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An n-dimensional pseudo-differential operator involving the Hankel transformation

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Abstract. An n-dimensional pseudo-differential operator (p.d.o.) involving the n-dimensional Hankel transformation is defined. The symbol class H^m is introduced. It is shown that p.d.o.'s associated with symbols belonging to this class are continuous linear mappings of the n-dimensional Zemanian space $H_\mu(I^n)$ into itself. An integral representation for the p.d.o. is obtained. Using the Hankel convolution, it is shown that the p.d.o. satisfies a certain L^1-norm inequality.

Keywords. Hankel transformation; Hankel convolution; pseudo-differential operator.

1. Introduction

In this paper, we shall use the following notations: $\mathbb{R}^n$ and $\mathbb{C}^n$ are respectively the real and complex n-dimensional Euclidean spaces. An n-tuple is denoted by $x = (x_1, \ldots, x_n)$. We shall restrict x and y to the first orthant of $\mathbb{R}^n$ which we denote by I^n. Thus, $I^n = \{x \in \mathbb{R}^n : 0 < x_i < \infty,\ i = 1, \ldots, n\}$ and the Euclidean norm $|x| = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$. A function on a subset of $\mathbb{R}^n$ is denoted by $\phi(x) = \phi(x_1 \ldots, x_n)$. By $[x]$, we mean the product $x_1 \ldots x_n$. Thus $[x^q] = x_1^{q_1} \ldots x_n^{q_n}$, where $q = (q_1, \ldots, q_n)$. The notation $x \leq y$ means $x_i \leq y_i \quad (i = 1, 2, \ldots, n)$. In what follows, the letters k and q are denoted by nonnegative integers in $\mathbb{R}^n$, i.e., k_i and q_i $(i = 1, 2, \ldots, n)$ are nonnegative integers. Letting $|k| = k_1 + \cdots + k_n$, we shall write

$$D_x^k := \frac{\partial^{|k|}}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}},$$

and denote

$$(x^{-1} D_x)^k := \prod_{i=1}^n \left(x_i^{-1} \frac{\partial}{\partial x_i}\right)^{k_i}.$$

Here we use the notation and terminology of [2, 4]. For $\mu \in \mathbb{R}$, the differential operators N_μ, M_μ and S_μ are defined by

$$\begin{aligned} N_{i\mu} &= x_i^{\mu+1/2}\frac{\partial}{\partial x_i}x_i^{-\mu-1/2}, \\ N_\mu &= N_{1\mu}\ldots N_{n\mu} = [x]^{\mu+1/2}\frac{\partial^n}{\partial x_1\ldots\partial x_n}[x]^{-\mu-1/2}, \\ M_{i\mu} &= x_i^{-\mu-1/2}\frac{\partial}{\partial x_i}x_i^{\mu+1/2}, \\ M_\mu &= M_{1\mu}\ldots M_{n\mu} = [x]^{-\mu-1/2}\frac{\partial^n}{\partial x_1\ldots\partial x_n}[x]^{\mu+1/2} \end{aligned}$$

and

$$\begin{aligned} S_\mu &= M_\mu N_\mu = [x]^{-\mu-1/2}\frac{\partial^n}{\partial x_1\ldots\partial x_n}[x]^{2\mu+1}\frac{\partial^n}{\partial x_1\ldots\partial x_n}[x]^{-\mu-1/2} \\ &= \prod_{i=1}^n\left(\frac{\partial^2}{\partial x_i^2}-\frac{4\mu^2-1}{4x_i^2}\right). \end{aligned}$$

Now, we define the test function space $H_\mu(I^n)$ to be the space of all smooth complex-valued functions $\phi(x)$ which are defined on I^n such that for each pair of nonnegative integers q and k in $\mathbb{N}_0^n$,

$$\gamma_{q,k}^\mu(\phi) = \sup_{x\in I^n}\left|[x^q](x^{-1}D_x)^k[x]^{-\mu-1/2}\phi(x)\right| < \infty. \tag{1.1}$$

The space $H_\mu(I^n)$ is topologized by the family of seminorms $\{\gamma_{q,k}^\mu\}_{q,k\in\mathbb{N}_0^n}$.

The n-dimensional classical μ-th order Hankel transformation h_μ is defined by

$$\begin{aligned} \hat{\phi}(y) = (h_\mu\phi)(y) &= \int_0^\infty\ldots\int_0^\infty \phi(x_1,\ldots,x_n) \\ &\quad\times\left(\prod_{i=1}^n(x_iy_i)^{1/2}J_\mu(x_iy_i)\right)\mathrm{d}x_1\ldots\mathrm{d}x_n, \quad \phi\in H_\mu(I^n), \end{aligned} \tag{1.2}$$

where $y \in I^n$ and J_μ is the Bessel function of the first kind and order μ can be extended by transposition to distributions belonging to $H_\mu'(I^n)$, the dual of the test function space $H_\mu(I^n)$, provided that $\mu \geq -1/2$ [2]. The inversion formula for (1.2) is given by

$$\begin{aligned} \phi(x) &= \int_0^\infty\ldots\int_0^\infty \hat{\phi}(y_1,\ldots,y_n) \\ &\quad\times\left(\prod_{i=1}^n(x_iy_i)^{1/2}J_\mu(x_iy_i)\right)\mathrm{d}y_1\ldots\mathrm{d}y_n, \quad x\in I^n. \end{aligned} \tag{1.3}$$

From [2] we have the following relations for any $\phi \in H_\mu(I^n)$ and $x, y \in I^n$:

$$h_{\mu+1}([-x]\phi) = N_\mu h_\mu \phi, \tag{1.4}$$

$$h_{\mu+1}(N_\mu \phi) = [-y] h_\mu \phi, \tag{1.5}$$

$$h_\mu(S_\mu \phi) = (-1)^n [y]^2 h_\mu \phi. \tag{1.6}$$

We have the Leibniz formula and Bessel differential operator of the rth order in $\mathbb{R}^n$ respectively as follows:

$$\prod_{i=1}^{n}((x_i^{-1}\partial/\partial x_i)^{k_i}(x_i^{-\mu-1/2}(\psi\phi)(x_1,\ldots,x_n)))$$
$$= \prod_{i=1}^{n}\left(\sum_{\nu_i=0}^{k_i}\binom{k_i}{\nu_i}(x_i^{-1}\partial/\partial x_i)^{\nu_i}\psi(x_1,\ldots,x_n)(x_i^{-1}\partial/\partial x_i)^{k_i-\nu_i}\right.$$
$$\left.\times\,(x_i^{-\mu-1/2}\phi(x_1,\ldots,x_n))\right), \tag{1.7}$$

$$S_{\mu,x}^{r}\phi(x_1,\ldots,x_n) = \prod_{i=1}^{n}\left(\sum_{j_i=0}^{r_i} b_{j_i} x_i^{2j_i+\mu+1/2}(x_i^{-1}\partial/\partial x_i)^{r_i+j_i}\right.$$
$$\left.(x_i^{-\mu-1/2}\phi(x_1,\ldots,x_n))\right), \tag{1.8}$$

where the b_{j_i} are constants depending only on μ.

For $i = 1, 2, \ldots, n$, let $\Delta(x_i, y_i, z_i)$ be the area of a triangle with sides x_i, y_i, z_i if such a triangle exists and zero otherwise. For fixed $\gamma > 0$, set

$$D_{\gamma-1/2}(x,y,z) = \prod_{i=1}^{n} D_{\gamma-1/2}(x_i,y_i,z_i)$$
$$= 2^{n(3\gamma-5/2)}\pi^{-n/2}(\Gamma(\gamma+1/2))^{2n}(\Gamma(\gamma))^{-n}$$
$$\times \prod_{i=1}^{n}((x_i y_i z_i)^{-2\gamma+1}\{\Delta(x_i,y_i,z_i)\}^{2\gamma-2}).$$

We note that the n-dimensional Delsarte kernel $D_{\gamma-1/2}(x, y, z)$ is nonnegative and symmetric in x, y, z. We have the formula

$$\int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{n}(j_{\gamma-1/2}(z_i t_i) D_{\gamma-1/2}(x_i,y_i,z_i)\mathrm{d}\sigma(z_i))$$
$$= \prod_{i=1}^{n}(j_{\gamma-1/2}(x_i t_i) j_{\gamma-1/2}(y_i t_i)), \tag{1.9}$$

where

$$\prod_{i=1}^{n} \mathrm{d}\sigma(x_i) = (2^{\gamma-1/2}\Gamma(\gamma+1/2))^{-n} \prod_{i=1}^{n} (x_i^{2\gamma}\mathrm{d}x_i), \tag{1.10}$$

$$\prod_{i=1}^{n} j_{\gamma-1/2}(x_i) = (2^{\gamma-1/2}\Gamma(\gamma+1/2))^{n} \prod_{i=1}^{n} (x_i^{1/2-\gamma} J_{\gamma-1/2}(x_i)). \tag{1.11}$$

Let $f \in L^1(I^n)$. Then the n-dimensional Hankel translate $\tau_x f$ of f by $x = (x_1, \ldots, x_n)$ is defined by

$$(\tau_x f)(y) = \int_0^\infty \cdots \int_0^\infty f(z_1, \ldots, z_n) \prod_{i=1}^{n} (D_{\gamma-1/2}(x_i, y_i, z_i)\mathrm{d}\sigma(z_i)), \quad x, y \in I^n. \tag{1.12}$$

Let f and g be functions in $L^1(I^n)$ and let their n-dimensional Hankel convolution $f\#g$ be defined by

$$(f\#g)(x) = \int_0^\infty \cdots \int_0^\infty (\tau_x f)(y_1, \ldots, y_n) g(y_1, \ldots, y_n)\mathrm{d}\sigma(y_1)\ldots\mathrm{d}\sigma(y_n), \quad x \in I^n. \tag{1.13}$$

The integral defining $(f\#g)(x)$ converges for almost all $x \in I^n$ and

$$\|f\#g\|_{L^1} \le \|f\|_{L^1} \; \|g\|_{L^1}. \tag{1.14}$$

The above result is a generalization of the corresponding one-dimensional case investigated by Hirschman in pp. 309–311 of [1]. Schwartz's theory of the Fourier transform of distributions in $\mathcal{S}'(\mathbb{R}^n)$ has been exploited by many authors in the study of pseudo-differential operators, see for instance, Zaidman [6, 5]. In this paper we have used the Zemanian's theory of the Hankel transform of distributions in $H'_\mu(I^n)$ to develop a theory of n-dimensional pseudo-differential operators corresponding to [3]. Unless otherwise stated, we shall always assume $\mu \ge -1/2$.

