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Precise asymptotics for complete moment convergence in Hilbert spaces

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Abstract. Let $\{X, X_n; n \geq 1\}$ be a sequence of i.i.d. random variables taking values in a real separable Hilbert space $(\mathbf{H}, \|\cdot\|)$ with covariance operator Σ. Set $S_n = \sum_{i=1}^{n} X_i$, $n \geq 1$. We prove that for $1 < p < 2$ and $r > 1 + p/2$,

$$\lim_{\varepsilon \searrow 0} \varepsilon^{(2r-p-2)/(2-p)} \sum_{n=1}^{\infty} n^{r/p-2-1/p} \mathsf{E}\{\|S_n\| - \sigma\varepsilon n^{1/p}\}_+ = \sigma^{-(2r-2-p)/(2-p)} \frac{p(2-p)}{(r-p)(2r-p-2)} \mathsf{E}\|Y\|^{2(r-p)/(2-p)},$$

where Y is a Gaussian random variable taking value in a real separable Hilbert space with mean zero and covariance operator Σ, and σ^2 is the largest eigenvalue of Σ.

Keywords. Complete convergence; complete moment convergence; convergence rates; Hilbert spaces; precise asymptotics.

1. Introduction and main result

Let $\{X, X_n; n \geq 1\}$ be a sequence of independent and identically distributed (i.i.d.) real-valued random variables and $S_n = \sum_{k=1}^{n} X_k$, for $n \geq 1$. The famous complete convergence exhibits that for $0 < p < 2$ and $r \geq p$,

$$\sum_{n=1}^{\infty} n^{r/p-2} \mathsf{P}(|S_n| \geq \varepsilon n^{1/p}) < \infty, \quad \forall\, \varepsilon > 0,$$

if and only if $\mathsf{E}|X|^r < \infty$ and when $r \geq 1$, $\mathsf{E}X = 0$. For $r = 2$, $p = 1$, the sufficiency was proved by Hsu and Robbins [10], and the necessity by Erdös [6,7]. For the case $r = p = 1$, we refer to Spitzer [16], and for the general result to Baum and Katz [2].

Also Chow [4] first discussed the complete moment convergence of i.i.d. random variables, and got the following result.

Theorem A. *Suppose that* $\mathsf{E}X = 0$. *For* $1 \le p < 2$ *and* $r > p$, *if* $\mathsf{E}(|X_1|^r + |X_1| \log(1 + |X_1|)) < \infty$, *then for any* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{r/p-2-1/p} \mathsf{E}\{|S_n| - \varepsilon n^{1/p}\}_+ < \infty,$$

where $\{x\}_+ = x \vee 0$.

Observing that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r/p-2-1/p} \mathsf{E}\big\{|S_n| - \varepsilon n^{1/p}\big\}_+ \\
&\quad = \sum_{n=1}^{\infty} n^{r/p-2} \int_0^{\infty} \mathsf{P}(|S_n| \ge (\varepsilon + x) n^{1/p}) \mathrm{d}x \\
&\quad = \int_0^{\infty} \sum_{n=1}^{\infty} n^{r/p-2} \mathsf{P}(|S_n| \ge (\varepsilon + x) n^{1/p}) \mathrm{d}x < \infty,
\end{aligned}$$

we can conclude the complete convergence from the complete moment convergence immediately. Due to its interest, Theorem A was generalized to many other cases. For example, Li and Zhang [13] investigated the complete moment convergence for partial sums of moving average processes with negatively associated and mixing innovations. Chen [3], Wang and Su [14] and Wang *et al.* [15] studied the complete moment convergence in Banach spaces. Further, the exact convergence rates of the complete moment convergence have also been studied, and one can refer to Jiang and Zhang [11] and Li [12] for details.

Recently, Huang and Zhang [8] obtained the precise rates in the law of the logarithm for Hilbert-space valued random variables. Inspired by them, in this paper we aim to study the precise asymptotics for complete moment convergence in Hilbert spaces setting. In what follows, let $\{X, X_n; n \ge 1\}$ be a sequence of i.i.d. random variables which take values in a real separable Hilbert space $(\mathbf{H}, \|\cdot\|)$ with mean zero and covariance Σ unless it is specially mentioned. Denote the largest eigenvalue of Σ by σ^2, i.e. $\sigma^2 = \sup\{\mathsf{E}[(X, y)^2] : \|y\| \le 1\}$, where $(\cdot, \cdot)$ denotes the scalar product in $\mathbf{H}$. Let l be the dimension of the corresponding eigenspace, and let σ_i^2, $1 \le i \le l'$ be the positive eigenvalues of Σ arranged in a nonincreasing order and take into account the multiplicities. Further, if $l' < \infty$, put $\sigma_i^2 = 0$, $i \ge l'$. Note that we always have $\sigma_i^2 = \sigma^2$, $1 \le i \le l$ and $\sigma_i^2 < \sigma^2$, $i > l$ [5]. Now we are in a position to state our main result.

