s, \\[2ex] \dfrac{(t-a)^{l-1}(b-s)^{l-j_1-1}}{(l-1)!(b-a)^{l-j_1-1}} - \dfrac{(t-s)^{l-1}}{(l-1)!}, & s \le t. \end{cases}$$

Lemma 2.1. *For* $(t,s)\in[a,b]\times[a,b]$, *we have*

$$G_l(t,s)\le G_l(b,s). \tag{2.4}$$

Proof. For $a\le t\le s\le b$, we have

$$\begin{aligned} G_l(t,s) &= \frac{(t-a)^{l-1}(b-s)^{l-j_1-1}}{(l-1)!(b-a)^{l-j_1-1}} \\ &\le \frac{(b-a)^{l-1}(b-s)^{l-j_1-1}}{(l-1)!(b-a)^{l-j_1-1}} \\ &= G_l(b,s). \end{aligned}$$

Similarly, for $a\le s\le t\le b$, we have

$$G_l(t,s)\le G_l(b,s).$$

Thus, we have

$$G_l(t,s)\le G_l(b,s), \qquad \text{for all } (t,s)\in[a,b]\times[a,b].$$

□

Lemma 2.2. *Let* $I=\left[\frac{3a+b}{4},\frac{a+3b}{4}\right]$. *For* $(t,s)\in I\times[a,b]$, *we have*

$$G_l(t,s)\ge\frac{1}{j_1\,.\,4^{l-1}}G_l(b,s). \tag{2.5}$$

Proof. The Green's function $G_l(t,s)$ for the BVP (2.1) and (2.3) clearly satisfies that

$$G_l(t,s)>0 \text{ on } (a,b)\times(a,b).$$

For $a\le t\le s\le b$ and $t\in I$, we have

$$\begin{aligned} \frac{G_l(t,s)}{G_l(b,s)} &= \left(\frac{t-a}{b-a}\right)^{l-1} \\ &\ge \frac{1}{4^{l-1}}. \end{aligned}$$

Similarly, for $a\le s\le t\le b$ and $t\in I$ we have

$$\begin{aligned} \frac{G_l(t,s)}{G_l(b,s)} &= \frac{(t-a)^{l-1}(b-s)^{l-j_1-1}-(t-s)^{l-1}(b-a)^{l-j_1-1}}{(b-a)^{l-1}(b-s)^{l-j_1-1}-(b-s)^{l-1}(b-a)^{l-j_1-1}} \\ &\ge \frac{(t-a)^{l-j_1-1}(b-s)^{l-j_1-1}[(t-a)^{j_1}-(t-s)^{j_1}]}{(b-a)^{l-1}(b-s)^{l-j_1-1}-(b-s)^{l-1}(b-a)^{l-j_1-1}} \\ &= \frac{1}{j_1}\left(\frac{t-a}{b-a}\right)^{l-2} \\ &\ge \frac{1}{j_1}\left(\frac{t-a}{b-a}\right)^{l-1} \\ &\ge \frac{1}{j_1\,.\,4^{l-1}}. \end{aligned}$$

Therefore

$$G_l(t,s) \geq \frac{1}{j_1 \,.\, 4^{l-1}} G_l(b,s), \quad \text{for } (t,s) \in I \times [a,b]. \qquad \square$$

Remark 2.1.

$$G_n(t,s) \geq \gamma G_n(b,s) \text{ and } G_m(t,s) \geq \gamma G_m(b,s),$$

for all $(t,s) \in I \times [a,b]$, where $\gamma = \min\left\{\frac{1}{q\,.\,4^{n-1}}, \frac{1}{p\,.\,4^{m-1}}\right\}$.

3. Existence of solution

In this section, we impose growth conditions on f_1 and f_2 which allows us to apply the Schauder fixed point theorem to establish an existence result for solution of the BVP (1.1) and (1.2).

Let $E = \{y : y \in C[a,b]\}$ with the norm $\|y\|_0 = \max_{t\in[a,b]}\{|y(t)|\}$. Let $B = E \times E$ and for $(y_1, y_2) \in B$, we denote the norm by $\|(y_1, y_2)\| = \|y_1\|_0 + \|y_2\|_0$, then B is a Banach space.

