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Remarks on Hausdorff measure and stability for the *p*-obstacle problem ($1 < p < 2$)

PEIHAO ZHAO and JUN ZHENG*

School of Mathematics and Statistics, Lanzhou University, Lanzhou, Gansu 730000, China
*Author to whom correspondence should be addressed
E-mail: zhengjun08@lzu.cn

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Abstract. In this paper, we consider the obstacle problem for the inhomogeneous p-Laplace equation

$$\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) = f \cdot \chi_{\{u>0\}}, \quad 1 < p < 2,$$

where f is a positive, Lipschitz function. We prove that the free boundary has finite $(N-1)$-Hausdorff measure and stability property, which completes previous works by Caffarelli (*J. Fourier Anal. Appl.* **4(4–5)** (1998) 383–402) for $p = 2$, and Lee and Shahgholian (*J. Differ. Equ.* **195** (2003) 14–24) for $2 < p < \infty$.

Keywords. Obstacle problem; Hausdorff measure; stability.

1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, $f \in L^\infty(\Omega)$, $g \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. We consider the p-obstacle problem governed by

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f \cdot \chi_{\{u>0\}}.$$

A function u in $K^p = \{v \in W^{1,p}(\Omega);\ v - g \in W_0^{1,p}(\Omega),\ v \geq 0, \text{ a.e. in } \Omega\}$ is a solution to the obstacle problem if

$$\int_\Omega (|\nabla u|^{p-2}\nabla u \cdot (\nabla v - \nabla u) + f(v-u))\mathrm{d}x \geq 0,$$

holds for all $v \in K^p$. According to a result established by Choe and Lewis [3] any solution u belongs to $W^{1,p}(\Omega) \bigcap C^{1,\alpha}(\Omega)$, provided $f \in L^q(\Omega)$ for some $q > N$. But the exact value of α is unknown. In 2000 and 2003, Karp *et al.* [6] and Lee and Shahgholian [7] proved that the growth rates of u and ∇u are $\frac{p}{p-1}$, $\frac{1}{p-1}(1 < p < \infty)$, respectively. Moreover, Lee and Shahgholian obtained the stability and finite $(N-1)$-Hausdorff measure of the free boundary when $p > 2$. However, they did not give the same results when $1 < p < 2$. In recent years, the obstacle problems for A-operator were considered by Challal and Lyaghfouri [2] and Challal *et al.* [3] in Orlicz-Soblev space. In [2], the authors established the exact growth of the solution for the A-obstacle problem near the free boundary, which extends the result for the p-obstacle problem. In [3], the authors gave the stability result for the A-obstacle problem, then proved that the free boundary has also finite $(N-1)$-Hausdorff measure (see §6 of [3]). However, it seems that there are some inaccurate statements in their proofs. We refer the reader to §6 of [3]. In fact, using their approximating equation (see page 41 of [3]), we cannot obtain the $(N-1)$-Hausdorff measure for the free boundary successfully. But it is very useful for us to obtain the desired result by using their idea. So in this paper, using an approximating equation given in [5] and the important idea of [3], we give the $(N-1)$-Hausdorff measure result for p-obstacle problem when $1 < p < 2$.

In this paper, we choose $f(x) \equiv 1$ and $g \in W^{1,p}(\Omega) \cap C^{1,\beta}(\partial\Omega)$ for some $\beta \in (0,1)$. Then the obstacle problem we analyze becomes

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \chi_{\{u>0\}}, \quad 1 < p < 2.$$

To describe the results obtained in this paper, we consider the following formulation of the obstacle problem.

DEFINITION 1.1 [2]

We say that a function u in $W^{1,p}(B_1)$, where $B_1 = B_1(0)$ is the unit ball in $\mathbb{R}^N$, belongs to the class $\mathscr{G} = \mathscr{G}(p, N)$ $(1 < p < 2)$ if

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \chi_{\{u>0\}}, & \text{in } B_1, \quad (1.1)\\ 0 \le u \le 1, & \text{in } B_1, \quad (1.2)\\ u(0) = 0. & \quad (1.3)\end{cases}$$

Condition (1.1) is understood in the weak sense, i.e. $\int |\nabla u|^{p-2}\nabla u \cdot \nabla\phi \mathrm{d}x = -\int \phi\chi_{\{u>0\}}\mathrm{d}x$ for all $\phi \in W_0^{1,p}(B_1)$. According to [4,6,8], (1.3) makes sense since (1.1) and (1.2) provided that $u \in C^{1,\alpha}(B_1)$ for some $\alpha \in (0,1)$.