Theorem 1.1. *Suppose that* $\mathsf{E}X = 0$ *and* $\mathsf{E}\|X\|^3 < \infty$. *Then for* $1 < p < 2$ *and* $r > 1 + p/2$, *we have that* $\mathsf{E}\|X\|^r < \infty$ *implies*

$$\begin{aligned}
&\lim_{\varepsilon \searrow 0} \varepsilon^{(2r-p-2)/(2-p)} \sum_{n=1}^{\infty} n^{r/p-2-1/p} \mathsf{E}\{\|S_n\| - \sigma \varepsilon n^{1/p}\}_+ \\
&\quad = \sigma^{-(2r-2-p)/(2-p)} \frac{p(2-p)}{(r-p)(2r-p-2)} \mathsf{E}\|Y\|^{2(r-p)/(2-p)},
\end{aligned}$$

where Y *is a Gaussian random variable taking value in a real separable Hilbert space with mean zero and covariance operator* Σ.

It is noted that in Theorem 1.1, the precise rate of moment convergence is derived, which extends the result of Jiang and Zhang [11] to the Hilbert space setting. As for the moment complete convergence, it has been studied extensively by many researchers since Chow [4]. Among them, Wang *et al.* [15] obtained the moment complete convergence in Banach spaces of Type 2 (including the Hilbert spaces), and thus our result can be viewed as a generalization of their moment convergence. In fact, from Theorem 1.1, we can conclude the following complete moment convergence and complete convergence for S_n readily, which exhibit that the precise moment convergence rate implies the complete convergence and moment complete convergence.

COROLLARY 1.1

Under the assumptions of Theorem 1.1, *then for* $1 < p < 2, r > 1 + p/2$ *and any* $\varepsilon > 0$, *we have*

$$\sum_{n=1}^{\infty} n^{r/p-2-1/p}\mathsf{E}\{\|S_n\| - \sigma\varepsilon n^{1/p}\}_+ < \infty$$

and

$$\sum_{n=1}^{\infty} n^{r/p-2}\mathsf{P}\{\|S_n\| \geq \sigma\varepsilon n^{1/p}\} < \infty.$$

2. Proofs

In this section, we shall show the validity of Theorem 1.1. In the sequel, let Y be a non-degenerate Gaussian random variable with mean zero and covariance Σ. Let C denote positive constants whose values can differ in different places and $a_n \sim b_n$ means that $a_n/b_n \to 1$ as $n \to \infty$. We first proceed with an important lemma, which, in fact, can be viewed as the convergence rates of partial sums to the Gaussian random variables.

Lemma 2.1 [5]. *Let* $\xi_1, \xi_2, \ldots, \xi_n$ *be independent* $\mathbf{H}$*-valued random variables with* $\mathsf{E}\xi_i = 0$ *and* $\mathsf{E}[\|\xi_i\|^3] < \infty$, *and let* $Y_1, Y_2, \ldots, Y_n$ *be independent Gaussian mean zero random variables with* $\mathsf{Cov}(\xi_i) = \mathsf{Cov}(Y_i)$, $i = 1, 2, \ldots, n$. *Here* $\mathsf{Cov}(\xi_i)$ *denotes the covariance operator of a* $\mathbf{H}$*-valued random variable. Then we have that, for any* $s, t > 0$,

$$\mathsf{P}\left\{\left\|\sum_{i=1}^{n}\xi_i\right\| \geq s\right\} \leq \mathsf{P}\left\{\left\|\sum_{i=1}^{n}Y_i\right\| \geq s - t\right\} + Ct^{-3}\sum_{i=1}^{n}\mathsf{E}[\|\xi_i\|^3] \tag{2.1}$$

and

$$\mathsf{P}\left\{\left\|\sum_{i=1}^{n}\xi_i\right\| \geq s\right\} \geq \mathsf{P}\left\{\left\|\sum_{i=1}^{n}Y_i\right\| \geq s + t\right\} - Ct^{-3}\sum_{i=1}^{n}\mathsf{E}[\|\xi_i\|^3], \tag{2.2}$$

where C is a universal constant.