Let us take

$$\epsilon_m = \max_{t\in[a,b]} \int_a^b G_m(t,s)\mathrm{d}s \quad \text{and} \quad \epsilon_n = \max_{t\in[a,b]} \int_a^b G_n(t,s)\mathrm{d}s.$$

Theorem 3.1. *Assume that the functions* $f_i(t, y_1, y_2)$ *for* $i = 1, 2$ *are continuous with respect to* $(y_1, y_2) \in \mathbb{R}^2$. *If* M *satisfies*

$$Q \leq M\epsilon,$$

where $\epsilon = \frac{1}{2\max\{\epsilon_m,\, \epsilon_n\}}$ *and* $Q > 0$ *satisfies*

$$Q \geq \max_{\|(y_1,y_2)\|\leq M} \{|f_1(t, y_1, y_2)|, |f_2(t, y_1, y_2)| : t \in [a,b]\},$$

then the BVP (1.1) *and* (1.2) *has a solution.*

Proof. Let $\mathcal{P} = \{(y_1, y_2) \in B : \|(y_1, y_2)\| \leq M\}$. Note that $\mathcal{P}$ is a closed, bounded and convex subset of B to which the Schauder fixed point theorem is applicable.

Define $T : \mathcal{P} \to B$ by

$$\begin{aligned} T(y_1, y_2)(t) &= \left(\int_a^b G_m(t,s) f_1(s, y_1, y_2)\mathrm{d}s,\ \int_a^b G_n(t,s) f_2(s, y_1, y_2)\mathrm{d}s\right) \\ &= (T_m(y_1, y_2)(t), T_n(y_1, y_2)(t)), \end{aligned}$$

for $t \in [a,b]$. Obviously the solution of the BVP (1.1) and (1.2) is the fixed point of operator T. It can be shown that $T : \mathcal{P} \to B$ is continuous.

We claim that $T : \mathcal{P} \to \mathcal{P}$. Let $(y_1, y_2) \in \mathcal{P}$. By using Lemma 2.1, we get

$$\begin{aligned}\|T(y_1, y_2)\| &= \|T_m(y_1, y_2)\|_0 + \|T_n(y_1, y_2)\|_0 \\ &= \max_{t\in[a,b]} |T_m(y_1, y_2)| + \max_{t\in[a,b]} |T_n(y_1, y_2)| \\ &\le (\epsilon_m + \epsilon_n)Q \\ &\le \frac{Q}{\epsilon},\end{aligned}$$

where

$$Q \ge \max_{\|(y_1,y_2)\|\le M} \{|f_1(t, y_1, y_2)|, |f_2(t, y_1, y_2)| : t \in [a, b]\}.$$

Thus we have

$$\|T(y_1, y_2)\| \le M,$$

where M satisfies $Q \le M\epsilon$.

It can be shown that $T : \mathcal{P} \to \mathcal{P}$ is a compact operator by the Arzela-Ascoli theorem. Hence T has a fixed point in $\mathcal{P}$ by the Schauder fixed point theorem.

Therefore the BVP (1.1) and (1.2) has a solution. □

Theorem 3.2. *If the functions* $f_i(t, y_1, y_2)$, *for* $i = 1, 2$ *are continuous and bounded on* $[a, b] \times \mathbb{R}^2$, *then the BVP* (1.1) *and* (1.2) *has a solution.*

Proof. Since the function $f(t, y_1, y_2)$ is bounded, it has a supremum for $t \in [a, b]$ and $(y_1, y_2) \in \mathbb{R}^2$. Let us choose $P > \sup\{|f_1(t, y_1, y_2)|,\ |f_2(t, y_1, y_2)| : t \in [a, b]\}$. Pick M large enough such that

$$P < \epsilon M,$$

where $\epsilon = \frac{1}{2\max\{\epsilon_m,\ \epsilon_n\}}$. Then there is a number $Q > 0$ such that $P > Q$, where

$$Q \ge \max_{\|(y_1,y_2)\|\le M} \{|f_1(t, y_1, y_2)|, |f_2(t, y_1, y_2)| : t \in [a, b]\}.$$

Hence

$$\frac{1}{\epsilon} < \frac{M}{P} \le \frac{M}{Q},$$

and thus the BVP (1.1) and (1.2) has a solution by Theorem 3.1. □

4. Existence of positive solutions

In this section, we consider the existence of at least one and two positive solutions for the system of BVP (1.1) and (1.2). We also assume throughout this section that $f_{1,2} : [a, b] \times \mathbb{R}^2 \to [0, \infty)$ are continuous and

$$f_i^0 = \lim_{y_1+y_2\to 0} \frac{f_i(t, y_1, y_2)}{y_1 + y_2}, \qquad f_i^\infty = \lim_{y_1+y_2\to \infty} \frac{f_i(t, y_1, y_2)}{y_1 + y_2}$$

exist.