2. Main results

In this section, we give an estimate on $\{|u(x)| \le \delta^{\frac{p}{p-1}}\}$ $(\delta < 1)$ based on the growth rates given in [6,7]. We use the same notations as [7],

$$O_\delta = \{|\nabla u(x)| \le \delta^{\frac{1}{p-1}}\} \quad \text{and} \quad O_{\delta_i} = \{|u_{x_i}(x)| \le \delta^{\frac{1}{p-1}}\}.$$

By the known results (see [6]), if $x_0 \in \partial\{u > 0\} \cap B_{1-\delta}$ then there exist $y_0 \in \{u > 0\}$ and $c > 0$ $(c = c(N,p))$ such that

$$B_{c\delta}(y_0) \subset B_\delta(x_0) \cap O_\delta \cap \{u > 0\}. \tag{2.1}$$

Let $\Lambda(u) := B_{\frac{1}{2}} \cap \{u = 0\}$ and $\Lambda(u_1) \vee \Lambda(u_2) := (\Lambda(u_1)\backslash\Lambda(u_2)) \cup (\Lambda(u_2)\backslash\Lambda(u_1))$.

Denote by $\mathscr{L}^N$ the N-dimensional Lebesgue measure and by $\mathscr{H}^{N-1}$ the $(N-1)$-Hausdorff measure. The main results in this paper are the following theorems.

Theorem 2.1. *For $u \in \mathscr{G}$, $x_0 \in \partial\{u > 0\} \cap B_{\frac{1}{2}}$ and $0 < r < \frac{1}{4}$, there holds*

$$\mathscr{H}^{N-1}(\partial\{u > 0\} \cap B_r(x_0)) < C_0 r^{N-1},$$

for a generic constant $C_0 = C_0(p, N)$.

Theorem 2.2. *Let $u_1, u_2 \in \mathscr{G}$ be two disjoint solutions satisfying*

$$\|u_1 - u_2\|_\infty < \delta^{\frac{p}{p-1}}. \tag{2.2}$$

Then

$$\mathscr{L}^N(\Lambda(u_1) \vee \Lambda(u_2)) \le C_1\delta,$$

and

$$(\Lambda(u_2))_{(-C_1\delta)} \subset \Lambda(u_1) \subset \{u_2 < \delta^{\frac{p}{p-1}}\},$$

where $C_1 = C_1(N, p)$ is large enough, and

$$(\Lambda(u_2))_{(-C_1\delta)} = \{x \in \Lambda(u_2);\ \operatorname{dist}(x, \{u_2 > 0\}) > \delta\}.$$

3. Main proofs

In order to prove Theorem 2.1, we need the following lemma given in [7] for $1 < p < \infty$.

Lemma 3.1 (*Lemma* 2.3 *of* [7]). *There is a positive constant $M = M(p, N)$ such that for every $u \in \mathscr{G}$, there holds*

$$|\nabla u(x)| \le M|x|^{\frac{1}{p-1}}, \quad \forall\, x \in B_1.$$

Now we introduce the following approximating equation (see [5]):

$$-\text{div}\,(|u_\epsilon|^{p-2}\nabla u_\epsilon) + \vartheta_\epsilon(u_\epsilon) = 0 \quad \text{in } B_1,\ \ u_\epsilon = u \quad \text{on } \partial B_1. \tag{3.1}$$

Here, for each $\epsilon > 0$, $\vartheta_\epsilon : \mathbb{R} \to [0, 1]$ is the nondecreasing Lipschitz function given by

$$\vartheta_\epsilon(t) = 0,\ t < 0, \quad \vartheta_\epsilon(t) = \frac{t}{\epsilon},\ 0 < t \le \epsilon \ \text{ and } \ \vartheta_\epsilon(t) = 1,\ t > \epsilon.$$