Now we start to introduce some Propositions, and the proof of our main result is based on them.

PROPOSITION 2.1

For $1 < p < 2$ *and* $r > 1 + p/2$, *we have*

$$\lim_{\varepsilon \searrow 0} \varepsilon^{(2r-p-2)/(2-p)} \sum_{n=1}^{\infty} n^{r/p-3/2-1/p} \mathsf{E}\{\|Y\| - \sigma\varepsilon n^{1/p-1/2}\}_+ = \sigma^{-(2r-2-p)/(2-p)} \frac{p(2-p)}{(r-p)(2r-p-2)} \mathsf{E}\|Y\|^{2(r-p)/(2-p)}.$$

Proof. Note that

$$\begin{aligned}
&\lim_{\varepsilon \searrow 0} \varepsilon^{(2r-p-2)/(2-p)} \sum_{n=1}^{\infty} n^{r/p-3/2-1/p} \mathsf{E}\{\|Y\| - \sigma\varepsilon n^{1/p-1/2}\}_+ \\
&\quad = \lim_{\varepsilon \searrow 0} \varepsilon^{(2r-p-2)/(2-p)} \sum_{n=1}^{\infty} n^{r/p-3/2-1/p} \int_{\sigma\varepsilon n^{1/p-1/2}}^{\infty} \mathsf{P}(\|Y\| \geq x)\mathrm{d}x \\
&\quad = \lim_{\varepsilon \searrow 0} \varepsilon^{(2r-p-2)/(2-p)} \int_1^{\infty} y^{r/p-3/2-1/p} \int_{\sigma\varepsilon y^{1/p-1/2}}^{\infty} \mathsf{P}(\|Y\| \geq x)\mathrm{d}x\mathrm{d}y \\
&\quad = \sigma^{-(2r-2-p)/(2-p)} \frac{2p}{2-p} \lim_{\varepsilon \searrow 0} \int_{\sigma\varepsilon}^{\infty} t^{2(r-2)/(2-p)} \int_t^{\infty} \mathsf{P}(\|Y\| \geq x)\mathrm{d}x\mathrm{d}t \\
&\quad = \sigma^{-(2r-2-p)/(2-p)} \frac{2p}{2-p} \lim_{\varepsilon \searrow 0} \int_{\sigma\varepsilon}^{\infty} \mathsf{P}(\|Y\| \geq x) \int_{\sigma\varepsilon}^{x} t^{2(r-2)/(2-p)}\mathrm{d}t\mathrm{d}x \\
&\quad = \sigma^{-(2r-2-p)/(2-p)} \frac{2p}{2-p} \left(\frac{2(r-2)}{2-p} + 1\right)^{-1} \\
&\qquad \times \lim_{\varepsilon \searrow 0} \int_{\sigma\varepsilon}^{\infty} \mathsf{P}(\|Y\| \geq x) x^{2(r-2)/(2-p)+1}\mathrm{d}x \\
&\quad = \sigma^{-(2r-2-p)/(2-p)} \frac{2p}{2r-2-p} \lim_{\varepsilon \searrow 0} \int_{\sigma\varepsilon}^{\infty} \mathsf{P}(\|Y\| \geq x) x^{2(r-2)/(2-p)+1}\mathrm{d}x \\
&\quad = \sigma^{-(2r-2-p)/(2-p)} \frac{p(2-p)}{(r-p)(2r-2-p)} \mathsf{E}\|Y\|^{2(r-p)/(2-p)}.
\end{aligned}$$

Thus the Proposition is proved. □

Without loss of generality, we assume that $\sigma = 1$ in the sequel. For each $n \geq 1$, $1 \leq i \leq n$, we define that

$$X'_{ni} = X_i I(\|X_i\| \leq n^{1/p}), \qquad \bar{X}'_{ni} = X'_{ni} - \mathsf{E}[X'_{ni}],$$

and

$$S'_{nj} = \sum_{i=1}^{j} X'_{nj}, \qquad \bar{S}'_{nj} = \sum_{i=1}^{j} \bar{X}'_{nj},$$

where $1 < p < 2$. Let $\{Y_{ni}, 1 \leq i \leq n\}$ be a sequence of independent $\mathbf{H}$-valued Gaussian random variables with mean zero and covariance operator $\mathsf{Cov}(Y_{ni}) = \mathsf{Cov}(\bar{X}'_{ni}) := \Sigma_n, 1 \leq i \leq n$.