2. The pseudo differential operator $h_{\mu,a}$

DEFINITION 2.1

Let $a(x_1, \ldots, x_n; y_1, \ldots, y_n)$ be a complex-valued function belonging to the space $C^\infty(I^n \times I^n)$, where $I = (0, \infty)$ and let its derivatives satisfy certain growth conditions such as (2.1) below. Then the pseudo-differential operator $h_{\mu,a}$ associated with the symbol $a(x_1, \ldots, x_n; y_1, \ldots, y_n)$ is defined by

$$(h_{\mu,a}\phi)(y_1, \ldots, y_n) = \int_0^\infty \cdots \int_0^\infty \left(\prod_{i=1}^{n} (x_i y_i)^{1/2} J_\mu(x_i y_i)\right) \times a(x_1, \ldots, x_n; y_1, \ldots, y_n)((h_\mu \phi)(x_1, \ldots, x_n))\mathrm{d}x_1 \ldots \mathrm{d}x_n.$$

DEFINITION 2.2

Let $m \in (-\infty, \infty)$. The function $a(x_1, \ldots, x_n; y_1, \ldots, y_n) \in C^\infty(I^n \times I^n)$ is said to belong to the class H^m if and only if for all $q, \nu, \alpha \in \mathbb{N}_0^n$, there exists $D > 0$ such that

$$\prod_{i=1}^n (1 + x_i)^{q_i} \left|(x_i^{-1}\partial/\partial x_i)^{\nu_i}(y_i^{-1}\partial/\partial y_i)^{\alpha_i} a(x_1, \ldots, x_n; y_1, \ldots, y_n)\right|$$
$$\leq D\left(1 + \prod_{i=1}^n y_i\right)^{m-|\alpha|}. \tag{2.1}$$

Theorem 2.1. *Let the symbol $a(x_1, \ldots, x_n; y_1, \ldots, y_n)$ belong to H^m. Then for $\mu \geq -1/2$ the pseudo-differential operator $h_{\mu,a}$ is a continuous linear mapping of $H_\mu(I^n)$ into itself.*

Proof. Let $\Phi(y_1, \ldots, y_n) = (h_{\mu,a}\phi)(y_1, \ldots, y_n), \phi \in H_\mu(I^n)$. Then using formulas (1.4) and (1.5) and using Zemanian's technique (p. 141 of [7]), for $i = 1, 2, \ldots, n$, we have

$$\begin{aligned} N_{i\mu}\Phi(y_1, \ldots, y_n) &= y_i^{\mu+1/2}(\partial/\partial y_i)y_i^{-\mu-1/2}\Phi(y_1, \ldots, y_n) \\ &= y_i^{\mu+1+1/2}(y_i^{-1}\partial/\partial y_i)y_i^{-\mu-1/2}\Phi(y_1, \ldots, y_n); \end{aligned}$$

$$\begin{aligned} N_{i(\mu+1)}N_{i\mu}\Phi(y_1, \ldots, y_n) &= y_i^{\mu+1+1/2}(\partial/\partial y_i)y_i^{-(\mu+1)-1/2}N_{i\mu} \\ &\quad \Phi(y_1, \ldots, y_n) \\ &= y_i^{\mu+2+1/2}(y_i^{-1}\partial/\partial y_i)y_i^{-\mu-3/2} \\ &\quad \times [y_i^{\mu+3/2}(y_i^{-1}\partial/\partial y_i)y_i^{-\mu-1/2}\Phi(y_1, \ldots, y_n)] \\ &= y_i^{\mu+2+1/2}(y_i^{-1}\partial/\partial y_i)^2 y_i^{-\mu-1/2}\Phi(y_1, \ldots, y_n). \end{aligned}$$

□

Similarly

$$\begin{aligned} &N_{i(\mu+k_i-1)} \ldots N_{i\mu}\Phi(y_1, \ldots, y_n) \\ &\quad = y_i^{\mu+k_i+1/2}(y_i^{-1}\partial/\partial y_i)^{k_i} y_i^{-\mu-1/2}\Phi(y_1, \ldots, y_n). \end{aligned}$$

Now, we have

$$\begin{aligned} &(N_{1(\mu+k_1-1)} \ldots N_{1\mu}) \ldots (N_{n\mu+k_n-1} \ldots N_{n\mu})\Phi(y_1, \ldots, y_n) \\ &\quad = (y_1^{\mu+k_1+1/2} \ldots y_n^{\mu+k_n+1/2})((y_1^{-1}\partial/\partial y_1)^{k_1} \ldots (y_n^{-1}\partial/\partial y_n)^{k_n}) \\ &\qquad (y_1^{-\mu-1/2} \ldots y_n^{-\mu-1/2})\Phi(y_1, \ldots, y_n). \end{aligned}$$

Therefore, using (1.3) and (1.7) we get

$$\prod_{i=1}^{n}(N_{i(\mu+k_i-1)}\dots N_{i\mu})\Phi(y_1,\dots,y_n)=[y^{\mu+k+1/2}](y^{-1}D_y)^k[y]^{-\mu-1/2}\Phi(y)$$

$$=[y^{\mu+k+1/2}](y^{-1}D_y)^k[y]^{-\mu-1/2}\int_0^\infty\dots\int_0^\infty\prod_{i=1}^{n}(x_iy_i)^{1/2}J_\mu(x_iy_i)$$

$$\times\, a(x_1,\dots,x_n;y_1,\dots,y_n)\hat{\phi}(x_1,\dots,x_n)\mathrm{d}x_1\dots\mathrm{d}x_n$$

$$=[y^{\mu+k+1/2}](y^{-1}D_y)^k\int_0^\infty\dots\int_0^\infty\prod_{i=1}^{n}(x_i)^{1/2}(y_i)^{-\mu}J_\mu(x_iy_i)$$

$$\times\, a(x_1,\dots,x_n;y_1,\dots,y_n)\hat{\phi}(x_1,\dots,x_n)\mathrm{d}x_1\dots\mathrm{d}x_n$$

$$=[y^{\mu+k+1/2}]\int_0^\infty\dots\int_0^\infty\left(\prod_{i=1}^{n}(x_i)^{1/2}\sum_{r_i=0}^{k_i}\binom{k_i}{r_i}(y_i^{-1}\partial/\partial y_i)^{k_i-r_i}(y_i)^{-\mu}\right.$$

$$\left.\times\, J_\mu(x_iy_i)(y_i^{-1}\partial/\partial y_i)^{r_i}a(x_1,\dots,x_n;y_1,\dots,y_n)\right)$$

$$\times\,\hat{\phi}(x_1,\dots,x_n)\mathrm{d}x_1\dots\mathrm{d}x_n$$

$$=[y^{\mu+k+1/2}]\int_0^\infty\dots\int_0^\infty[x]^{1/2}\left(\prod_{i=1}^{n}\sum_{r_i=0}^{k_i}\binom{k_i}{r_i}(-x_i)^{k_i-r_i}(y_i)^{-\mu-k_i-r_i}\right.$$

$$\left.\times\, J_{\mu+k_i-r_i}(x_iy_i)(y_i^{-1}\partial/\partial y_i)^{r_i}a(x_1,\dots,x_n;y_1,\dots,y_n)\right)$$

$$\times\,\hat{\phi}(x_1,\dots,x_n)\mathrm{d}x_1\dots\mathrm{d}x_n. \tag{2.2}$$

Therefore

$$\prod_{i=1}^{n}N_{i\mu+k_i-1}\dots N_{i\mu}\Phi(y_1,\dots,y_n)=\sum_{r_1=0}^{k_1}\cdots\sum_{r_n=0}^{k_n}\binom{k_1}{r_1}\dots\binom{k_n}{r_n}$$

$$\times\int_0^\infty\dots\int_0^\infty[y^{r+1/2}][x]^{1/2}(y^{-1}D_y)^r a(x_1,\dots,x_n;y_1,\dots,y_n)$$

$$\times\,\hat{\phi}(x_1,\dots,x_n)[-x^{k-r}]\prod_{i=1}^{n}J_{\mu+k_i-r_i}(x_iy_i)\mathrm{d}x_1\dots\mathrm{d}x_n$$

$$=\sum_{r_1=0}^{k_1}\cdots\sum_{r_n=0}^{k_n}\binom{k_1}{r_1}\dots\binom{k_n}{r_n}[y^r]\int_0^\infty\dots\int_0^\infty\prod_{i=1}^{n}(x_iy_i)^{1/2}$$

$$\times\, J_{\mu+k_i-r_i}(x_iy_i)(y^{-1}D_y)^r a(x_1,\dots,x_n;y_1,\dots,y_n)[-x^{k-r}]$$

$$\times\,\hat{\phi}(x_1,\dots,x_n)\mathrm{d}x_1\dots\mathrm{d}x_n$$

$$=\sum_{r_1=0}^{k_1}\cdots\sum_{r_n=0}^{k_n}\binom{k_1}{r_1}\dots\binom{k_n}{r_n}h_{\mu+k-r}([y^r](y^{-1}D_y)^r$$

$$a(x_1,\dots,x_n;y_1,\dots,y_n)[-x^{k-r}]\hat{\phi}(x_1,\dots,x_n)(y_1,\dots,y_n).$$