According to [5], there exists a unique solution u_ϵ to (3.1) which converge to the solution u to (1.1) in $C^{1,\theta}(\bar{B}_1)$ for some θ, $0 < \theta < 1$. Moreover, we have the following lemmas:

Lemma 3.2. *There is a positive constant $M_0 = M_0(p, N)$ such that for small ϵ, there holds*

$$\int_{B_{\frac{r}{2}}} [|\nabla u_\epsilon|^{p-2}|D^2 u_\epsilon(x)|]^2 \mathrm{d}x \le M_0 r^{N-2}, \qquad \forall\, 0 < r < 1.$$

Proof. Let $G_\epsilon(t) = (\epsilon + t^2)^{\frac{p-2}{2}} t$, $t \in (-\infty, +\infty)$. $\Phi = G(u_{\epsilon x_i})\varphi^2$, where $\varphi \in \mathcal{D}(B_{\frac{3r}{4}})$ satisfies

$$\begin{cases} 0 \le \varphi \le 1, & \text{in } B_{\frac{3r}{4}}, \\ \varphi = 1, & \text{in } B_{\frac{r}{2}}, \\ |\nabla\varphi| \le \frac{4}{r}, & \text{in } B_{\frac{3r}{4}}. \end{cases}$$

Now differentiate eq. (3.1) with respect to x_i, and multiply it by Φ and by integrating $B_{\frac{3r}{4}}$, we get

$$\int_{B_{\frac{3r}{4}}} (|\nabla u_\epsilon|^{p-2}\nabla u_\epsilon)_{x_i} \cdot \nabla\Phi \mathrm{d}x = -\int_{B_{\frac{3r}{4}}} (\vartheta_\epsilon(u_\epsilon))_{x_i}\Phi \mathrm{d}x. \tag{3.2}$$

The left-hand side of (3.2) becomes

$$\begin{aligned} I^i &= \int_{B_{\frac{3r}{4}}} (|\nabla u_\epsilon|^{p-2}\nabla u_\epsilon)_{x_i} \cdot \nabla\Phi \mathrm{d}x \\ &= \int_{B_{\frac{3r}{4}}} \varphi^2 G'(u_{\epsilon x_i})[(p-2)|\nabla u_\epsilon|^{p-4}|\nabla u_\epsilon \cdot \nabla u_{\epsilon x_i}|^2 \\ &\qquad\qquad + |\nabla u_\epsilon|^{p-2}|\nabla u_{\epsilon x_i}|^2]\mathrm{d}x \\ &\quad + \int_{B_{\frac{3r}{4}}} 2\varphi G(u_{\epsilon x_i})[(p-2)|\nabla u_\epsilon|^{p-4}(\nabla u_\epsilon \cdot \nabla u_{\epsilon x_i})\nabla u_\epsilon \\ &\qquad\qquad + |\nabla u_\epsilon|^{p-2}\nabla u_{\epsilon x_i}] \cdot \nabla\varphi \mathrm{d}x \\ &=: I_1^i + I_2^i. \end{aligned} \tag{3.3}$$

Since $1 < p < 2$, it follows that

$$\begin{aligned} I_1^i &= \int_{B_{\frac{3r}{4}}} \varphi^2[(p-2)u_{\epsilon x_i}^2(\epsilon + u_{\epsilon x_i}^2)^{\frac{p-4}{2}} + (\epsilon + u_{\epsilon x_i}^2)^{\frac{p-2}{2}}] \\ &\quad \cdot [(p-2)|\nabla u_\epsilon|^{p-4}|\nabla u_\epsilon \cdot \nabla u_{\epsilon x_i}|^2 + |\nabla u_\epsilon|^{p-2}|\nabla u_{\epsilon x_i}|^2]\mathrm{d}x \\ &\geq (p-1)^2 \int_{B_{\frac{3r}{4}}} \varphi^2(\epsilon + u_{\epsilon x_i}^2)^{\frac{p-2}{2}}|\nabla u_\epsilon|^{p-2}|\nabla u_{\epsilon x_i}|^2\mathrm{d}x. \end{aligned} \tag{3.4}$$