PROPOSITION 2.2

Denote $\beta_n = O((\log n)^{-\delta})$ *for some* $\delta > 0$. *For* $1 < p < 2, r > 1 + p/2$ *and* $2 < q < 2p/(r-1)$, *if* $\mathsf{E}\|X\|^3 < \infty$ *and* $\mathsf{E}\|X\|^r < \infty$, *then*

$$n^{-1/2}\mathsf{E}\{\|\bar{S}'_{nn}\| - \varepsilon n^{1/p}\}_+ \leq \frac{1}{1-\beta_n}\mathsf{E}\{\|Y\| - \varepsilon n^{1/p-1/2} - (1-\beta_n)n^{1/q-1/2}\}_+ + I_n, \tag{2.3}$$

and when $l < \infty$,

$$n^{-1/2}\mathsf{E}\{\|\bar{S}'_{nn}\| - \varepsilon n^{1/p}\}_+ \geq \frac{1}{\gamma_n(1+\beta_n)}\mathsf{E}\{\|Y\| - \varepsilon n^{1/p-1/2}\gamma_n - \gamma_n(1+\beta_n)n^{1/q-1/2}\}_+ - I_n, \tag{2.4}$$

where

$$\gamma_n := \|(\Sigma_n^{1/2}\Sigma^{-1/2})^{-1}\| = \sup_{y\neq 0}\frac{\|(\Sigma_n^{1/2}\Sigma^{-1/2})^{-1}y\|}{\|y\|}$$

and

$$I_n \geq 0 \ \ \textit{satisfies} \ \sum_{n=1}^{\infty} n^{r/p-3/2-1/p} I_n < \infty.$$

Proof. Observe that

$$n^{-1/2}\mathsf{E}\{\|\bar{S}'_{nn}\| - \varepsilon n^{1/p}\}_+ = n^{-1/2}\int_0^{n^{1/q}} \mathsf{P}(\|\bar{S}'_{nn}\| \geq \varepsilon n^{1/p} + y)\mathrm{d}y + n^{-1/2}\int_{n^{1/q}}^{\infty} \mathsf{P}(\|\bar{S}'_{nn}\| \geq \varepsilon n^{1/p} + y)\mathrm{d}y.$$

Set $I_{n1} := n^{-1/2}\int_0^{n^{1/q}} \mathsf{P}(\|\bar{S}'_{nn}\| \geq \varepsilon n^{1/p} + y)\mathrm{d}y$. Since $r > p$ and $q > p$, which are deduced from the fact that $1 < p < 2, r > 1 + p/2$ and $2 < q < 2p/(r-1)$, by virtue of the traditional complete convergence in Hilbert spaces, we have that $\mathsf{E}\|X\|^r < \infty$ implies

$$\sum_{n=1}^{\infty} n^{r/p-2-1/p+1/2} I_{n1} \leq \sum_{n=1}^{\infty} n^{r/p-2-1/p+1/q}\mathsf{P}(\|\bar{S}'_{nn}\| \geq \varepsilon n^{1/p}) \leq \sum_{n=1}^{\infty} n^{r/p-2}\mathsf{P}(\|S_n\| \geq \varepsilon n^{1/p}) < \infty.$$

Also from Lemma 2.1, it follows that there exists a constant C and a sequence of independent **H**-valued Gaussian mean zero random variables $\{Y_{ni}, 1 \leq i \leq n\}$ with

$\mathsf{Cov}(Y_{ni}) = \mathsf{Cov}(\bar{X}'_{ni}) = \Sigma_n$, $1 \le i \le n$, such that

$$n^{-1/2}\int_{n^{1/q}}^{\infty} \mathsf{P}(\|\bar{S}'_{nn}\| \ge \varepsilon n^{1/p} + y)\mathrm{d}y$$
$$\le n^{-1/2}\int_{n^{1/q}}^{\infty}\left(\mathsf{P}\left(\left\|\sum_{i=1}^{n} Y_{ni}\right\| \ge \varepsilon n^{1/p} + y - \beta_n y\right)\mathrm{d}y + C(\beta_n y)^{-3}\sum_{i=1}^{n}\mathsf{E}\|\bar{X}'_{ni}\|^3\right)\mathrm{d}y$$