To prove the existence of at least one and two positive solutions of the BVP (1.1) and (1.2), we will use the following theorem which can be found in Krasnoselskii's book [18] and in Deimling's book [5].

Theorem 4.1. *Let B be a Banach space, $\mathcal{P} \subseteq B$ be a cone, and suppose that Ω_1, Ω_2 are open subsets of B with $0 \in \Omega_1$ and $\bar{\Omega}_1 \subset \Omega_2$. Suppose further that $T : \mathcal{P} \cap (\bar{\Omega}_2 \backslash \Omega_1) \to \mathcal{P}$ is a completely continuous operator such that either*

(i) $\|Tu\| \leq \|u\|,\ u \in \mathcal{P} \cap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|,\quad u \in \mathcal{P} \cap \partial\Omega_2,$ *or*
(ii) $\|Tu\| \geq \|u\|,\ u \in \mathcal{P} \cap \partial\Omega_1$ *and* $\|Tu\| \leq \|u\|,\quad u \in \mathcal{P} \cap \partial\Omega_2$

holds. Then T has a fixed point in $\mathcal{P} \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

Theorem 4.2. *The BVP* (1.1) *and* (1.2) *has at least one positive solution in the case*

(i) $f_1^0 = f_2^0 = 0$ *and* $f_1^\infty = \infty$ *or* $f_2^\infty = \infty$ (*or*) (*superlinear*),
(ii) $f_1^\infty = f_2^\infty = 0$ *and* $f_1^0 = \infty$ *or* $f_2^0 = \infty$ (*sublinear*).

Proof. We consider the Banach space $B = E \times E$ where $E = \{y : y \in C[a, b]\}$ with the norm $\|y\|_0 = \max_{t\in[a,b]}\{|y(t)|\}$ and for $(y_1, y_2) \in B$, we denote the norm by $\|(y_1, y_2)\| = \|y_1\|_0 + \|y_2\|_0$. Then define a cone $\mathcal{P}$ in B by

$$\mathcal{P} = \Big\{(y_1, y_2) \in B : y_1(t) \geq 0, y_2(t) \geq 0 \text{ and } \max_{t\in I}(y_1(t) + y_2(t)) \geq \gamma\|(y_1, y_2)\|\Big\}.$$

For convenience, we denote

$$T_n(y_1, y_2)(t) = \int_a^b G_n(t, s) f_1(s, y_1, y_2)\mathrm{d}s,$$
$$T_m(y_1, y_2)(t) = \int_a^b G_m(t, s) f_1(s, y_1, y_2)\mathrm{d}s,$$

for $t \in [a, b]$ and the operator

$$T(y_1, y_2)(t) = (T_n(y_1, y_2)(t), T_m(y_1, y_2)(t)).$$

We now show that $T : \mathcal{P} \to \mathcal{P}$.

First note that $(y_1, y_2) \in \mathcal{P}$ implies that $T_n(y_1, y_2)(t) \geq 0$ and $T_m(y_1, y_2)(t) \geq 0$ on $[a, b]$ and

$$\begin{aligned}
&\min_{t\in I}(T_n(y_1, y_2)(t) + T_m(y_1, y_2)(t))\\
&\quad= \min_{t\in I}\left(\int_a^b G_n(t, s) f_1(s, y_1, y_2)\mathrm{d}s + \int_a^b G_m(t, s) f_2(s, y_1, y_2)\mathrm{d}s\right)\\
&\quad\geq \gamma \int_a^b \max_{t\in[a,b]} |G_n(t, s)| f_1(s, y_1, y_2)\mathrm{d}s\\
&\qquad+ \gamma \int_a^b \max_{t\in[a,b]} |G_m(t, s)| f_2(s, y_1, y_2)\mathrm{d}s\\
&\quad= \gamma\|T_n(y_1, y_2)\|_0 + \gamma\|T_m(y_1, y_2)\|_0\\
&\quad= \gamma\|T(y_1, y_2)\|.
\end{aligned}$$

Also

$$\|T(y_1, y_2)(\omega)\| \geq \int_a^b G_n(\omega, s) f_1(s, y_1, y_2)\mathrm{d}s + \int_a^b G_m(\omega, s) f_2(s, y_1, y_2)\mathrm{d}s$$
$$\geq \gamma \|T_n(y_1, y_2)\|_0 + \gamma \|T_m(y_1, y_2)\|_0$$
$$= \gamma \|T(y_1, y_2)\|.$$

Hence $T(y_1, y_2) \in \mathcal{P}$ and so $T : \mathcal{P} \to \mathcal{P}$ which is what we want to prove. Therefore T is completely continuous.