Using the fact that $|p-2| < p$ and Young inequality, we have

$$\begin{aligned} \left|I_2^i\right| &= \int_{B_{\frac{3r}{4}}} 2\varphi(\epsilon + u_{\epsilon x_i}^2)^{\frac{p-2}{2}} u_{\epsilon x_i}[(p-2)|\nabla u_\epsilon|^{p-4}(\nabla u_\epsilon \cdot \nabla u_{\epsilon x_i})\nabla u_\epsilon \\ &\qquad\qquad + |\nabla u_\epsilon|^{p-2}\nabla u_{\epsilon x_i}] \cdot \nabla\varphi \mathrm{d}x \\ &\leq \frac{8}{r}(p+1)\int_{B_{\frac{3r}{4}}} \varphi(\epsilon + u_{\epsilon x_i}^2)^{\frac{p-2}{2}}|u_{\epsilon x_i}||\nabla u_\epsilon|^{p-2}|\nabla u_{\epsilon x_i}|\mathrm{d}x \\ &= \int_{B_{\frac{3r}{4}}} [(\epsilon+u_{\epsilon x_i}^2)^{\frac{p-2}{2}}|\nabla u_\epsilon|^{p-2}][(p-1)\varphi|\nabla u_{\epsilon x_i}|]\left[\frac{8(p+1)}{r(p-1)}|u_{\epsilon x_i}|\right]\mathrm{d}x \\ &\leq \frac{(p-1)^2}{2}\int_{B_{\frac{3r}{4}}} \varphi^2(\epsilon + u_{\epsilon x_i}^2)^{\frac{p-2}{2}}|\nabla u_\epsilon|^{p-2}|\nabla u_{\epsilon x_i}|^2\mathrm{d}x \\ &\quad + \frac{32(p+1)^2}{r^2(p-1)^2}\int_{B_{\frac{3r}{4}}} (\epsilon + u_{\epsilon x_i}^2)^{\frac{p-2}{2}}|\nabla u_\epsilon|^{p-2}|u_{\epsilon x_i}|^2\mathrm{d}x \end{aligned}$$

$$\leq \frac{(p-1)^2}{2}\int_{B_{\frac{3r}{4}}} \varphi^2(\epsilon+u_{\epsilon x_i}^2)^{\frac{p-2}{2}}|\nabla u_\epsilon|^{p-2}|\nabla u_{\epsilon x_i}|^2\mathrm{d}x$$

$$+\frac{32(p+1)^2}{r^2(p-1)^2}\int_{B_{\frac{3r}{4}}} |\nabla u_\epsilon|^{p-2}|u_{\epsilon x_i}|^p\mathrm{d}x$$

$$\leq \frac{(p-1)^2}{2}\int_{B_{\frac{3r}{4}}} \varphi^2(\epsilon+u_{\epsilon x_i}^2)^{\frac{p-2}{2}}|\nabla u_\epsilon|^{p-2}|\nabla u_{\epsilon x_i}|^2\mathrm{d}x$$

$$+\frac{32(p+1)^2}{r^2(p-1)^2}\int_{B_{\frac{3r}{4}}} |\nabla u_\epsilon|^{2(p-1)}\mathrm{d}x. \tag{3.5}$$

The right-hand side of (3.2) becomes

$$I^i = -\int_{B_{\frac{3r}{4}}} \vartheta_\epsilon'(u_\epsilon)u_{\epsilon x_i}(\epsilon+u_{\epsilon x_i}^2)^{\frac{p-2}{2}}u_{\epsilon x_i}\varphi^2\mathrm{d}x$$

$$\leq 0. \tag{3.6}$$

By (3.2)–(3.6) we have

$$\int_{B_{\frac{r}{2}}} (\epsilon+u_{\epsilon x_i}^2)^{\frac{p-2}{2}}|\nabla u_\epsilon|^{p-2}|\nabla u_{\epsilon x_i}|^2\mathrm{d}x \leq \frac{64(p+1)^2}{r^2(p-1)^4}\int_{B_{\frac{3r}{4}}} |\nabla u_\epsilon|^{2(p-1)}\mathrm{d}x.$$