and

$$n^{-1/2}\int_{n^{1/q}}^{\infty} \mathsf{P}(\|\bar{S}'_{nn}\| \ge \varepsilon n^{1/p} + y)\mathrm{d}y$$
$$\ge n^{-1/2}\int_{n^{1/q}}^{\infty}\left(\mathsf{P}\left(\left\|\sum_{i=1}^{n} Y_{ni}\right\| \ge \varepsilon n^{1/p} + y + \beta_n y\right)\mathrm{d}y - C(\beta_n y)^{-3}\sum_{i=1}^{n}\mathsf{E}\|\bar{X}'_{ni}\|^3\right)\mathrm{d}y.$$

Define

$$\Lambda_{n1} := n^{-1/2}\int_{n^{1/q}}^{\infty} \mathsf{P}\left(\left\|\sum_{i=1}^{n} Y_{ni}\right\| \ge \varepsilon n^{1/p} + y - \beta_n y\right)\mathrm{d}y,$$

$$\Lambda_{n2} := n^{-1/2}\int_{n^{1/q}}^{\infty} \mathsf{P}\left(\left\|\sum_{i=1}^{n} Y_{ni}\right\| \ge \varepsilon n^{1/p} + y + \beta_n y\right)\mathrm{d}y$$

and

$$I_{n2} := n^{-1/2}\int_{n^{1/q}}^{\infty} C(\beta_n y)^{-3}\sum_{i=1}^{n}\mathsf{E}\|\bar{X}'_{ni}\|^3\mathrm{d}y. \tag{2.5}$$

Recall that $2 < q < 2p/(r-1)$, and then we have

$$\sum_{n=1}^{\infty} n^{r/p-2-1/p+1/2} I_{n2}$$
$$= C\sum_{n=1}^{\infty} n^{r/p-2-1/p}\beta_n^{-3}\int_{n^{1/q}}^{\infty} y^{-3}\sum_{i=1}^{n}\mathsf{E}\|\bar{X}'_{ni}\|^3\mathrm{d}y$$
$$\le C\sum_{n=1}^{\infty} n^{r/p-1-1/p-2/q}(\log n)^{3\delta}\mathsf{E}\|X\|^3 I(\|X\| \le n^{1/p})$$
$$\le C\sum_{n=1}^{\infty} n^{r/p-1-1/p-2/q}(\log n)^{3\delta}\sum_{j=1}^{n}\mathsf{E}\|X\|^3 I((j-1)^{1/p} < \|X\| \le j^{1/p})$$
$$\le C\sum_{j=1}^{\infty}\mathsf{E}\|X\|^3 I((j-1)^{1/p} < \|X\| \le j^{1/p})\sum_{n=j}^{\infty} n^{r/p-1-1/p-2/q}(\log n)^{3\delta}$$
$$\le C\sum_{j=1}^{\infty}\mathsf{E}\|X\|^3 I((j-1)^{1/p} < \|X\| \le j^{1/p}) j^{(r-1)/p-2/q}(\log j)^{3\delta}$$
$$\le C\mathsf{E}\|X\|^3 < \infty.$$

Denote $I_n = I_{n1} + I_{n2}$. Thus, by the above discussion, it follows that

$$\Lambda_{n2} - I_n \le n^{-1/2}\mathsf{E}\{\|\bar{S}'_{nn}\| - \varepsilon n^{1/p}\}_+ \le \Lambda_{n1} + I_n, \tag{2.6}$$

where I_n satisfies

$$\sum_{n=1}^{\infty} n^{r/p-2-1/p+1/2} I_n = \sum_{n=1}^{\infty} n^{r/p-2-1/p+1/2}(I_{n1} + I_{n2}) < \infty.$$

Now by applying Corollary 3 of [1], we have that for any $x \in \mathbb{R}$,

$$\mathsf{P}\left(\left\|\sum_{i=1}^{n} Y_{ni}\right\| \le x\right) \ge \mathsf{P}(\|Y\| \le x/\sqrt{n}), \quad n \ge 1.$$