(i) Since $f_1^0 = f_2^0 = 0$, there exists $H_1 > 0$ so that

$$f_i(t, y_1, y_2) < \epsilon(y_1 + y_2), \quad i = 1, 2$$

whenever $y_1 + y_2 < H_1$, where $\epsilon = \frac{1}{2\max\{\epsilon_n, \epsilon_m\}}$.

Let $\Omega_1 = \{(y_1, y_2) : (y_1, y_2) \in E, \|(y_1, y_2)\| < H_1\}$. Then for $(y_1, y_2) \in \mathcal{P} \cap \partial\Omega_1$ and $t \in [a, b]$, we have

$$\begin{aligned} T_n(y_1, y_2)(t) &= \int_a^b G_n(t, s) f_1(s, y_1, y_2)\mathrm{d}s \\ &\leq \int_a^b G_n(t, s)\epsilon(y_1 + y_2)\mathrm{d}s \\ &\leq \epsilon H_1 \max_{t\in[a,b]} \int_a^b G_n(t, s)\mathrm{d}s \\ &= \epsilon H_1 \epsilon_n \\ &\leq \frac{H_1}{2} \\ &= \frac{\|(y_1, y_2)\|}{2}. \end{aligned}$$

Thus we have $\|T_n(y_1, y_2)\|_0 \leq \frac{\|(y_1, y_2)\|}{2}$.

Similarly $\|T_m(y_1, y_2)\|_0 \leq \frac{\|(y_1, y_2)\|}{2}$.

Therefore for all $(y_1, y_2) \in \mathcal{P} \cap \partial\Omega_1$ and $t \in [a, b]$, we get

$$\begin{aligned} \|T(y_1, y_2)\| &= \|(T_n(y_1, y_2), T_m(y_1, y_2))\| \\ &= \|T_n(y_1, y_2)\|_0 + \|T_m(y_1, y_2)\|_0 \leq \|(y_1, y_2)\|. \end{aligned}$$

Further since $f_1^\infty = \infty$, there exists $\bar{H}_2 > 0$ so that $f_1(t, y_1, y_2) > \zeta(y_1 + y_2)$ whenever $y_1 + y_2 > \bar{H}_2$ where $\zeta > 0$ is chosen such that

$$\zeta\gamma \int_{s\in I} G_n(t_0, s)\mathrm{d}s \geq 1.$$

If we define $H_2 = \max\{2H_1, \frac{\bar{H}_2}{\gamma}\}$ and $\Omega_2 = \{(y_1, y_2) : (y_1, y_2) \in E, \|(y_1, y_2)\| < H_2\}$, then for $(y_1, y_2) \in \mathcal{P} \cap \partial\Omega_2$, we have

$$\min_{t\in I}\{y_1(t) + y_2(t)\} \geq \gamma\|(y_1, y_2)\| = \gamma H_2 \geq \bar{H}_2,$$

and

$$\begin{aligned}\|T(y_1, y_2)\| &= \|T_n(y_1, y_2)\|_0 + \|T_m(y_1, y_2)\|_0\\ &\geq \|T_n(y_1, y_2)\|_0\\ &= \max_{t\in[a,b]}\left|\int_a^b G_n(t_0, s) f_1(s, y_1, y_2)\mathrm{d}s\right|\\ &\geq \int_{s\in I} G_n(t_0, s) f_1(s, y_1, y_2)\mathrm{d}s\\ &\geq \zeta \int_{s\in I} G_n(t_0, s)(y_1 + y_2)(s)\mathrm{d}s\\ &\geq \zeta\gamma\|(y_1, y_2)\| \int_{s\in I} G_n(t_0, s)\mathrm{d}s\\ &\geq \|(y_1, y_2)\| = H_2,\end{aligned}$$

and so $\|T(y_1, y_2)\| \geq \|(y_1, y_2)\|$ for all $(y_1, y_2) \in \mathcal{P} \cap \partial\Omega_2$.