Since $p<2$, it follows that

$$\int_{B_{\frac{r}{2}}} (\epsilon+|\nabla u_\epsilon|^2)^{\frac{p-2}{2}}|\nabla u_\epsilon|^{p-2}|\nabla u_{\epsilon x_i}|^2\mathrm{d}x$$

$$\leq \frac{64(p+1)^2}{r^2(p-1)^4}\int_{B_{\frac{3r}{4}}} |\nabla u_\epsilon|^{2(p-1)}\mathrm{d}x. \tag{3.7}$$

Summing up (3.7) from $i=1$ to N, we get

$$\int_{B_{\frac{r}{2}}} [(\epsilon+|\nabla u_\epsilon|^2)^{\frac{p-2}{2}}|D^2u_\epsilon|]^2\mathrm{d}x \leq \frac{64(p+1)^2}{r^2(p-1)^4}\int_{B_{\frac{3r}{4}}} |\nabla u_\epsilon|^{2(p-1)}\mathrm{d}x. \tag{3.8}$$

Since $u_\epsilon \to u$ in $C^{1,\theta}(\bar{B}_1)$, from (3.8) and Lemma 3.1, we get $D^2u_\epsilon \in L^2(B_{\frac{r}{2}})$. Furthermore, we can deduce that $D^2u \in L^2(B_{\frac{r}{2}})$. Moreover, $(\epsilon+|\nabla u_\epsilon|^2)^{\frac{p-2}{2}}|D^2u_\epsilon| \rightharpoonup |\nabla u|^{p-2}|D^2u|$ in $L^2(B_{\frac{r}{2}})$ and $|\nabla u_\epsilon|^{p-2}|D^2u_\epsilon| \rightharpoonup |\nabla u|^{p-2}|D^2u|$ in $L^2(B_{\frac{r}{2}})$. Then we have

$$\int_{B_{\frac{r}{2}}} \{[(\epsilon+|\nabla u_\epsilon|^2)^{\frac{p-2}{2}}|D^2u_\epsilon|]^2-(|\nabla u_\epsilon|^{p-2}|D^2u_\epsilon|)^2\}\mathrm{d}x \to 0 \text{ as } \epsilon\to 0. \tag{3.9}$$

So for ϵ small enough, by (3.8), (3.9) and Lemma 3.1, we can obtain the desired result.

Lemma 3.3. $\frac{|\vartheta_\epsilon(u_\epsilon)|^2}{(p+1)^2} \le [|\nabla u_\epsilon|^{p-2}|D^2u_\epsilon|]^2$ *in* B_1.

Proof.

$$\begin{aligned} |\vartheta_\epsilon(u_\epsilon)|^2 &= (\mathrm{div}|\nabla u_\epsilon|^{p-2}\nabla u_\epsilon)^2 \\ &= \left[\sum_{i,j=1}^{N}(|\nabla u_\epsilon|^{p-2}\delta_{ij} + (p-2)|\nabla u_\epsilon|^{p-4}u_{\epsilon x_i}u_{\epsilon x_j})u_{\epsilon x_i x_j}\right]^2 \\ &\le (p+1)^2[|\nabla u_\epsilon|^{p-2}|D^2u_\epsilon|]^2. \end{aligned}$$

Lemma 3.4. *For any ball* $B_r(x_0) \subset B_{\frac{1}{2}}$, *with* $x_0 \in \partial\{u > 0\} \cap B_{\frac{1}{2}}$ *and* $r < \frac{1}{2}$, *there holds*

$$\int_0^1 \mathscr{L}^N(O_\delta \cap B_{rs}(x_0) \cap \{u > 0\})\mathrm{d}s \le C_2\delta r^{N-1},$$

where $0 < \delta < 1$, $C_2 = C_2(p, N)$ *is a constant.*

Proof. Firstly we define

$$O_\epsilon = \left\{|\nabla u_\epsilon| \le 2\delta^{\frac{1}{p-1}}\right\} \text{ and } O_{\epsilon_i} = \left\{|u_{\epsilon x_i}| \le 2\delta^{\frac{1}{p-1}}\right\}.$$