From equation (2.2) for $i = 1$,

$$
\begin{aligned}
&(N_{1(\mu+k_1-1)}\dots N_{1\mu})\Phi(y_1,\dots,y_n)\\
&= y_1^{\mu+k_1+1/2}(y_1^{-1}\partial/\partial y_1)^{k_1} y_1^{-\mu-1/2}\Phi(y_1,\dots,y_n)\\
&= y_1^{\mu+k_1+1/2}(y_1^{-1}\partial/\partial y_1)^{k_1} y_1^{-\mu-1/2}\int_0^\infty (x_1y_1)^{1/2}J_\mu(x_1y_1)\\
&\quad\times a(x_1,\dots,x_n;y_1,\dots,y_n)\hat{\phi}(x_1,\dots,x_n)\mathrm{d}x_1\\
&= y_1^{\mu+k_1+1/2}(y_1^{-1}\partial/\partial y_1)^{k_1}\int_0^\infty (x_1)^{1/2}(y_1)^{-\mu}J_\mu(x_1y_1)\\
&\quad\times a(x_1,\dots,x_n;y_1,\dots,y_n)\hat{\phi}(x_1,\dots,x_n)\mathrm{d}x_1\\
&= y_1^{\mu+k_1+1/2}\int_0^\infty x_1^{1/2}\sum_{r_1=0}^{k_1}\binom{k_1}{r_1}(y_1^{-1}\partial/\partial y_1)^{k_1-r_1}(y_1)^{-\mu}J_\mu(x_1y_1)\\
&\quad\times (y_1^{-1}\partial/\partial y_1)^{r_1}a(x_1,\dots,x_n;y_1,\dots,y_n)\hat{\phi}(x_1,\dots,x_n)\mathrm{d}x_1\\
&= y_1^{\mu+k+1/2}\int_0^\infty x_1^{1/2}\sum_{r_1=0}^{k_1}\binom{k_1}{r_1}(-x_1)^{k_1-r_1}y_1^{-\mu-k_1-r_1}J_{\mu+k_1+r_1}(x_1y_1)\\
&\quad\times (y_1^{-1}\partial/\partial y_1)^{r_1}a(x_1,\dots,x_n;y_1,\dots,y_n)\hat{\phi}(x_1,\dots,x_n)\mathrm{d}x_1\\
&= \sum_{r_1=0}^{k_1}\binom{k_1}{r_1}\int_0^\infty y_1^{r_1+1/2}x_1^{1/2}(y_1^{-1}\partial/\partial y_1)^{r_1}a(x_1,\dots,x_n;y_1,\dots,y_n)\\
&\quad\times \hat{\phi}(x_1,\dots,x_n)(-x_1)^{k_1-r_1}J_{\mu+k_1-r_1}(x_1y_1)\mathrm{d}x_1\\
&= \sum_{r_1=0}^{k_1}\binom{k_1}{r_1}y_1^{r_1}\int_0^\infty (x_1y_1)^{1/2}J_{\mu+k_1-r_1}(x_1y_1)(y_1^{-1}\partial/\partial y_1)^{r_1}\\
&\quad a(x_1,\dots,x_n;y_1,\dots,y_n)(-x_1)^{k_1-r_1}\hat{\phi}(x_1,\dots,x_n)\,\mathrm{d}x_1\\
&= \sum_{r_1=0}^{k_1}\binom{k_1}{r_1}h_{\mu+k_1-r_1}\{(y_1^{r_1}(y_1^{-1}\partial/\partial y_1)^{r_1}a(x_1,\dots,x_n;y_1,\dots,y_n)\\
&\quad\times (-x_1)^{k_1-r_1}\hat{\phi}(x_1,\dots,x_n))(y_1,\dots,y_n)\}.
\end{aligned}
$$

Using the formula $(-y_1)h_\mu\Phi = h_{\mu+1}(N_\mu\Phi)$ in the above, we have

$$
\begin{aligned}
&(-y_1)(N_{1(\mu+k_1-1)}\dots N_{1\mu})\Phi(y_1,\dots,y_n) = \sum_{r_1=0}^{k_1}\binom{k_1}{r_1}(-y_1)h_{\mu+k_1-r_1}\\
&(y_1^{r_1}(y_1^{-1}\partial/\partial y_1)^{r_1}a(x_1,\dots,x_n;y_1,\dots,y_n)(-x_1)^{k_1-r_1}\hat{\phi}(x_1,\dots,x_n))\\
&\times(y_1,\dots,y_n) = \sum_{r_1=0}^{k_1}\binom{k_1}{r_1}h_{\mu+k_1-r_1+1}N_{\mu+k_1-r_1}(y_1^{r_1}(y_1^{-1}\partial/\partial y_1)^{r_1}\\
&a(x_1,\dots,x_n;y_1,\dots,y_n)(-x_1)^{k_1-r_1}\hat{\phi}(x_1,\dots,x_n))(y_1,\dots,y_n)
\end{aligned}
$$

$$= \sum_{r_1=0}^{k_1} \binom{k_1}{r_1} y_1^{r_1} \int_0^\infty (x_1 y_1)^{1/2} J_{\mu+k_1-r_1+1} N_{\mu+k_1-r_1}$$

$$((y_1^{-1}\partial/\partial y_1)^{r_1} a(x_1, \ldots, x_n; y_1, \ldots, y_n)(-x_1)^{k_1-r_1} \hat{\phi}(x_1, \ldots, x_n)) \, \mathrm{d}x_1$$

$$= \sum_{r_1=0}^{k_1} \binom{k_1}{r_1} \int_0^\infty y_1^{r_1} (x_1 y_1)^{1/2} J_{\mu+k_1-r_1+1}(x_1 y_1)$$

$$\times x_1^{\mu+k_1-r_1+1/2} (\partial/\partial x_1) x_1^{-\mu-k_1+r_1-1/2}$$

$$((y_1^{-1}\partial/\partial y_1)^{r_1} a(x_1, \ldots, x_n; y_1, \ldots, y_n)(-x_1)^{k_1-r_1} \hat{\phi}(x_1, \ldots, x_n)) \, \mathrm{d}x_1$$

$$= \sum_{r_1=0}^{k_1} \binom{k_1}{r_1} (-1)^{k_1-r_1} \int_0^\infty y_1^{r_1+1/2} x_1^{\mu+k_1-r_1+2} (x_1^{-1}\partial/\partial x_1)$$

$$(x_1^{-\mu-1/2} (y_1^{-1}\partial/\partial y_1)^{r_1} a(x_1, \ldots, x_n; y_1, \ldots, y_n)$$

$$\times (-x_1)^{k_1-r_1} \hat{\phi}(x_1, \ldots, x_n) J_{\mu+k_1-r_1+1}(x_1 y_1)) \, \mathrm{d}x_1$$

$$= \sum_{r_1=0}^{k_1} \binom{k_1}{r_1} (-1)^{k_1-r_1} y_1^{r_1} \int_0^\infty (x_1 y_1)^{1/2}$$

$$\times J_{\mu+k_1-r_1+1}(x_1 y_1) x_1^{\mu+k_1-r_1+1/2} (x_1^{-1}\partial/\partial x_1)$$

$$(x_1^{-\mu-1/2} (y_1^{-1}\partial/\partial y_1)^{r_1} a(x_1, \ldots, x_n; y_1, \ldots, y_n)$$

$$\times (-x_1)^{k_1-r_1} \hat{\phi}(x_1, \ldots, x_n)) \, \mathrm{d}x_1$$

$$= \sum_{r_1=0}^{k_1} \binom{k_1}{r_1} (-1)^{k_1-r_1} y_1^{r_1} h_{\mu+k_1-r_1+1} \{x_1^{\mu+k_1-r_1+1/2} (x_1^{-1}\partial/\partial x_1)$$

$$(x_1^{-\mu-1/2} (y_1^{-1}\partial/\partial y_1)^{r_1} a(x_1, \ldots, x_n; y_1, \ldots, y_n)(-x_1)^{k_1-r_1} \hat{\phi}(x_1, \ldots, x_n))\}.$$

Using the formula $(-y_1) h_\mu \Phi = h_{\mu+1} N_\mu \Phi$ repeatedly, we obtain

$$(-y_1)^{t_1} (N_{1(\mu+k_1-1)} \ldots N_{1\mu}) \Phi(y_1, \ldots, y_n)$$

$$= \sum_{r_1=0}^{k_1} \binom{k_1}{r_1} (-1)^{k_1-r_1} \int_0^\infty y_1^{r_1+1/2} x_1^{\mu+k_1-r_1+t_1+1} (x_1^{-1}\partial/\partial x_1)^{t_1}$$

$$(x_1^{-\mu-1/2} (y_1^{-1}\partial/\partial y_1)^{r_1} a(x_1, \ldots, x_n; y_1, \ldots, y_n)$$

$$\times (-x_1)^{k_1-r_1} \hat{\phi}(x_1, \ldots, x_n)) J_{\mu+k_1-r_1+t_1}(x_1 y_1) \mathrm{d}x_1.$$

Similarly, for $i = 2, \ldots, n$,

$$(-y_i)^{t_i} (N_{i(\mu+k_i-1)} \ldots N_{i\mu}) \Phi(y_1, \ldots, y_n)$$

$$= \sum_{r_i=0}^{k_i} \binom{k_i}{r_i} (-1)^{k_i-r_i} \int_0^\infty y_i^{r_i+1/2} x_i^{\mu+k_i-r_i+t_i+1} (x_i^{-1}\partial/\partial x_i)^{t_i}$$

$$(x_i^{-\mu-1/2}(y_i^{-1}\partial/\partial y_i)^{r_i}a(x_1,\dots,x_n;y_1,\dots,y_n)$$
$$\times(-x_i)^{k_i-r_i}\hat{\phi}(x_1,\dots,x_n))J_{\mu+k_i-r_i+t_i}(x_iy_i)\mathrm{d}x_i.$$