This of course provides

$$\begin{aligned}
\Lambda_{n1} &= n^{-1/2}\int_{n^{1/q}}^{\infty} \mathsf{P}\left(\left\|\sum_{i=1}^{n} Y_{ni}\right\| \ge \varepsilon n^{1/p} + y - \beta_n y\right)\mathrm{d}y \\
&\le n^{-1/2}\int_{n^{1/q}}^{\infty} \mathsf{P}\left(\|Y\| \ge \varepsilon n^{1/p-1/2} + \frac{1-\beta_n}{\sqrt{n}} y\right)\mathrm{d}y \\
&= \frac{1}{1-\beta_n}\int_{(1-\beta_n)n^{1/q-1/2}}^{\infty} \mathsf{P}(\|Y\| \ge \varepsilon n^{1/p-1/2} + t)\mathrm{d}t \\
&= \frac{1}{1-\beta_n}\mathsf{E}\{\|Y\| - \varepsilon n^{1/p-1/2} - (1-\beta_n)n^{1/q-1/2}\}_+.
\end{aligned} \tag{2.7}$$

For $l < \infty$, notice that Σ^{-1} exists since its eigenvalues are positive, and also $\Sigma_n \to \Sigma$, as $n \nearrow \infty$. Thus we also assume that Σ_n^{-1} exists for all $n \ge 1$, and then we have that

$$\begin{aligned}
\|Y\| &= \|(\Sigma_n^{1/2}\Sigma^{-1/2})^{-1}(\Sigma_n^{1/2}\Sigma^{-1/2})Y\| \\
&\le \|(\Sigma_n^{1/2}\Sigma^{-1/2})^{-1}\| \cdot \|\Sigma_n^{1/2}\Sigma^{-1/2}Y\| := \gamma_n \|\Sigma_n^{1/2}\Sigma^{-1/2}Y\|,
\end{aligned}$$

and for any $x > 0$,

$$\begin{aligned}
\mathsf{P}\left(\left\|\sum_{i=1}^{n} Y_{ni}\right\| \ge x\sqrt{n}\right) &= \mathsf{P}(\|Y_{ni}\| \ge x) \\
&= \mathsf{P}(\|\Sigma_n^{1/2}\Sigma^{-1/2}Y\| \ge x) \ge \mathsf{P}(\|Y\| \ge x\gamma_n),
\end{aligned}$$

which leads to

$$\begin{aligned}
\Lambda_{n2} &= n^{-1/2}\int_{n^{1/q}}^{\infty} \mathsf{P}\left(\left\|\sum_{i=1}^{n} Y_{ni}\right\| \ge \varepsilon n^{1/p} + y + \beta_n y\right)\mathrm{d}y \\
&\ge n^{-1/2}\int_{n^{1/q}}^{\infty} \mathsf{P}\left(\|Y\| \ge \varepsilon\gamma_n n^{1/p-1/2} + \frac{1+\beta_n}{\sqrt{n}} y\gamma_n\right)\mathrm{d}y \\
&= \frac{1}{(1+\beta_n)\gamma_n}\int_{(1+\beta_n)\gamma_n n^{1/q-1/2}}^{\infty} \mathsf{P}(\|Y\| > \varepsilon\gamma_n n^{1/p-1/2} + t)\mathrm{d}t \\
&= \frac{1}{(1+\beta_n)\gamma_n}\mathsf{E}\{\|Y\| - \varepsilon\gamma_n n^{1/p-1/2} - (1+\beta_n)\gamma_n n^{1/q-1/2}\}_+.
\end{aligned} \tag{2.8}$$

Combining (2.6)–(2.8), the conclusion follows, as desired, and the proof of Proposition 2.2 is complete. □

PROPOSITION 2.3

For $1 < p < 2$ *and* $r > 1 + p/2$, *we have that* $\mathsf{E}\|X\|^r < \infty$ *implies*

$$\sum_{n=1}^{\infty} n^{r/p-2-1/p}\mathsf{E}\|X\|I(\|X\| > n^{1/p}) < \infty. \tag{2.9}$$

Proof.

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r/p-2-1/p}\mathsf{E}\|X\|I(\|X\| > n^{1/p}) \\
&\quad = \sum_{n=1}^{\infty} n^{r/p-2-1/p} \sum_{j=1}^{n} \mathsf{E}\|X\|I(j^{1/p} < \|X\| \le (j+1)^{1/p}) \\
&\quad = \sum_{j=1}^{\infty} \mathsf{E}\|X\|I(j^{1/p} < \|X\| \le (j+1)^{1/p}) \sum_{n=1}^{j} n^{r/p-2-1/p} \\
&\quad \le C\sum_{j=1}^{\infty} \mathsf{E}\|X\|I(j^{1/p} < \|X\| \le (j+1)^{1/p}) j^{r/p-1-1/p} \\
&\quad \le C\mathsf{E}\|X\|^r < \infty,
\end{aligned}$$

which means the Proposition holds true. □

Proof of Theorem 1.1. Write

$$II_n := n^{-1/2}\mathsf{E}\left\| \sum_{i=1}^{n} X_i I(\|X_i\| > n^{1/p}) \right\| \le n^{1/2}\mathsf{E}\|X\|I(\|X\| > n^{1/p}).$$