Then we have

$$O_\delta \cap B_{\frac{1}{2}} \subset O_\epsilon \cap B_{\frac{1}{2}}. \tag{3.10}$$

Indeed, there exists $\epsilon_0 = \epsilon_0(\delta, p)$, such that $\forall\, \epsilon \in (0, \epsilon_0)$. There holds

$$\|\nabla u_\epsilon - \nabla u\|_{\infty, \bar{B}_{\frac{1}{2}}} < \delta^{\frac{1}{p-1}}.$$

On the other hand,

$$|\nabla u_\epsilon| \le |\nabla u_\epsilon - \nabla u| + |\nabla u| \le \delta^{\frac{1}{p-1}} + \delta^{\frac{1}{p-1}} = 2\delta^{\frac{1}{p-1}}.$$

Now differentiating eq. (3.1) with respect to x_i gives

$$\begin{aligned} &-\mathrm{div}\,[\nabla u_{\epsilon x_i}|\nabla u_\epsilon|^{p-2} + (p-2)|\nabla u_\epsilon|^{p-4}\nabla u_\epsilon(\nabla u_{\epsilon x_i} \cdot \nabla u_\epsilon)] \\ &\quad +\vartheta'_\epsilon(u_\epsilon)u_{\epsilon x_i} = 0. \end{aligned} \tag{3.11}$$

Let

$$F(\eta) = \begin{cases} 2\delta^{\frac{1}{p-1}}(\epsilon + 4\delta^{\frac{2}{p-1}})^{\frac{p-2}{2}}, & \eta > 2\delta^{\frac{1}{p-1}}, \\ (\epsilon + \eta^2)^{\frac{p-2}{2}}\eta, & |\eta| \le 2\delta^{\frac{1}{p-1}}, \\ -2\delta^{\frac{1}{p-1}}(\epsilon + 4\delta^{\frac{2}{p-1}})^{\frac{p-2}{2}}, & \eta < -2\delta^{\frac{1}{p-1}}. \end{cases}$$

Then $F'(\eta) = [(p-2)\eta^2(\epsilon + \eta^2)^{\frac{p-4}{2}} + (\epsilon + \eta^2)^{\frac{p-2}{2}}]\chi_{\{|\eta| < 2\delta^{\frac{1}{p-1}}\}}$.

Multiplying (3.11) by $F(u_{\epsilon x_i})$ and integrating over $B_{rs}(x_0)$, we get

$$\begin{aligned}&\int_{B_{rs}(x_0)} [|\nabla u_\epsilon|^{p-2}\nabla u_{\epsilon x_i} + (p-2)|\nabla u_\epsilon|^{p-4}\nabla u_\epsilon(\nabla u_{\epsilon x_i}\cdot\nabla u_\epsilon)]\\ &\qquad \cdot\nabla F(u_{\epsilon x_i}) + \vartheta_\epsilon'(u_\epsilon)u_{\epsilon x_i}F(u_{\epsilon x_i})\mathrm{d}x\\ &\quad = \int_{\partial B_{rs}(x_0)} [|\nabla u_\epsilon|^{p-2}D_\nu u_{\epsilon x_i} + (p-2)|\nabla u_\epsilon|^{p-4}D_\nu u_\epsilon(\nabla u_{\epsilon x_i}\cdot\nabla u_\epsilon)]\\ &\qquad \times F(u_{\epsilon x_i})\mathrm{d}S, \end{aligned} \tag{3.12}$$

where D_ν is the outward normal derivative.