Therefore

$$((-y_1)^{t_1}N_{1(\mu+k_1-1)}\dots N_{1\mu})\dots((-y_n)^{t_n}N_{n(\mu+k_n-1)}\dots N_{n\mu})\Phi(y_1,\dots,y_n)$$
$$=\sum_{r_1=0}^{k_1}\binom{k_1}{r_1}\cdots\sum_{r_n=0}^{k_n}\binom{k_n}{r_n}((-1)^{k_1-r_1}\dots(-1)^{k_n-r_n})$$
$$\times\int_0^\infty\dots\int_0^\infty(y_1^{r_1+1/2}\dots y_n^{r_n+1/2})$$
$$\times(x_1^{\mu+k_1-r_1+t_1+1}\dots x_n^{\mu+k_n-r_n+t_n+1})((x_1^{-1}\partial/\partial x_1)^{t_1}\dots(x_n^{-1}\partial/\partial x_n)^{t_n})$$
$$((x_1^{-\mu-1/2}\dots x_n^{-\mu-1/2})((y_1^{-1}\partial/\partial y_1)^{r_1}\dots(y_n^{-1}\partial/\partial y_n)^{r_n})$$
$$a(x_1,\dots,x_n;y_1,\dots,y_n)(-x_1)^{k_1-r_1}\dots(-x_n)^{k_n-r_n}\hat{\phi}(x_1,\dots,x_n))$$
$$\times(J_{\mu+k_1-r_1+t_1}(x_1y_1)\dots J_{\mu+k_n-r_n+t_n}(x_ny_n))\mathrm{d}x_1\dots\mathrm{d}x_n;$$

or

$$\prod_{i=1}^n((-y_i)^{t_i}N_{i\mu+k_i-1}\dots N_{i\mu})\Phi(y_1,\dots,y_n)$$
$$=\prod_{i=1}^n\sum_{r_i=0}^{k_i}\binom{k_i}{r_i}(-1)^{k_i-r_i}\int_0^\infty\dots\int_0^\infty[y^{r+1/2}][x^{\mu+k-r+t+1}]$$
$$\times(x^{-1}D_x)^t[x]^{-\mu-1/2}\hat{\phi}(x_1,\dots,x_n)(y^{-1}D_y)^r a(x_1,\dots,x_n;y_1,\dots,y_n)$$
$$\times\left(\prod_{i=1}^n J_{\mu+k_i-r_i+t_i}(x_iy_i)\right)\mathrm{d}x_1\dots\mathrm{d}x_n$$
$$=\prod_{i=1}^n\sum_{r_i=0}^{k_i}\binom{k_i}{r_i}(-1)^{k_i-r_i}\int_0^\infty\dots\int_0^\infty[y^{r+1/2}][x^{\mu+k-r+t+1}]$$
$$\times\left(\sum_{\nu_1=0}^{t_1}\cdots\sum_{\nu_n=0}^{t_n}\binom{t_1}{\nu_1}\dots\binom{t_n}{\nu_n}\right)(x^{-1}D_x)^\nu(y^{-1}D_y)^r$$
$$a(x_1,\dots,x_n;y_1,\dots,y_n)(x^{-1}D_x)^{t-\nu}[x]^{-\mu-1/2}\hat{\phi}(x_1,\dots,x_n)$$
$$\times\prod_{i=1}^n J_{\mu+k_i-r_i+t_i}(x_iy_i)\mathrm{d}x_1\dots\mathrm{d}x_n. \tag{2.3}$$

Now, multiplying both sides of (2.2) by $\prod_{i=1}^{n}(-y_i)^{t_i}$, we get

$$\prod_{i=1}^{n}((-y_i)^{t_i} N_{i\mu+k_i-1}\dots N_{i\mu})\Phi(y_1,\dots,y_n)$$
$$= (-1)^{|t|}[y^{\mu+k+t+1/2}](y^{-1}D_y)^k[y]^{-\mu-1/2}\Phi(y). \tag{2.4}$$

Comparing (2.3) and (2.4), we have

$$(-1)^{|t|}[y^{\mu+k+t+1/2}](y^{-1}D_y)^k[y]^{-\mu-1/2}\Phi(y)$$
$$= \prod_{i=1}^{n}\sum_{r_i=0}^{k_i}\binom{k_i}{r_i}(-1)^{k_i-r_i}\int_0^\infty\dots\int_0^\infty[y^{r+1/2}]$$
$$\times[x^{\mu+k-r+t+1}]\left(\sum_{\nu_1=0}^{t_1}\cdots\sum_{\nu_n=0}^{t_n}\binom{t_1}{\nu_1}\cdots\binom{t_n}{\nu_n}\right)$$
$$\times(x^{-1}D_x)^\nu a_r(x_1,\dots,x_n;y_1,\dots,y_n)(x^{-1}D_x)^{t-\nu}$$
$$[x]^{-\mu-1/2}\hat{\phi}(x_1,\dots,x_n)\prod_{i=1}^{n}J_{\mu+k_i-r_i+t_i}(x_iy_i)\mathrm{d}x_1\dots\mathrm{d}x_n.$$

Therefore

$$(-1)^{|t|}[y^t](y^{-1}D_y)^k[y]^{-\mu-1/2}\Phi(y)$$
$$= \prod_{i=1}^{n}\sum_{r_i=0}^{k_i}\binom{k_i}{r_i}(-1)^{k_i-r_i}\int_0^\infty\dots\int_0^\infty[y^{-(\mu+k-r)}]$$
$$\times[x^{\mu+k-r+t+1}]\left(\sum_{\nu_1=0}^{t_1}\cdots\sum_{\nu_n=0}^{t_n}\binom{t_1}{\nu_1}\cdots\binom{t_n}{\nu_n}\right)$$
$$\times(x^{-1}D_x)^\nu a_r(x_1,\dots,x_n;y_1,\dots,y_n)$$
$$\times(x^{-1}D_x)^{t-\nu}[x]^{-\mu-1/2}\hat{\phi}(x_1,\dots,x_n)\prod_{i=1}^{n}J_{\mu+k_i-r_i+t_i}(x_iy_i)\mathrm{d}x_1\dots\mathrm{d}x_n$$
$$= \prod_{i=1}^{n}\sum_{r_i=0}^{k_i}\binom{k_i}{r_i}(-1)^{k_i-r_i}\int_0^\infty\dots\int_0^\infty[x^{2\lambda+t+1}]$$
$$\times\left(\sum_{\nu_1=0}^{t_1}\cdots\sum_{\nu_n=0}^{t_n}\binom{t_1}{\nu_1}\cdots\binom{t_n}{\nu_n}\right)(x^{-1}D_x)^\nu a_r(x_1,\dots,x_n;y_1,\dots,y_n)$$
$$\times(x^{-1}D_x)^{t-\nu}[x^{-\mu-1/2}]\hat{\phi}(x_1,\dots,x_n)$$
$$\times\prod_{i=1}^{n}(x_iy_i)^{-\lambda_i}J_{\lambda_i+t_i}(x_iy_i)\mathrm{d}x_1\dots\mathrm{d}x_n$$

where $\lambda_i = \mu+k_i-r_i,\ i=1,2,\dots,n$.

Setting $t_i = p_i + s_i$, respectively $t_i = p_i$, $i = 1, 2, \dots, n$, in the above expression and using the property (2.1) with $q = (0, \dots, 0)$ and taking into account that $\mu \geq -1/2$, we arrive at the following estimate:

$$\begin{aligned}
&(1+[y^s])|[y^p](y^{-1}D_y)^k[y]^{-\mu-1/2}\Phi(y)| \\
&\quad = |[y^p](y^{-1}D_y)^k[y]^{-\mu-1/2}\Phi(y)| + |[y^{p+s}](y^{-1}D_y)^k[y]^{-\mu-1/2}\Phi(y)| \\
&\quad \leq D\sum_{r_1=0}^{k_1}\cdots\sum_{r_n=0}^{k_n}\binom{k_1}{r_1}\cdots\binom{k_n}{r_n}\int_0^\infty\cdots\int_0^\infty(1+[y])^{m-|r|} \\
&\qquad \times\Bigg([x^{2\lambda+p+1}]\sum_{\nu_1=0}^{p_1}\cdots\sum_{\nu_n=0}^{p_n}\binom{p_1}{\nu_1}\cdots\binom{p_n}{\nu_n} \\
&\qquad \times(x^{-1}D_x)^{p-\nu}[x]^{-\mu-1/2}\hat{\phi}(x_1,\dots,x_n) \\
&\qquad +[x^{2\lambda+p+s+1}]\sum_{\nu_1=0}^{p_1+s_1}\cdots\sum_{\nu_n=0}^{p_n+s_n}\binom{p_1+s_1}{\nu_1}\cdots\binom{p_n+s_n}{\nu_n} \\
&\qquad \times(x^{-1}D_x)^{p+s-\nu}[x]^{-\mu-1/2}\hat{\phi}(x_1,\dots,x_n)\Bigg)\mathrm{d}x_1\dots\mathrm{d}x_n.
\end{aligned}$$

Since $(1+[y])^{m-|r|} \leq (1+[y])^m$, we have

$$\begin{aligned}
&(1+[y^s])|[y^p](y^{-1}D_y)^k[y]^{-\mu-1/2}\Phi(y)| \\
&\quad \leq D\sum_{r_1=0}^{k_1}\cdots\sum_{r_n=0}^{k_n}\binom{k_1}{r_1}\cdots\binom{k_n}{r_n}\int_0^\infty\cdots\int_0^\infty(1+[y])^{m} \\
&\qquad \times\Bigg([x^{2\lambda+p+s+1}]\sum_{\nu_1=0}^{p_1+s_1}\cdots\sum_{\nu_n=0}^{p_n+s_n}\binom{p_1+s_1}{\nu_1}\cdots\binom{p_n+s_n}{\nu_n} \\
&\qquad \times(x^{-1}D_x)^{p+s-\nu}[x]^{-\mu-1/2}\hat{\phi}(x_1,\dots,x_n)+[x^{2\lambda+p+1}]\sum_{\nu_1=0}^{p_1}\cdots\sum_{\nu_n=0}^{p_n}\binom{p_1}{\nu_1} \\
&\qquad \times\dots\binom{p_n}{\nu_n}(x^{-1}D_x)^{p-\nu}[x]^{-\mu-1/2}\hat{\phi}(x_1,\dots,x_n)\Bigg)\mathrm{d}x_1\dots\mathrm{d}x_n.
\end{aligned}$$