Consequently, by part (i) of Theorem 4.1, it follows that T has a fixed point (y_1, y_2) in $\mathcal{P} \cap (\bar{\Omega}_2 \backslash \Omega_1)$ and this implies that the BVP (1.1) and (1.2) has a positive solution (y_1, y_2) which satisfies $H_1 \leq \|(y_1, y_2)\| \leq H_2$.

(ii) Since $f_1^0 = \infty$, there exists $H_1 > 0$ so that

$$f_1(t, y_1, y_2) \geq \zeta(y_1 + y_2),$$

whenever $y_1 + y_2 < H_1$, where $\zeta > 0$ is chosen such that

$$\zeta\gamma \int_{s\in I} G_n(t_0, s)\mathrm{d}s \geq 1.$$

Let $\Omega_1 = \{(y_1, y_2) : (y_1, y_2) \in E, \|(y_1, y_2)\| < H_1\}$, then for $(y_1, y_2) \in \mathcal{P} \cap \partial\Omega_1$ and $t \in [a, b]$, we have

$$\begin{aligned}\|T(y_1, y_2)\| &= \|T_n(y_1, y_2)\|_0 + \|T_m(y_1, y_2)\|_0\\ &\geq \|T_n(y_1, y_2)\|_0\\ &= \max_{t\in[a,b]}\left|\int_a^b G_n(t_0, s) f_1(s, y_1, y_2)\mathrm{d}s\right|\\ &\geq \int_{s\in I} G_n(t_0, s) f_1(s, y_1, y_2)\mathrm{d}s\\ &\geq \zeta \int_{s\in I} G_n(t_0, s)(y_1 + y_2)(s)\mathrm{d}s\\ &\geq \zeta\gamma\|(y_1, y_2)\| \int_{s\in I} G_n(t_0, s)\mathrm{d}s\\ &\geq \|(y_1, y_2)\|,\end{aligned}$$

and so $\|T(y_1, y_2)\| \geq \|(y_1, y_2)\|$ for all $(y_1, y_2) \in \mathcal{P} \cap \partial\Omega_1$.

We assume that $f_1^\infty = f_2^\infty = 0$.
Let us define

$$g_1(r) = \max\{f_1(t, y_1, y_2) : t \in [a, b],\ 0 \le y_1 + y_2 \le r\},$$

$$g_2(r) = \max\{f_2(t, y_1, y_2) : t \in [a, b],\ 0 \le y_1 + y_2 \le r\},$$

then $g_1(r), g_2(r)$ are increasing functions. By $f_1^\infty = f_2^\infty = 0$, we have

$$\lim_{r\to\infty} \frac{g_1(r)}{r} = 0, \quad \lim_{r\to\infty} \frac{g_2(r)}{r} = 0,$$

so there exists $H_2 > 0$ such that $r > H_2 > 2H_1$ and $g_1(r) \le \epsilon r$, $\ g_2(r) \le \epsilon r$, where $\epsilon = \frac{1}{2\max\{\epsilon_n, \epsilon_m\}}$.

Let $\Omega_2 = \{(y_1, y_2) : (y_1, y_2) \in E, \|(y_1, y_2)\| < H_2\}$. If for $(y_1, y_2) \in \mathcal{P} \cap \partial\Omega_2$ then $r = H_2 = \|(y_1, y_2)\|$, so we have

$$\begin{aligned}
\|T_n(y_1, y_2)\|_0 &= \max_{t\in[a,b]} \left| \int_a^b G_n(t, s) f_1(s, y_1, y_2)\mathrm{d}s \right| \\
&\le \int_a^b G_n(t, s) g_1(H_2)\mathrm{d}s \\
&\le \epsilon H_2 \int_a^b G_n(t, s)\mathrm{d}s \\
&\le \epsilon_n \epsilon H_2 \\
&\le \frac{H_2}{2} = \frac{\|(y_1, y_2)\|}{2}.
\end{aligned}$$

In a similar way, we can easily see that

$$\|T_m(y_1, y_2)\|_0 \le \frac{\|(y_1, y_2)\|}{2}.$$

Therefore

$$\|T(y_1, y_2)\| \le \|(y_1, y_2)\|, \qquad \text{for all } (y_1, y_2) \in \mathcal{P} \cap \partial\Omega_2.$$

By part (ii) of Theorem 4.1, T has a fixed point (y_1, y_2) in $\mathcal{P} \cap (\bar{\Omega}_2 \backslash \Omega_1)$, so that the BVP (1.1) and (1.2) has a positive solution (y_1, y_2) and $H_1 \le \|(y_1, y_2)\| \le H_2$. Thus, the proof is completed. □