On one hand, by the choice of F and Lemma 3.2, we have

$$\begin{aligned}&\int_0^1\int_{\partial B_{rs}(x_0)} [|\nabla u_\epsilon|^{p-2}D_\nu u_{\epsilon x_i} + (p-2)|\nabla u_\epsilon|^{p-4}D_\nu u_\epsilon(\nabla u_{\epsilon x_i}\cdot\nabla u_\epsilon)]F(u_{\epsilon x_i})\mathrm{d}S\mathrm{d}s\\ &\quad \le (p+1)\int_{B_r(x_0)} |\nabla u_\epsilon|^{p-2}|D^2u_\epsilon||F(u_{\epsilon x_i})|\mathrm{d}x\\ &\quad \le (p+1)\left(\int_{B_r(x_0)}[|\nabla u_\epsilon|^{p-2}|D^2u_\epsilon|]^2\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{B_r(x_0)}|F(u_{\epsilon x_i})|^2\mathrm{d}x\right)^{\frac{1}{2}}\\ &\quad \le C_3\delta r^{N-1}, \end{aligned} \tag{3.13}$$

where C_3 is a positive constant depending only on p, N.

On the other hand, from (3.10), we can obtain

$$\begin{aligned}&\sum_{i=1}^N\int_{B_{rs}(x_0)} [|\nabla u_\epsilon|^{p-2}\nabla u_{\epsilon x_i} + (p-2)|\nabla u_\epsilon|^{p-4}\nabla u_\epsilon\nabla u_\epsilon\cdot\nabla u_{\epsilon x_i}]\cdot\nabla F(u_{\epsilon x_i})\mathrm{d}x\\ &\quad = \sum_{i=1}^N\int_{B_{rs}(x_0)\bigcap O_{\epsilon_i}} [|\nabla u_\epsilon|^{p-2}|\nabla u_{\epsilon x_i}|^2 + (p-2)|\nabla u_\epsilon|^{p-4}|\nabla u_\epsilon\cdot\nabla u_{\epsilon x_i}|^2]\\ &\qquad \cdot\left[(p-2)u_{\epsilon x_i}^2(\epsilon+u_{\epsilon x_i}^2)^{\frac{p-4}{2}} + (\epsilon+u_{\epsilon x_i}^2)^{\frac{p-2}{2}}\right]\mathrm{d}x\\ &\quad \ge (p-1)^2\sum_{i=1}^N\int_{B_{rs}(x_0)\cap O_{\epsilon_i}} |\nabla u_\epsilon|^{p-2}(\epsilon+u_{\epsilon x_i}^2)^{\frac{p-2}{2}}|\nabla u_{\epsilon x_i}|^2\mathrm{d}x\\ &\quad \ge (p-1)^2\int_{B_{rs}(x_0)\cap O_\epsilon} (\epsilon+|\nabla u_\epsilon|^2)^{p-2}\sum_{i=1}^N|\nabla u_{\epsilon x_i}|^2\mathrm{d}x\\ &\quad = (p-1)^2\int_{B_{rs}(x_0)\cap O_\epsilon} (\epsilon+|\nabla u_\epsilon|^2)^{p-2}|D^2u_\epsilon|^2\mathrm{d}x\\ &\quad \ge (p-1)^2\int_{B_{rs}(x_0)\cap O_\delta} (\epsilon+|\nabla u_\epsilon|^2)^{p-2}|D^2u_\epsilon|^2\mathrm{d}x. \end{aligned} \tag{3.14}$$

Since $\vartheta_\epsilon(t)$ is nondecreasing in t and $F(\eta)\eta \ge 0$, we have

$$\int_{B_{rs}(x_0)} \vartheta_\epsilon'(u_\epsilon)u_{\epsilon x_i}F(u_{\epsilon x_i})\mathrm{d}x \ge 0. \tag{3.15}$$

Thus, for ϵ small enough, by (3.9) and (3.12)–(3.15), we get

$$\int_0^1 \int_{B_{rs}(x_0)\cap O_\delta} |\nabla u_\epsilon|^{2(p-2)}|D^2 u_\epsilon|^2 \mathrm{d}x\mathrm{d}s \le C_2\delta r^{N-1}.$$

By Lemma 3.3, we have

$$\int_0^1 \int_{B_{rs}(x_0)\cap O_\delta} |\vartheta_\epsilon(u_\epsilon)|^2 \mathrm{d}x \le C_2\delta r^{N-1}.$$