Hence

$$\begin{aligned}
&|[y^p](y^{-1}D_y)^k[y]^{-\mu-1/2}\Phi(y)| \\
&\quad \leq D\sum_{r_1=0}^{k_1}\cdots\sum_{r_n=0}^{k_n}\binom{k_1}{r_1}\cdots\binom{k_n}{r_n}\int_0^\infty\cdots\int_0^\infty\frac{(1+[y])^{m}}{(1+[y^s])} \\
&\qquad \times\Bigg([x^{2\lambda+p+s+1}]\sum_{\nu_1=0}^{p_1+s_1}\cdots\sum_{\nu_n=0}^{p_n+s_n}\binom{p_1+s_1}{\nu_1}\cdots\binom{p_n+s_n}{\nu_n} \\
&\qquad \times(x^{-1}D_x)^{p+s-\nu}[x]^{-\mu-1/2}\hat{\phi}(x_1,\dots,x_n)+[x^{2\lambda+p+1}]\sum_{\nu_1=0}^{p_1}\cdots\sum_{\nu_n=0}^{p_n}\binom{p_1}{\nu_1} \\
&\qquad \times\dots\binom{p_n}{\nu_n}(x^{-1}D_x)^{p-\nu}[x]^{-\mu-1/2}\hat{\phi}(x_1,\dots,x_n)\Bigg)\mathrm{d}x_1\dots\mathrm{d}x_n. \qquad (2.5)
\end{aligned}$$

We note that if $m \leq 0$, we have $(1+[y])^m \leq 1$, and then

$$\frac{(1+[y])^m}{(1+[y^s])} \leq 1, \tag{2.6}$$

while if $m > 0$, then

$$\frac{(1+[y])^m}{(1+[y^s])} \leq 2^m \frac{(1+[y]^m)}{(1+[y^s])}. \tag{2.7}$$

Since s is an arbitrary n-tuple of nonnegative integers we can choose $s = (s_1, \ldots, s_n)$ such that

$$\frac{(1+[y]^m)}{(1+[y^s])} \leq 1. \tag{2.8}$$

From equations (2.7) and (2.8), we have

$$\frac{(1+[y])^m}{(1+[y^s])} \leq 2^m, \tag{2.9}$$

so that from (2.6) and (2.9), we get

$$\frac{(1+[y])^m}{(1+[y^s])} \leq \max(1, 2^m) = D_m. \tag{2.10}$$

Using (2.10) in (2.5), we have

$$\begin{aligned}
&|[y^p](y^{-1}D_y)^k[y]^{-\mu-1/2}\Phi(y)| \\
&\leq D_m D \sum_{r_1=0}^{k_1} \cdots \sum_{r_n=0}^{k_n} \binom{k_1}{r_1} \cdots \binom{k_n}{r_n} \\
&\quad \int_0^\infty \cdots \int_0^\infty \Bigg([x^{2\lambda+p+s+1}] \sum_{\nu_1=0}^{p_1+s_1} \cdots \sum_{\nu_n=0}^{p_n+s_n} \binom{p_1+s_1}{\nu_1} \cdots \binom{p_n+s_n}{\nu_n} \\
&\quad \times (x^{-1}D_x)^{p+s-\nu}[x]^{-\mu-1/2}\hat{\phi}(x_1, \ldots, x_n) \\
&\quad + [x^{2\lambda+p+1}] \sum_{\nu_1=0}^{p_1} \cdots \sum_{\nu_n=0}^{p_n} \binom{p_1}{\nu_1} \cdots \binom{p_n}{\nu_n} \\
&\quad \times (x^{-1}D_x)^{p-\nu}[x]^{-\mu-1/2}\hat{\phi}(x_1, \ldots, x_n) \Bigg) \mathrm{d}x_1 \ldots \mathrm{d}x_n.
\end{aligned}$$

Now, let N_i be a nonnegative integer such that $N_i > 2(\mu + k_i) + p_i + s_i + 3,\ i = 1, 2, \ldots, n$. Then

$$
\begin{aligned}
&|[y^p](y^{-1}D_y)^k[y]^{-\mu-1/2}\Phi(y)| \\
&\le D' \sum_{r_1=0}^{k_1}\cdots\sum_{r_n=0}^{k_n}\binom{k_1}{r_1}\cdots\binom{k_n}{r_n}\int_0^\infty\cdots\int_0^\infty \prod_{i=1}^{n}(1+x_i)^{N_i-2} \\
&\quad\times\Bigg(\sum_{\nu_1=0}^{p_1+s_1}\cdots\sum_{\nu_n=0}^{p_n+s_n}\binom{p_1+s_1}{\nu_1}\cdots\binom{p_n+s_n}{\nu_n} \\
&\quad\times|(x^{-1}D_x)^{p+s-\nu}[x]^{-\mu-1/2}\hat{\phi}(x_1,\ldots,x_n)| \\
&\quad+\sum_{\nu_1=0}^{p_1}\cdots\sum_{\nu_n=0}^{p_n}\binom{p_1}{\nu_1}\cdots\binom{p_n}{\nu_n} \\
&\quad\times|(x^{-1}D_x)^{p-\nu}[x]^{-\mu-1/2}\hat{\phi}(x_1,\ldots,x_n)|\Bigg)\mathrm{d}x_1\ldots\mathrm{d}x_n \\
&\le D'' \sum_{r_1=0}^{k_1}\cdots\sum_{r_n=0}^{k_n}\binom{k_1}{r_1}\cdots\binom{k_n}{r_n} \\
&\quad\times\Bigg(\sum_{\nu_1=0}^{p_1+s_1}\cdots\sum_{\nu_n=0}^{p_n+s_n}\binom{p_1+s_1}{\nu_1}\cdots\binom{p_n+s_n}{\nu_n}\sum_{j_1=0}^{N_1}\cdots\sum_{j_n=0}^{N_n}\binom{N_1}{j_1}\cdots\binom{N_n}{j_n} \\
&\quad\times\gamma^{\mu}_{j,p+s-\nu}\,\hat{\phi}+\sum_{\nu_1=0}^{p_1}\cdots\sum_{\nu_n=0}^{p_n}\binom{p_1}{\nu_1}\cdots\binom{p_n}{\nu_n} \\
&\quad\times\sum_{j_1=0}^{N_1}\cdots\sum_{j_n=0}^{N_n}\binom{N_1}{j_1}\cdots\binom{N_n}{j_n}\gamma^{\mu}_{j,p-\nu}\,\hat{\phi}\Bigg).
\end{aligned}
$$

Therefore in view of eq. (1.1),

$$
\begin{aligned}
\gamma^{\mu}_{p,k}\,(\Phi) &\le D'' \sum_{r_1=0}^{k_1}\cdots\sum_{r_n=0}^{k_n}\binom{k_1}{r_1}\cdots\binom{k_n}{r_n}\sum_{j_1=0}^{N_1}\cdots\sum_{j_n=0}^{N_n}\binom{N_1}{j_1}\cdots\binom{N_n}{j_n} \\
&\times\Bigg(\sum_{\nu_1=0}^{p_1+s_1}\cdots\sum_{\nu_n=0}^{p_n+s_n}\binom{p_1+s_1}{\nu_1}\cdots\binom{p_n+s_n}{\nu_n}\gamma^{\mu}_{j,p+s-\nu}\,\hat{\phi} \\
&+\sum_{\nu_1=0}^{p_1}\cdots\sum_{\nu_n=0}^{p_n}\binom{p_1}{\nu_1}\cdots\binom{p_n}{\nu_n}\gamma^{\mu}_{j,p-\nu}\,\hat{\phi}\Bigg),
\end{aligned}
$$

where D'' is a positive constant. From above, the continuity of $h_{\mu,a}$ follows.

3. An integral representation

The function $a_\eta(y_1, \ldots, y_n)$, where $\eta = (\eta_1, \ldots, \eta_n)$, associated with the symbol $a(x_1, \ldots, x_n; y_1, \ldots, y_n)$ and defined by

$$a_\eta(y_1, \ldots, y_n) = \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^n (x_i y_i)^{1/2} J_\mu(x_i y_i) \times \{(x_i \eta_i)^{1/2} J_\mu(x_i \eta_i) a(x_1, \ldots, x_n; \eta_1, \ldots, \eta_n)\} \mathrm{d}x_1 \ldots \mathrm{d}x_n, \tag{3.1}$$

will play a fundamental role in our investigation. An estimate for $a_\eta(y_1, \ldots, y_n)$ is given by

Lemma 3.1. *Let the symbol $a(x_1, \ldots, x_n; y_1, \ldots, y_n)$ belong to H^m. Then the function $a_\eta(y_1, \ldots, y_n)$ defined by* (3.1) *satisfies the inequality*

$$|a_\eta(y_1, \ldots, y_n)| \le A(1 + [\eta])^{\mu+m+4r+1/2}(1 + [y])^{\mu+1/2}(1 + [y^{2r}])^{-1},$$

where A is a positive constant, $\eta = (\eta_1, \ldots, \eta_n)$ and $r \in \mathbb{N}_0^n$ with $r > (0, \ldots, 0)$.