Theorem 4.3. *Assume that f_i satisfies the following conditions*:

(i) $f_1^0 = \infty$ *or* $f_2^0 = \infty$ *and* $f_1^\infty = \infty$ *or* $f_2^\infty = \infty$ *(or)*,
(ii) *There exists* $l > 0$ *such that, if* $0 \le y_1 \le l$ *and* $0 \le y_2 \le l$, *then* $f_i(t, y_1, y_2) < \epsilon l$, *for* $i = 1, 2$ *where* $\epsilon = \frac{1}{2\max\{\epsilon_n, \epsilon_m\}}$.

Then, the BVP (1.1) *and* (1.2) *has at least two positive solutions* (y_1, y_2) *and* $\left(y_1', y_2'\right)$ *which belongs to* $\mathcal{P}$ *such that*

$$0 \leq \|(y_1, y_2)\| < l < \|\left(y_1', y_2'\right)\|.$$

Proof. Since $f_1^0 = \infty$, there is an $r \in (0, l)$ such that

$$f_1(t, y_1, y_2) \geq \zeta(y_1 + y_2)$$

whenever $y_1 + y_2 < r$, where $\zeta > 0$ is chosen such that $\zeta\gamma \int_{s\in I} G_n(t_0, s)\mathrm{d}s \geq 1$.

Let $\Omega_1 = \{(y_1, y_2) : (y_1, y_2) \in E, \| (y_1, y_2) \| < r\}$, then for $(y_1, y_2) \in \mathcal{P} \cap \partial\Omega_1$ and $t \in [a, b]$, we have

$$\begin{aligned}
\| T(y_1, y_2)\| &= \|T_n(y_1, y_2)\|_0 + \|T_m(y_1, y_2)\|_0 \\
&\geq \|T_n(y_1, y_2)\|_0 \\
&= \max_{t\in[a,b]} \left| \int_a^b G_n(t_0, s) f_1(s, y_1, y_2)\mathrm{d}s \right| \\
&\geq \int_{s\in I} G_n(t_0, s) f_1(s, y_1, y_2)\mathrm{d}s \\
&\geq \zeta \int_{s\in I} G_n(t_0, s)(y_1 + y_2)(s)\mathrm{d}s \\
&\geq \zeta\gamma \|(y_1, y_2)\| \int_{s\in I} G_n(t_0, s)\mathrm{d}s \\
&\geq \|(y_1, y_2)\|.
\end{aligned}$$

Thus, we have

$$\|T(y_1, y_2)\| \geq \| (y_1, y_2)\|, \quad \text{for all} \quad (y_1, y_2) \in \mathcal{P} \cap \partial\Omega_1. \tag{4.1}$$

Now consider $(y_1, y_2) \in \mathcal{P}$ with $\|(y_1, y_2)\| = l$. From condition (ii),

$$\begin{aligned}
\|T_n(y_1, y_2)\|_0 &= \max_{t\in[a,b]} \left| \int_a^b G_n(t, s) f_1(s, y_1, y_2)\mathrm{d}s \right| \\
&\leq \epsilon l \int_a^b G_n(t, s)\mathrm{d}s \\
&\leq \epsilon_n \epsilon l \\
&\leq \frac{l}{2} = \frac{\|(y_1, y_2)\|}{2}.
\end{aligned}$$

In a similar way, we can easily see that

$$\|T_m(y_1, y_2)\|_0 \leq \frac{\|(y_1, y_2)\|}{2}.$$

Thus, we have

$$\|T(y_1, y_2)\| \leq \|(y_1, y_2)\|, \quad \text{for all } (y_1, y_2) \in \mathcal{P} \cap \partial\Omega_2 \tag{4.2}$$

where $\Omega_2 = \{(y_1, y_2) : (y_1, y_2) \in E, \|(y_1, y_2)\| < l\}$. By (ii) of Theorem 4.1, together with (4.1) and (4.2), T has a fixed point (y_1, y_2) in $\mathcal{P} \cap (\bar{\Omega}_2 \backslash \Omega_1)$, so the BVP (1.1) and (1.2) has a positive solution (y_1, y_2) and $r \leq \|(y_1, y_2)\| \leq l$.