According to Theorem 2 of [5], $u_\epsilon \ge u$. By the definition of ϑ_ϵ, we have

$$\int_0^1 \int_{B_{rs}(x_0)\cap O_\delta\cap\{u\ge\epsilon\}} \mathrm{d}x\mathrm{d}s \le C_2\delta r^{N-1}.$$

Now letting $\epsilon \to 0$ implies that

$$\int_0^1 \mathscr{L}^N(B_{rs}(x_0)\cap O_\delta \cap \{u>0\})\mathrm{d}s \le C_2\delta r^{N-1}.$$

This completes the proof of Lemma 3.4.

Due to the above lemmas, we use the same technique as [7] for the proofs of Theorems 2.1 and 2.2.

Proof of Theorems 2.1 *and* 2.2 (see [7]). Under the conditions of Lemma 3.4, firstly we conclude that there exists a positive constant $C_4 = C_4(p, N)$ such that

$$\mathscr{L}^N(O_\delta \cap B_r(x_0)\cap \{u>0\}) \le C_4\delta r^{N-1} \text{ for all } r < \frac{1}{4}.$$

If not, then there exists a ball $B_r(x_0)$ with center on the free boundary such that for any $k \in \mathbb{R}$,

$$\mathscr{L}^N(O_\delta \cap B_r(x_0)\cap \{u>0\}) \ge k\delta r^{N-1}.$$

However, by Lemma 3.4 we have

$$\begin{aligned} 2^N C_2\delta r^N &\ge 2^N \int_0^1 \mathscr{L}^N(O_\delta \cap B_{2rs}(x_0)\cap\{u>0\})\mathrm{d}s \\ &\ge \frac{1}{2}\mathscr{L}^N(O_\delta\cap B_r(x_0)\cap\{u>0\}) \\ &\ge \frac{1}{2}k\delta r^{N-1}, \end{aligned}$$

which is a contradiction for large k.

Secondly, due to Besicovitch covering theorem, let $\{B_\delta(x^i)\}_{i\in I}$ be finite coverings of $\partial\{u>0\}\cap B_r(x_0)$ with $x^i \in \partial\{u>0\}$, with at most $n = n(N)$ overlapping at each point. Then by (2.1) we have

$$\begin{aligned} \sum_{i\in I}(C\delta)^N &\le \sum_{i\in I}\mathscr{L}^N(O_\delta\cap B_\delta(x^i)\cap\{u>0\}) \\ &\le nC\mathscr{L}^N(O_\delta\cap B_r(x_0)\cap\{u>0\}) \le C'\delta r^{N-1}. \end{aligned}$$

where C, C' are positive constants. This proves Theorem 2.1.

From Theorem 2.1 in [6] and Lemma 3.1, we can deduce that for appropriate C,

$$\left\{0 < u < C\delta^{\frac{p}{p-1}}\right\} \cap B_r(x_0) \subset O_\delta \cap B_r(x_0). \tag{3.16}$$

By (3.16) and (2.2), we get directly

$$\mathscr{L}^N(\Lambda(u_1)\backslash\Lambda(u_2)) \leq C_1\delta, \quad \mathscr{L}^N(\Lambda(u_2)\backslash\Lambda(u_1)) \leq C_1\delta,$$

and

$$\Lambda(u_1) \subset \left\{u_2 < \delta^{\frac{p}{p-1}}\right\}.$$

If $x \in (\Lambda(u_2))_{(-C_1\delta)}$, $x\bar{\in}\Lambda(u_1)$, i.e. $x \in \{u_1 > 0\}$, $u_2(x) = 0$. By (2.2), we have

$$\|u_1 - u_2\|_\infty = \|u_1\|_\infty < \delta^{\frac{p}{p-1}}. \tag{3.17}$$

Since $x \in \overline{\{u_1 > 0\}}$, by Lemma 3.1 in [6], we have

$$\sup_{B_{C_1\delta}(x)} u_1 \geq C_4(C\delta)^{\frac{p}{p-1}}$$

for large C, which is a contradiction with (3.17). This proves Theorem 2.2.

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