Proof. For $r \in \mathbb{N}_0^n$, using formulas (1.6) and (1.8), we have

$$\left(\prod_{i=1}^n (-y_i^2)^{r_i}\right) a_\eta(y_1, \ldots, y_n) = \int_0^\infty \cdots \int_0^\infty \left(\prod_{i=1}^n (x_i y_i)^{1/2} J_\mu(x_i y_i) \times S_\mu^{r_i}\{(x_i \eta_i)^{1/2} J_\mu(x_i \eta_i) a(x_1, \ldots, x_n; \eta_1, \ldots, \eta_n)\}\right) \mathrm{d}x_1 \ldots \mathrm{d}x_n$$

$$= \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^n \{(x_i y_i)^{1/2} J_\mu(x_i y_i)\} \times \left(\sum_{j_i=0}^{r_i} b_{j_i}(x_i)^{2j_i+\mu+1/2}(x_i^{-1}\partial/\partial x_i)^{r_i+j_i} \times ((x_i)^{-\mu-1/2}\{(x_i \eta_i)^{1/2} J_\mu(x_i \eta_i)\} a(x_1, \ldots, x_n; \eta_1, \ldots, \eta_n))\right) \mathrm{d}x_1 \ldots \mathrm{d}x_n.$$

Using (1.7), we get

$$\left(\prod_{i=1}^n (-y_i^2)^{r_i}\right) a_\eta(y_1, \ldots, y_n) = \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^n \{(x_i y_i)^{1/2} J_\mu(x_i y_i)\} \times \left(\sum_{j_i=0}^{r_i} b_{j_i}(x_i)^{2j_i+\mu+1/2} \sum_{e_i=0}^{r_i+j_i} \binom{r_i + j_i}{e_i} (x_i^{-1}\partial/\partial x_i)^{r_i+j_i-e_i} \{(x_i)^{-\mu-1/2}(x_i \eta_i)^{1/2} J_\mu(x_i \eta_i)\} \times (x_i^{-1}\partial/\partial x_i)^{e_i} a(x_1, \ldots, x_n, \eta_1, \ldots, \eta_n)\right) \mathrm{d}x_1 \ldots \mathrm{d}x_n$$

to which an application of the formula

$$(x_i^{-1}\partial/\partial x_i)^{q_i} x_i^{-\mu} J_\mu(x_i\eta_i) = (-\eta_i)^{q_i} x_i^{-\mu-q_i} J_{\mu+q_i}(x_i\eta_i), \qquad i = 1, 2, \ldots, n,$$

yields

$$\begin{aligned}
&\left|\left(\prod_{i=1}^{n}(y_i^2)^{r_i}\right) a_\eta(y_1,\ldots,y_n)\right| \le \int_0^\infty \cdots \int_0^\infty \left|\prod_{i=1}^{n}\{(x_i y_i)^{1/2} J_\mu(x_i y_i)\}\right.\\
&\quad \times \left(\sum_{j_i=0}^{r_i} b_{j_i}(x_i)^{2j_i+\mu+1/2} \sum_{e_i=0}^{r_i+j_i} \binom{r_i+j_i}{e_i} (x_i^{-1}\partial/\partial x_i)^{e_i}\right.\\
&\qquad \left. a(x_1,\ldots,x_n;\eta_1,\ldots,\eta_n)\right) [\eta^{\mu+2r+2j-2i+1/2}]\\
&\quad \times \left.\prod_{i=1}^{n}(x_i\eta_i)^{-\mu-r_i-j_i+e_i} J_{\mu+r_i+j_i-e_i}(x_i\eta_i)\right| \mathrm{d}x_1\ldots\mathrm{d}x_n\\
&\le [y]^{\mu+1/2}\int_0^\infty \cdots \int_0^\infty [x]^{\mu+1/2}\left|\prod_{i=1}^{n}(x_i y_i)^{-\mu} J_\mu(x_i y_i)\right|\\
&\quad \times \left|\prod_{i=1}^{n}\sum_{j_i=0}^{r_i} b_{j_i}(x_i)^{2j_i+\mu+1/2}\sum_{e_i=0}^{r_i+j_i}\binom{r_i+j_i}{e_i}(x_i^{-1}\partial/\partial x_i)^{e_i}\right.\\
&\qquad \left. a(x_1,\ldots,x_n;\eta_1,\ldots,\eta_n)[\eta^{\mu+2r+2j-2i+1/2}]\right|\\
&\quad \times \left|\prod_{i=1}^{n}(x_i\eta_i)^{-\mu-r_i-j_i+e_i} J_{\mu+r_i+j_i-e_i}(x_i\eta_i)\right| \mathrm{d}x_1\ldots\mathrm{d}x_n\\
&\le B[y]^{\mu+1/2}\prod_{i=1}^{n}\left(\sum_{j_i=0}^{r_i}\sum_{e_i=0}^{r_i+j_i}\binom{r_i+j_i}{e_i}|b_{j_i}|\right)[\eta^{\mu+2r+2j-2i+1/2}]D'(1+[\eta])^m\\
&\quad \times \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{n}\left((1+x_i)^{-q_i} x_i^{2j+2\mu+1}\right)\mathrm{d}x_1\ldots\mathrm{d}x_n\\
&\le B[y]^{\mu+1/2}\prod_{i=1}^{n}\left(\sum_{j_i=0}^{r_i}\sum_{e_i=0}^{r_i+j_i}\binom{r_i+j_i}{e_i}|b_{j_i}|\right)[\eta^{\mu+2r+2j-2i+1/2}]\\
&\quad \times D'(1+[\eta])^m\,(B(2\mu+2j_i+2, q_i-2\mu-2j_i-2))\\
&\le B[y]^{\mu+1/2}\prod_{i=1}^{n}\left(\sum_{j_i=0}^{r_i}\sum_{e_i=0}^{r_i+j_i}\binom{r_i+j_i}{e_i}|b_{j_i}|\right)[\eta^{\mu+2r+2j-2i+1/2}]\\
&\quad \times D'(1+[\eta])^m\left(\frac{\Gamma(2\mu+2j_i+2)\Gamma(q_i-2\mu-2j_i-2)}{\Gamma(q_i)}\right).
\end{aligned}$$

Therefore

$$|a_\eta(y_1, \dots, y_n)| \leq A(1 + [y])^{\mu+1/2}(1 + [y^{2r}])^{-1}(1 + [\eta])^{\mu+m+4r+1/2},$$
$$\forall\, r > (0, \dots, 0) \in \mathbb{N}_0^n.$$

An integral representation for the pseudo-differential operator $h_{\mu,a}$ is now obtained. □

Theorem 3.1. *For any symbol $a(x_1, \dots, x_n; y_1, \dots, y_n) \in H^m$ the associated operator $h_{\mu,a}$ can be represented by*

$$\begin{aligned}
&(h_{\mu,a}\phi)(x_1, \dots, x_n)\\
&= \int_0^\infty \cdots \int_0^\infty \left(\prod_{i=1}^n (x_i y_i)^{1/2} J_\mu(x_i y_i)\right)\\
&\quad \times \left(\int_0^\infty \cdots \int_0^\infty a_\eta(y_1, \dots, y_n)\hat{\phi}(\eta_1, \dots, \eta_n)\mathrm{d}\eta_1 \dots \mathrm{d}\eta_n\right) \mathrm{d}y_1 \dots \mathrm{d}y_n,\\
&\qquad \phi \in H_\mu(I^n) \qquad (3.2)
\end{aligned}$$

where $\hat{\phi}(\eta_1, \dots, \eta_n) = (h_\mu \phi)(\eta_1, \dots, \eta_n)$ and all integrals are convergent.

Proof. Since

$$\begin{aligned}
a_\eta(y_1, \dots, y_n) &= \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^n ((x_i y_i)^{1/2} J_\mu(x_i y_i))\\
&\times \left(\prod_{i=1}^n (x_i \eta_i)^{1/2} J_\mu(x_i \eta_i) a(x_1, \dots, x_n; \eta_1, \dots, \eta_n)\right) \mathrm{d}x_1 \dots \mathrm{d}x_n,
\end{aligned}$$

by inversion, we have

$$\begin{aligned}
&\int_0^\infty \cdots \int_0^\infty a_\eta(y_1, \dots, y_n) \prod_{i=1}^n (x_i y_i)^{1/2} J_\mu(x_i y_i)\mathrm{d}y_1 \dots \mathrm{d}y_n\\
&= \prod_{i=1}^n (x_i \eta_i)^{1/2} J_\mu(x_i \eta_i) a(x_1, \dots, x_n; \eta_1, \dots, \eta_n).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&(h_{\mu,a}\phi)(x_1, \dots, x_n)\\
&= \int_0^\infty \cdots \int_0^\infty \left(\prod_{i=1}^n (x_i \eta_i)^{1/2} J_\mu(x_i \eta_i)\right)\\
&\quad \times a(x_1, \dots, x_n; \eta_1, \dots, \eta_n)\hat{\phi}(\eta_1, \dots, \eta_n)\mathrm{d}\eta_1 \dots \mathrm{d}\eta_n\\
&= \int_0^\infty \cdots \int_0^\infty \hat{\phi}(\eta_1, \dots, \eta_n)\mathrm{d}\eta_1 \dots \mathrm{d}\eta_n \int_0^\infty \cdots \int_0^\infty a_\eta(y_1, \dots, y_n)\\
&\quad \times \left(\prod_{i=1}^n (x_i y_i)^{1/2} J_\mu(x_i y_i)\right) \mathrm{d}y_1 \dots \mathrm{d}y_n\\
&= \int_0^\infty \cdots \int_0^\infty \left(\prod_{i=1}^n (x_i y_i)^{1/2} J_\mu(x_i y_i)\right) \mathrm{d}y_1 \dots \mathrm{d}y_n \int_0^\infty \cdots \int_0^\infty \hat{\phi}(\eta_1, \dots, \eta_n)\\
&\quad \times a_\eta(y_1, \dots, y_n)\mathrm{d}\eta_1 \dots \mathrm{d}\eta_n.
\end{aligned}$$

Now, using the estimate for $a_\eta(y_1, \ldots, y_n)$ given in Lemma 3.1, the above change in the order of integration can be justified and the existence of the last integral can be proved. Since $\hat{\phi}(\eta_1, \ldots, \eta_n) \in H_\mu(I^n)$, we have

$$\left|\hat{\phi}(\eta_1, \ldots, \eta_n)\right| \leq C[\eta]^{\mu+1/2}(1+[\eta])^{-l}, \qquad \forall\, l > 0.$$