Returning to condition (i), from $f_1^{\infty} = \infty$, there exists an $H_1 > 0$ such that $f_1(t, y_1, y_2) > N(y_1 + y_2)$, where we can choose $N > 0$ such that

$$N\gamma \int_{s\in I} G_n(t_0, s)\mathrm{d}s \geq 1.$$

Let us define $H_2 = \max\{2l, \frac{H_1}{\gamma}\}$ and $\Omega_3 = \{(y_1, y_2) : (y_1, y_2) \in E, \|(y_1, y_2)\| < H_2\}$, then for $(y_1, y_2) \in \mathcal{P} \cap \partial\Omega_3$ we have

$$\min_{t\in I}\{y_1(t) + y_2(t)\} \geq \gamma\|(y_1, y_2)\| = \gamma H_1,$$

and

$$\begin{aligned}
\|T(y_1, y_2)\| &= \|T_n(y_1, y_2)\|_0 + \|T_m(y_1, y_2)\|_0 \\
&\geq \|T_n(y_1, y_2)\|_0 \\
&= \max_{t\in[a,b]} \left| \int_a^b G_n(t_0, s) f_1(s, y_1, y_2)\mathrm{d}s \right| \\
&\geq \int_{s\in I} G_n(t_0, s) f_1(s, y_1, y_2)\mathrm{d}s \\
&\geq N \int_{s\in I} G_n(t_0, s)(y_1 + y_2)(s)\mathrm{d}s \\
&\geq N\gamma\|(y_1, y_2)\| \int_{s\in I} G_n(t_0, s)\mathrm{d}s \\
&\geq \|(y_1, y_2)\|,
\end{aligned}$$

and so

$$\|T(y_1, y_2)\| \geq \|(y_1, y_2)\| \text{ for all } (y_1, y_2) \in \mathcal{P} \cap \partial\Omega_3.$$

Theorem 4.1, together with (4.1) and (4.2), implies that T has a fixed point (y_1', y_2') in $\mathcal{P} \cap (\bar{\Omega}_2 \backslash \Omega_3)$ such that $l \leq \|(y_1', y_2')\| \leq H_1$. Thus, the proof is completed. □

Theorem 4.4. *Assume that f_i satisfies the following conditions*:

(i) $f_1^0 = 0$ *or* $f_2^0 = 0$ *and* $f_1^{\infty} = 0$ *or* $f_2^{\infty} = 0$ (*or*)
(ii) *There exists $d > 0$ such that, if $\gamma d \leq y_1 \leq d$ and $\gamma d \leq y_2 \leq d$, then $f_1(t, y_1, y_2) \geq \gamma d$ or $f_2(t, y_1, y_2) \geq \gamma d$.*

Then the BVP (1.1) *and* (1.2) *has at least two positive solutions* (y_1, y_2) *and* (y_1', y_2') *which belongs to $\mathcal{P}$ such that*

$$0 \leq \|(y_1, y_2)\| < d < \|(y_1', y_2')\|.$$

Proof. Since $f_1^0 = 0$, there is an $r \in (0, d)$ such that

$$f_1(t, y_1, y_2) \le \epsilon(y_1 + y_2),$$

whenever $y_1 + y_2 < r$.

Let $\Omega_1 = \{(y_1, y_2) : (y_1, y_2) \in E, \|(y_1, y_2)\| < r\}$, then for $(y_1, y_2) \in \mathcal{P} \cap \partial\Omega_1$ and $t \in [a, b]$, we have

$$\begin{aligned}
T_n(y_1, y_2)(t) &= \max_{t\in[a,b]} \left| \int_a^b G_n(t, s) f_1(s, y_1, y_2)\mathrm{d}s \right| \\
&\le \int_a^b G_n(t, s)\epsilon(y_1 + y_2)\mathrm{d}s \\
&\le \epsilon(y_1 + y_2) \max_{t\in[a,b]} \int_a^b G_n(t, s)\mathrm{d}s \\
&= \epsilon_n \epsilon(y_1 + y_2) \\
&\le \epsilon_n \epsilon r \\
&\le \frac{r}{2} \\
&= \frac{\|(y_1, y_2)\|}{2}.
\end{aligned}$$

Thus we have $\|T_n(y_1, y_2)\|_0 \le \frac{\|(y_1, y_2)\|}{2}$.

Similarly we can see $\|T_m(y_1, y_2)\|_0 \le \frac{\|(y_1, y_2)\|}{2}$.