Since $\{X, X_n; n \ge 1\}$ is a sequence of i.i.d. random variables with mean zero, we can obtain that

$$\begin{aligned}
&n^{-1/2}\mathsf{E}\{\|S_n\| - \varepsilon n^{1/p}\}_+ \\
&\quad = n^{-1/2}\mathsf{E}\Bigg\{\Bigg\| \bar{S}'_{nn} - \sum_{i=1}^{n} \mathsf{E}X_i I(\|X_i\| > n^{1/p}) \\
&\qquad\qquad + \sum_{i=1}^{n} X_i I(\|X_i\| > n^{1/p}) \Bigg\| - \varepsilon n^{1/p} \Bigg\}_+
\end{aligned}$$

and

$$\begin{aligned}
&n^{-1/2}\mathsf{E}\{\|\bar{S}'_{nn}\| - \varepsilon n^{1/p}\}_+ - 2II_n \\
&\quad \le n^{-1/2}\mathsf{E}\{\|S_n\| - \varepsilon n^{1/p}\}_+ \\
&\quad \le n^{-1/2}\mathsf{E}\{\|\bar{S}'_{nn}\| - \varepsilon n^{1/p}\}_+ + 2II_n.
\end{aligned}$$

Then it follows from Proposition 2.2 that

$$n^{-1/2}\mathsf{E}\{\|S_n\| - \varepsilon n^{1/p}\}_+ \\ \leq \frac{1}{1-\beta_n}\mathsf{E}\{\|Y\| - \varepsilon n^{1/p-1/2} - (1-\beta_n)n^{1/q-1/2}\}_+ + I_n + 2II_n, \tag{2.10}$$

and when $l < \infty$, we have

$$n^{-1/2}\mathsf{E}\{\|S_n\| - \varepsilon n^{1/p}\}_+ \\ \geq \frac{1}{\gamma_n(1+\beta_n)}\mathsf{E}\{\|Y\| - \gamma_n \varepsilon n^{1/p-1/2} - \gamma_n(1+\beta_n)n^{1/q-1/2}\}_+ - I_n - 2II_n, \tag{2.11}$$

where $\beta_n = O((\log n)^{-\delta})$, $\delta > 0$, and I_n, II_n satisfy that

$$\sum_{n=1}^{\infty} n^{r/p-2-1/p} I_n < \infty$$

and

$$\sum_{n=1}^{\infty} n^{r/p-2-1/p} II_n \leq \sum_{n=1}^{\infty} n^{r/p-2-1/p}\mathsf{E}\|X\|I(\|X\| > n^{1/p}) < \infty.$$

Therefore, we get

$$\limsup_{\varepsilon\searrow 0} \varepsilon^{(2r-p-2)/(2-p)} \sum_{n=1}^{\infty} n^{r/p-2-1/p}\mathsf{E}\{\|S_n\| - \varepsilon n^{1/p}\}_+ \\ \leq \limsup_{\varepsilon\searrow 0} \varepsilon^{(2r-p-2)/(2-p)} \sum_{n=1}^{\infty} n^{r/p-3/2-1/p}\frac{1}{1-\beta_n} \\ \times \mathsf{E}\{\|Y\| - \varepsilon n^{1/p-1/2} - (1-\beta_n)n^{1/q-1/2}\}_+ \\ = \limsup_{\varepsilon\searrow 0} \varepsilon^{(2r-p-2)/(2-p)} \sum_{n=1}^{\infty} n^{r/p-3/2-1/p}\frac{1}{1-\beta_n} \\ \times \mathsf{E}\{\|Y\| - \big(\varepsilon + (1-\beta_n)n^{1/q-1/p}\big)n^{1/p-1/2}\}_+.$$