Hence

$$\begin{aligned}
&|(h_{\mu,a}\phi)(x_1, \ldots, x_n)| \\
&\quad \leq \int_0^\infty \cdots \int_0^\infty \Bigg(\int_0^\infty \cdots \int_0^\infty [xy]^{\mu+1/2} \left| \prod_{i=1}^n (x_i y_i)^{-\mu} J_\mu(x_i y_i) \right| \\
&\qquad \times A\, C(1+[y])^{\mu+1/2}(1+[y^{2r}])^{-1}(1+[\eta])^{\mu+m+4r+1/2}[\eta]^{\mu+1/2} \\
&\qquad \times (1+[\eta])^{-l} \mathrm{d}\eta_1 \ldots \mathrm{d}\eta_n \Bigg) \mathrm{d}y_1 \ldots \mathrm{d}y_n \\
&\quad \leq L[x]^{\mu+1/2} \int_0^\infty \cdots \int_0^\infty (1+[y])^{2\mu+1}(1+[y^{2r}])^{-1} \mathrm{d}y_1 \ldots \mathrm{d}y_n \\
&\qquad \times \int_0^\infty \cdots \int_0^\infty (1+[\eta])^{2\mu+m+4r+1-l} \mathrm{d}\eta_1 \ldots \mathrm{d}\eta_n. \qquad (3.3)
\end{aligned}$$

The above integrals are convergent since $\mu \geq -1/2$, and $l(> r)$ can be chosen sufficiently large. Indeed, one can show that

$$\begin{aligned}
&\int_0^\infty \cdots \int_0^\infty \frac{(1+y_1 \ldots y_n)^{2\mu+1}}{1+y_1^{2r_1} \ldots y_n^{2r_n}} \mathrm{d}y_1 \ldots \mathrm{d}y_n \\
&\quad \leq \int_0^1 \cdots \int_0^1 \frac{(1+1)^{2\mu+1}}{1} \mathrm{d}y_1 \ldots \mathrm{d}y_n \\
&\qquad + \int_1^\infty \cdots \int_1^\infty \frac{(2y_1 \ldots y_n)^{2\mu+1}}{y_1^{2r_1} \ldots y_n^{2r_n}} \mathrm{d}y_1 \ldots \mathrm{d}y_n \\
&\quad = 2^{2\mu+1} + 2^{2\mu+1} \int_1^\infty \cdots \int_1^\infty y_1^{2\mu+1-2r_1} \ldots y_n^{2\mu+1-2r_n} \mathrm{d}y_1 \ldots \mathrm{d}y_n < \infty, \\
&\qquad \forall\, r_i > \mu+1,\ i = 1, 2, \ldots, n.
\end{aligned}$$

Similarly, the second integral in (3.3) can also be shown to converge. □

4. An L^1-norm inequality

In the proof of Theorem 4.1, we shall need the following estimate for the Hankel transform of $[x]^{\mu+1/2}a(x_1, \ldots, x_n; \eta_1, \ldots, \eta_n)$. We write

$$A_\eta(y_1, \ldots, y_n) = h_\mu\left\{[x]^{\mu+1/2}a(x_1, \ldots, x_n; \eta_1, \ldots, \eta_n)\right\}(y_1, \ldots, y_n).$$

Lemma 4.1. *For* $\mu \geq -1/2$ *and* $r \in \mathbb{N}_0^n$, $r > (0, \ldots, 0)$, *there exists a constant* $C > 0$ *such that*

$$\left|A_\eta(y_1, \ldots, y_n)\right| \leq C(1+[\eta])^m[y]^{\mu+1/2}(1+[y^{2r}])^{-1}. \qquad (4.1)$$

Proof. As in the proof of Lemma 3.1, we have

$$\left(\prod_{i=1}^{n}(-y_i^2)^{r_i}\right) A_\eta(y_1,\ldots,y_n) = \int_0^\infty \cdots \int_0^\infty \left(\prod_{i=1}^{n}(x_i y_i)^{1/2} J_\mu(x_i y_i)\right.$$

$$\left. \times\ S_\mu^{r_i}([x]^{\mu+1/2} a(x_1,\ldots,x_n;\eta_1,\ldots,\eta_n))\right) \mathrm{d}x_1 \ldots \mathrm{d}x_n$$

$$= \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{n}\{(x_i y_i)^{1/2} J_\mu(x_i y_i)\}$$

$$\times \left(\sum_{j_i=0}^{r_i} b_{j_i}(x_i)^{2j_i+\mu+1/2}(x_i^{-1}\partial/\partial x_i)^{r_i+j_i}\right.$$

$$\left. a(x_1,\ldots,x_n;\eta_1,\ldots,\eta_n)\right) \mathrm{d}x_1 \ldots \mathrm{d}x_n;$$

so that

$$\left|\left(\prod_{i=1}^{n}(y_i^2)^{r_i}\right) A_\eta(y_1,\ldots,y_n)\right| \leq \int_0^\infty \cdots \int_0^\infty \left|\prod_{i=1}^{n}\{(x_i y_i)^{1/2} J_\mu(x_i y_i)\}\right|$$

$$\times \left(\prod_{i=1}^{n}\sum_{j_i=0}^{r_i} |b_{j_i}|(x_i)^{2j_i+\mu+1/2}\right)$$

$$\times \left|(x_i^{-1}\partial/\partial x_i)^{r_i+j_i} a(x_1,\ldots,x_n;\eta_1,\ldots,\eta_n)\right| \mathrm{d}x_1 \ldots \mathrm{d}x_n$$

$$\leq \int_0^\infty \cdots \int_0^\infty \left|\prod_{i=1}^{n}\{(x_i y_i)^{1/2} J_\mu(x_i y_i)\}\right| \left(\prod_{i=1}^{n}\sum_{j_i=0}^{r_i} |b_{j_i}|\right)$$

$$\times [x^{2j+\mu+1/2}] D(1+[\eta])^m \prod_{i=1}^{n}(1+x_i)^{-q_i}\ \mathrm{d}x_1 \ldots \mathrm{d}x_n$$

$$\leq \int_0^\infty \cdots \int_0^\infty [y]^{\mu+1/2}[x]^{\mu+1/2} \left|\prod_{i=1}^{n}\{(x_i y_i)^{-\mu} J_\mu(x_i y_i)\}\right|$$

$$\times \left(\prod_{i=1}^{n}\sum_{j_i=0}^{r_i} |b_{j_i}|\right) [x^{2j+\mu+1/2}] D(1+[\eta])^m$$

$$\times \prod_{i=1}^{n}(1+x_i)^{-q_i}\ \mathrm{d}x_1 \ldots \mathrm{d}x_n$$

$$\leq \prod_{i=1}^{n}\sum_{j_i=0}^{r_i} [y]^{\mu+1/2} B(1+[\eta])^m$$

$$\times \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{n}\left((1+x_i)^{-q_i} x_i^{2j+2\mu+1}\right) \mathrm{d}x_1 \ldots \mathrm{d}x_n$$

$$\leq \prod_{i=1}^{n}\sum_{j_i=0}^{r_i} [y]^{\mu+1/2} B(1+[\eta])^m\ (B(2\mu+2j_i+2, q_i-2\mu-2j_i-2))$$

$$\leq \prod_{i=1}^{n}\sum_{j_i=0}^{r_i} [y]^{\mu+1/2} B(1+[\eta])^m \left(\frac{\Gamma(2\mu+2j_i+2)\Gamma(q_i-2\mu-2j_i-2)}{\Gamma(q_i)}\right).$$

Therefore,

$$\left|A_\eta(y_1, \ldots, y_n)\right| \leq C(1+[\eta])^m [y]^{\mu+1/2}(1+[y^{2r}])^{-1},$$

where C is a positive constant. □

We shall use the above inequality in obtaining a Sobolev norm inequality for a subspace of $H_\mu(I^n)$.

DEFINITION 4.1 (Sobolev type space)

The space $G_\mu^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, is defined to be the set of all those elements $\phi \in H_\mu(I^n)$ which satisfy

$$\|\phi\|_{G_\mu^s} = \left\|\eta^{s-\mu-1/2} h_\mu \phi\right\|_{L^1} < \infty. \tag{4.2}$$

Theorem 4.1. *Let $\mu > -1/2$. Then for all $v \in \mathbb{N}_0^n$ $\exists\, C > 0$ such that*

$$\left\|h_{\mu,a}\phi\right\|_{G_\mu^0} \leq \prod_{i=1}^{n}\sum_{l_i=0}^{v_i}\binom{v_i}{l_i}\|\phi\|_{G_\mu^{l_i}}, \quad \phi \in H_\mu(I^n). \tag{4.3}$$

Proof. Taking the Hankel transform with respect to x of (3.2), we get

$$\int_0^\infty \cdots \int_0^\infty \left(\prod_{i=1}^{n}(x_i y_i)^{1/2} J_\mu(x_i y_i)\right)(h_{\mu,a}\phi)(x_1, \ldots, x_n)\mathrm{d}x_1 \ldots \mathrm{d}x_n$$
$$= \int_0^\infty \cdots \int_0^\infty a_\eta(y_1, \ldots, y_n)\hat{\phi}(\eta_1, \ldots, \eta_n)\mathrm{d}\eta_1 \ldots \mathrm{d}\eta_n.$$

Now, multiplying both sides by $[y]^{-\mu-1/2}$ and using (1.11), we get

$$\begin{aligned}
&[y]^{-\mu-1/2} h_\mu(h_{\mu,a}\phi)(y_1, \ldots, y_n) = \int_0^\infty \cdots \int_0^\infty \hat{\phi}(\eta_1, \ldots, \eta_n)[y]^{-\mu-1/2} \\
&\quad \times \Bigg(\int_0^\infty \cdots \int_0^\infty \left\{\prod_{i=1}^{n}(x_i y_i)^{1/2} J_\mu(x_i y_i)\right\}\left\{\prod_{\nu=1}^{n}(x_\nu y_\nu)^{1/2} J_\mu(x_\nu y_\nu)\right\} \\
&\quad \times a(x_1, \ldots, x_n; \eta_1, \ldots, \eta_n)\mathrm{d}x_1 \ldots \mathrm{d}x_n\Bigg)\mathrm{d}\eta_1 \ldots \mathrm{d}\eta_n \\
&= \frac{1}{(2^\mu \Gamma(\mu+1))^{2n}} \int_0^\infty \cdots \int_0^\infty \hat{\phi}(\eta_1, \ldots, \eta_n)[y]^{-\mu-1/2} \\
&\quad \times \Bigg(\int_0^\infty \cdots \int_0^\infty \left\{\prod_{i=1}^{n}(x_i y_i)^{1/2}(x_i y_i)^\mu j_\mu(x_i y_i)\right\} \\
&\quad \times \left\{\prod_{\nu=1}^{n}(x_\nu y_\nu)^{1/2}(x_\nu y_\nu)^\mu j_\mu(x_\nu y_\nu)\right\} \\
&\quad \times a(x_1, \ldots, x_n; \eta_1, \ldots, \eta_n)\mathrm{d}x_1 \ldots \mathrm{d}x_n\Bigg)\mathrm{d}\eta_1 \ldots \mathrm{d}\eta_n.
\end{aligned}$$