Therefore for all $(y_1, y_2) \in \mathcal{P} \cap \partial\Omega_1$ and $t \in [a, b]$, we get

$$\|T(y_1, y_2)\| \le \|(y_1, y_2)\|. \tag{4.3}$$

Now consider $(y_1, y_2) \in \mathcal{P}$ with $\|(y_1, y_2)\| < d$. From condition (ii),

$$\begin{aligned}
\|T(y_1, y_2)\| &= \|T_n(y_1, y_2)\|_0 + \|T_m(y_1, y_2)\|_0 \\
&\ge \|T_n(y_1, y_2)\|_0 \\
&= \max_{t\in[a,b]} \left| \int_a^b G_n(t_0, s) f_1(s, y_1, y_2)\mathrm{d}s \right| \\
&\ge \int_{s\in I} G_n(t_0, s)) f_1(s, y_1, y_2)\mathrm{d}s \\
&\ge \gamma d \int_{s\in I} G_n(t_0, s)\mathrm{d}s \\
&\ge d = \|(y_1, y_2)\|,
\end{aligned}$$

and so $\|T(y_1, y_2)\| \ge \|(y_1, y_2)\|$. If we define $\Omega_2 = \{(y_1, y_2) : (y_1, y_2) \in E, \|(y_1, y_2)\| < d\}$, then

$$\|T(y_1, y_2)\| \ge \|(y_1, y_2)\|, \quad \text{for all } (y_1, y_2) \in \mathcal{P} \cap \partial\Omega_2. \tag{4.4}$$

Theorem 4.1(ii), together with (4.3) and (4.4) implies that T has a fixed point (y_1, y_2) in $\mathcal{P} \cap (\bar{\Omega}_2 \backslash \Omega_1)$, so the BVP (1.1) and (1.2) has a positive solution (y_1, y_2) and $r \le \|(y_1, y_2)\| \le d$.

Returning to condition (i), we know that for any $\mu > 0$ there exists an $M > 0$ such that $f_1(t, y_1, y_2) \leq M + \mu(y_1 + y_2)$ for $y_1, y_2 > 0$, and so

$$\begin{aligned} T_n(y_1, y_2)(t) &= \max_{t\in[a,b]} \left| \int_a^b G_n(t,s) f_1(s, y_1, y_2) \mathrm{d}s \right| \\ &\leq \int_a^b G_n(t,s)\{M + \mu(y_1 + y_2)\}\mathrm{d}s \\ &\leq \{M + \mu\|(y_1 + y_2)\|\}\epsilon_n \\ &\leq \{M + \mu\|(y_1 + y_2)\|\}\frac{1}{2\epsilon}, \end{aligned}$$

and similarly

$$T_m(y_1, y_2)(t) \leq \{M + \mu\|(y_1 + y_2)\|\}\frac{1}{2\epsilon}.$$

We choose $\epsilon > 0$ sufficiently small and $L > \frac{M}{\epsilon}$ sufficiently large, so that for all $(y_1, y_2) \in \mathcal{P} \cap \partial\Omega_3$,

$$\|T(y_1, y_2)\| \leq L = \|(y_1, y_2)\|, \tag{4.5}$$

where $\Omega_3 = \{(y_1, y_2) : (y_1, y_2) \in E, \|(y_1, y_2)\| < L\}$. Theorem 4.1, together with (4.4) and (4.5), implies that T has a fixed point (y_1', y_2') in $\mathcal{P}\cap(\bar{\Omega}_2\backslash\Omega_3)$ and this implies that the BVP (1.1) and (1.2) has a positive solution (y_1', y_2') which satisfies $d \leq \|(y_1', y_2')\| \leq L$. Thus the proof is completed. □

5. Example

Now, we give an example to demonstrate the result. Consider the following system of BVP,

$$\begin{cases} y_1'' + 2(y_1 + y_2)^2 e^{-\left(y_1 + y_2 - \frac{1}{2}\right)^2} = 0, & t \in [0, 1] \\ y_2'' + 2(y_1 + y_2)^3 e^{-\left(y_1 + y_2 - \frac{1}{2}\right)^2} = 0, & t \in [0, 1] \end{cases} \tag{5.1}$$

subject to the two-point boundary conditions

$$\begin{cases} y_1(0) = 0 = y_1'(1), \\ y_2(0) = y_2'(0) = y_2'(1) = 0. \end{cases} \tag{5.2}$$

From Theorem 4.2, the BVP (5.1) and (5.2) has at least one positive solution.

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