By virtue of $\beta_n \to 0$ and $(1-\beta_n)n^{1/q-1/p} \to 0$ as $n \to \infty$, since $q > p$, fixing $1 < \theta_1 < 2$ and $0 < \theta_2 < 1$, we have for n large enough and any $\varepsilon > 0$,

$$\frac{1}{1-\beta_n}\mathsf{E}\{\|Y\| - \big(\varepsilon + (1-\beta_n)n^{1/q-1/p}\big)n^{1/p-1/2}\}_+ \\ \leq \theta_1\mathsf{E}\{\|Y\| - \theta_2\varepsilon n^{1/p-1/2}\}_+.$$

This entails

$$\limsup_{\varepsilon\searrow 0} \varepsilon^{(2r-p-2)/(2-p)} \sum_{n=1}^{\infty} n^{r/p-2-1/p}\mathsf{E}\{\|S_n\| - \varepsilon n^{1/p}\}_+ \\ = \theta_1\theta_2{}^{-(2r-2-p)/(2-p)}\frac{p(2-p)}{(r-p)(2r-p-2)}\mathsf{E}\|Y\|^{2(r-p)/(2-p)},$$

and then by letting $\theta_1 \searrow 1$ and $\theta_2 \nearrow 1$, the upper bound follows, as desired.

Now, we consider the lower bound when $l' < \infty$. Notice that $\beta_n \to 0$, $\gamma_n \to 1$ and $(1+\beta_n)n^{1/q-1/p} \to 0$ as $n \to \infty$, since $q > p$. Then as above, we obtain

$$\begin{aligned}
&\liminf_{\varepsilon\searrow 0} \varepsilon^{(2r-p-2)/(2-p)} \sum_{n=1}^{\infty} n^{r/p-2-1/p} \mathsf{E}\{\|S_n\| - \varepsilon n^{1/p}\}_+ \\
&\quad\geq \liminf_{\varepsilon\searrow 0} \varepsilon^{(2r-p-2)/(2-p)} \sum_{n=1}^{\infty} n^{r/p-3/2-1/p} \\
&\qquad\times \mathsf{E}\{\|Y\| - \gamma_n \varepsilon n^{1/p-1/2} - \gamma_n(1+\beta_n)n^{1/q-1/2}\}_+ \\
&\quad= \liminf_{\varepsilon\searrow 0} \varepsilon^{(2r-p-2)/(2-p)} \sum_{n=1}^{\infty} n^{r/p-3/2-1/p} \frac{1}{\gamma_n(1+\beta_n)} \\
&\qquad\times \mathsf{E}\{\|Y\| - \big(\gamma_n\varepsilon + \gamma_n(1+\beta_n)n^{1/q-1/p}\big)n^{1/p}\}_+ \\
&\quad= \liminf_{\varepsilon\searrow 0} \varepsilon^{(2r-p-2)/(2-p)} \sum_{n=1}^{\infty} n^{r/p-3/2-1/p} \\
&\qquad\times \mathsf{E}\{\|Y\| - \varepsilon n^{1/p}\}_+ \\
&\quad= \frac{p(2-p)}{(r-p)(2r-p-2)} \mathsf{E}\|Y\|^{2(r-p)/(2-p)}.
\end{aligned}$$

Next we consider the infinite case. Assume $l' = \infty$. For any $l'' \geq l$, let $Q : \mathbf{H} \to \mathbf{H}$ be the mapping onto the l''-dimensional engenspace of σ_i^2, $i = 1, \cdots, l''$, i.e. $Q(y) = \sum_{i=1}^{l''}(y, e_i)e_i$, $y \in \mathbf{H}$. Since $\|Q(y)\| \leq \|y\|$, $y \in \mathbf{H}$, from the special case above, it leads to

$$\begin{aligned}
&\liminf_{\varepsilon\searrow 0} \varepsilon^{(2r-p-2)/(2-p)} \sum_{n=1}^{\infty} n^{r/p-2-1/p} \mathsf{E}\{\|S_n\| - \varepsilon n^{1/p}\}_+ \\
&\quad\geq \liminf_{\varepsilon\searrow 0} \varepsilon^{(2r-p-2)/(2-p)} \sum_{n=1}^{\infty} n^{r/p-2-1/p} \mathsf{E}\{\|Q(S_n)\| - \varepsilon n^{1/p}\}_+ \\
&\quad\geq \frac{p(2-p)}{(r-p)(2r-p-2)} \mathsf{E}\|Q(Y)\|^{2(r-p)/(2-p)}.
\end{aligned}$$

By letting $l'' \to \infty$, we then complete the proof of Theorem 1.1. □

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