Now, applying eqs (1.9) and (1.10), we can write the right-hand side of the above expression in the form

$$
\begin{aligned}
& R\int_0^\infty \cdots \int_0^\infty \hat{\phi}(\eta_1,\ldots,\eta_n)[y]^{-\mu-1/2}\left(\int_0^\infty \cdots \int_0^\infty [x]^{2\mu+1}[\eta]^{\mu+1}[y]^{\mu+1/2}\right.\\
&\quad\times\left(\int_0^\infty \cdots \int_0^\infty [z]^{\mu+1}\left\{\prod_{i=1}^n (x_i)^{-\mu} J_\mu(z_i x_i) D_\mu(\eta_i, y_i, z_i)\right\} \mathrm{d}z_1\ldots\mathrm{d}z_n\right)\\
&\quad\left.\times\, a(x_1,\ldots,x_n;\eta_1,\ldots,\eta_n)\mathrm{d}x_1\ldots\mathrm{d}x_n\right)\mathrm{d}\eta_1\ldots\mathrm{d}\eta_n\\
&= R\int_0^\infty \cdots \int_0^\infty [x]^{\mu+1}[\eta]^{\mu+1/2}\hat{\phi}(\eta_1,\ldots,\eta_n)\mathrm{d}\eta_1\ldots\mathrm{d}\eta_n\\
&\quad\times\left(\int_0^\infty \cdots \int_0^\infty [z]^{\mu+1/2}\left\{\prod_{i=1}^n D_\mu(\eta_i, y_i, z_i)\right\} \mathrm{d}z_1\ldots\mathrm{d}z_n\right)\\
&\quad\times\int_0^\infty \cdots \int_0^\infty a(x_1,\ldots,x_n;\eta_1,\ldots,\eta_n)[z]^{1/2}\left(\prod_{i=1}^n J_\mu(z_i x_i)\right)\mathrm{d}x_1\ldots\mathrm{d}x_n\\
&= R\int_0^\infty \cdots \int_0^\infty [x]^{\mu+1/2}[\eta]^{\mu+1/2}\hat{\phi}(\eta_1,\ldots,\eta_n)\mathrm{d}\eta_1\ldots\mathrm{d}\eta_n\\
&\quad\times\left(\int_0^\infty \cdots \int_0^\infty [z]^{\mu+1/2}\left\{\prod_{i=1}^n D_\mu(\eta_i, y_i, z_i)\right\} \mathrm{d}z_1\ldots\mathrm{d}z_n\right)\\
&\quad\times\int_0^\infty \cdots \int_0^\infty a(x_1,\ldots,x_n;\eta_1,\ldots,\eta_n)\\
&\quad\times\left(\prod_{i=1}^n (z_i x_i)^{1/2} J_\mu(z_i x_i)\right)\mathrm{d}x_1\ldots\mathrm{d}x_n\\
&= R\int_0^\infty \cdots \int_0^\infty [\eta]^{\mu+1/2}\hat{\phi}(\eta_1,\ldots,\eta_n)\mathrm{d}\eta_1\ldots\mathrm{d}\eta_n\int_0^\infty \cdots \int_0^\infty [z]^{\mu+1/2}\\
&\quad\times\left(\prod_{i=1}^n D_\mu(\eta_i, y_i, z_i)\right)\mathrm{d}z_1\ldots\mathrm{d}z_n\int_0^\infty \cdots\\
&\quad\times\int_0^\infty [x]^{\mu+1/2} a(x_1,\ldots,x_n;\eta_1,\ldots,\eta_n)\\
&\quad\times\left(\prod_{i=1}^n (z_i x_i)^{1/2} J_\mu(z_i x_i)\right)\mathrm{d}x_1\ldots\mathrm{d}x_n,
\end{aligned}
$$

where $R = 1/(2^\mu\Gamma(\mu+1))^{2n}$. Therefore,

$$
\begin{aligned}
&[y]^{-\mu-1/2} h_\mu(h_{\mu,a})(y_1,\ldots,y_n)\\
&\quad\le R\int_0^\infty \cdots \int_0^\infty [\eta]^{\mu+1/2}\hat{\phi}(\eta_1,\ldots,\eta_n)\mathrm{d}\eta_1\ldots\mathrm{d}\eta_n\int_0^\infty \cdots \int_0^\infty [z]^{\mu+1/2}\\
&\quad\times\left(\prod_{i=1}^n D_\mu(\eta_i, y_i, z_i)\right)\mathrm{d}z_1\ldots\mathrm{d}z_n\\
&\quad\times\int_0^\infty \cdots \int_0^\infty [x]^{\mu+1/2} a(x_1,\ldots,x_n;\eta_1,\ldots,\eta_n)\\
&\quad\times\left(\prod_{i=1}^n (z_i x_i)^{1/2} J_\mu(z_i x_i)\right)\mathrm{d}x_1\ldots\mathrm{d}x_n. \qquad (4.4)
\end{aligned}
$$

By an application of the estimates (4.1) to (4.4), we have

$$\begin{aligned}
&\left|[y]^{-\mu-1/2}h_\mu(h_{\mu,a}\phi)(y_1,\dots,y_n)\right| \\
&\le CR\int_0^\infty\dots\int_0^\infty(1+[\eta])^m[\eta]^{\mu+1/2}\hat{\phi}(\eta_1,\dots,\eta_n)\mathrm{d}\eta_1\dots\mathrm{d}\eta_n \\
&\quad\times\int_0^\infty\dots\int_0^\infty[z]^{2\mu+1}(1+[z^{2r}])^{-1}\left(\prod_{i=1}^n D_\mu(\eta_i,y_i,z_i)\right)\mathrm{d}z_1\dots\mathrm{d}z_n \\
&\le D\sum_{l_1=0}^{\nu_1}\cdots\sum_{l_n=0}^{\nu_n}\binom{\nu_1}{l_1}\cdots\binom{\nu_n}{l_n} \\
&\quad\times\int_0^\infty\dots\int_0^\infty[\eta^{l+\mu+1/2}]\hat{\phi}(\eta_1,\dots,\eta_n)\mathrm{d}\eta_1\dots\mathrm{d}\eta_n \\
&\quad\times\int_0^\infty\dots\int_0^\infty[z]^{2\mu+1}(1+[z^{2r}])^{-1}\left(\prod_{i=1}^n D_\mu(\eta_i,y_i,z_i)\right)\mathrm{d}z_1\dots\mathrm{d}z_n. \qquad (4.5)
\end{aligned}$$

Now, we set

$$f(z_1,\dots,z_n)=(1+[z^{2r}])^{-1}\in L^1(I^n) \qquad \text{for } r_i>0, \quad i=1,2,\dots,n$$

and

$$g(\eta_1,\dots,\eta_n)=(2^\mu\Gamma(\mu+1))^n[\eta^{l-\mu-1/2}]\,\hat{\phi}(\eta_1,\dots,\eta_n)\in L^1(I^n), \qquad \forall\, l\in\mathbb{N}_0^n$$

such that $l_i\le\nu_i$, $i=1,2,\dots,n$.

Then, according to (1.12) and (1.13), we have

$$\begin{aligned}
(\tau_y f)(\eta_1,\dots,\eta_n) \;=\; &\int_0^\infty\dots\int_0^\infty f(z_1,\dots,z_n)\left(\prod_{i=1}^n D_\mu(\eta_i,y_i,z_i)\right) \\
&\times[z]^{2\mu+1}(2^\mu(\Gamma(\mu+1))^{-n}\mathrm{d}z_1\dots\mathrm{d}z_n
\end{aligned}$$

and

$$\begin{aligned}
(f\#g)(y) \;=\; &\int_0^\infty\dots\int_0^\infty(\tau_y f)(\eta_1,\dots,\eta_n)\,g(\eta_1,\dots,\eta_n)[\eta]^{2\mu+1} \\
&\times(2^\mu\Gamma(\mu+1))^{-n}\mathrm{d}\eta_1\dots\mathrm{d}\eta_n.
\end{aligned}$$

Therefore, applying (1.14) to (4.5), we get

$$\begin{aligned}
&\left\|[y]^{-\mu-1/2}h_\mu(h_{\mu,a}\phi)(y_1,\dots,y_n)\right\|_{L^1}\le D\sum_{l_1=0}^{\nu_1}\cdots\sum_{l_n=0}^{\nu_n}\binom{\nu_1}{l_1}\cdots\binom{\nu_n}{l_n} \\
&\quad\times\left\|[\eta^{l-\mu-1/2}]\,\hat{\phi}(\eta_1,\dots,\eta_n)\right\|_{L^1}\left\|(1+[z^{2r}])^{-1}\right\|_{L^1} \\
&\le C\sum_{l_1=0}^{\nu_1}\cdots\sum_{l_n=0}^{\nu_n}\binom{\nu_1}{l_1}\cdots\binom{\nu_n}{l_n}\left\|[\eta^{l-\mu-1/2}]\,\hat{\phi}(\eta_1,\dots,\eta_n)\right\|_{L^1}
\end{aligned}$$

from which inequality (4.3) follows. □

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