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A note on TI-subgroups of finite groups

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Abstract. A subgroup H of a finite group G is called a TI-subgroup if $H \cap H^x = 1$ or H for any $x \in G$. In this short note, the finite groups all of whose nonabelian subgroups are TI-subgroups are classified.

Keywords. TI-subgroups; Frobenius groups.

1. Introduction

All groups considered in this paper are finite. A subgroup H of a group G satisfying $H^x \cap H = 1$ or H for any $x \in G$ will be called a TI-subgroup (trivial intersection). A topic of interest is the classification of groups in which certain subgroups are assumed to be TI-subgroups. For example, Walls [5] described the groups all of whose subgroups are TI-subgroups. Hochheim and Timmesfeld [3] investigated the groups with an abelian TI-2-subgroup. Recently, Guo *et al* [2] classified the groups all of whose abelian subgroups are TI-subgroups. As inspired by the above results, the aim of this paper is to study the finite NATI-groups, that is, all of whose nonabelian subgroups are TI-subgroups. We obtain a classification of the finite NATI-groups in Theorem 2.2 (nilpotent case) and Theorem 2.4 (nonnilpotent case).

Let $G = NH$ be a Frobenius group with a kernel N and a complement H. It is well-known that N is nilpotent and any Sylow subgroup of H is either a cyclic group or a generalized quaternion group (see [4]).

The Frobenius group has the following generalization:

DEFINITION 1.1

A *quasi-Frobenius* group is a group G such that the factor group $G/Z(G)$ is a Frobenius group. A *kernel* and a *complement* of a quasi-Frobenius group G are the preimages of a kernel and a complement of the Frobenius group $G/Z(G)$, respectively.

Lemma 1.2 [1]. *A group G is quasi-Frobenius if and only if G possesses a noncentral subgroup H such that $H \cap H^g \leq Z(G)$ for all $g \in G - H$. In this case, H is a complement of the quasi-Frobenius group G and $N = G - \bigcup_{g \in G} (H - Z(G))^g$ is its kernel.*

2. Main results

Theorem 2.1. *Let G be a NATI-group. Then G is solvable.*

Proof. Assume that the result is not true and let G be a counterexample of the smallest order. Then G has a maximal subgroup H such that $H = N_G(H)$. Suppose that $H_G \neq 1$. Then H_G and the factor group G/H_G satisfy the hypotheses. So, by the minimality of G, we see that H_G and G/H_G are both solvable. This implies that G is solvable.

In what follows, we assume that $H_G = 1$. If H is abelian, then H is a self-normalizing TI-subgroup of G. In fact, if $t \in H^g \cap H$ for some $g \in G - H$, then t centralizes $\langle H^g, H\rangle = G$. So $t \in H_G = 1$. If H is nonabelian, then, by our hypotheses, we have the same conclusion. It follows that G is a Frobenius group with complement H. Let N be the kernel of G. Then N is nilpotent. Again, by the minimality of G, G/N is solvable. Therefore, G is solvable. □

Theorem 2.2. *Let G be a nilpotent group. Then G is a NATI-group if and only if every nonabelian subgroup of G is a normal subgroup of G.*

Proof. If every nonabelian subgroup of G is a normal subgroup of G, then it is obvious that G is a NATI-group. Conversely, let G be a NATI-group and H any nonabelian subgroup of G. Assume that H is not normal in G. Then, by the hypotheses, $H^g \cap H = 1$ for some $g \in G$. Let Z be a subgroup of $Z(G)$ with order p, p being a prime. It is clear that HZ is a nonabelian subgroup of G and so HZ is a TI-subgroup of G. Hence $(HZ)^x \cap (HZ) = 1$ or HZ for any $x \in G$. Since $Z \leq (HZ)^x \cap (HZ)$, we conclude that $(HZ)^x = HZ$ and so HZ is normal in G. It follows that $H^g \leq HZ$ and $|H| = |H : H^g \cap H| = |HH^g : H^g| = |HZ : H^g| = p$, which is a contradiction. □

Let G be a NATI-group. By Theorem 2.1, G is solvable. It is well-known that every solvable group possesses composition factor of prime order. So we have the following useful proposition.

PROPOSITION 2.3

Let G be a NATI-group. Then every nonabelian subnormal subgroup of G is normal in G.

Theorem 2.4. *Let G be a nonnilpotent NATI-group. Then one of the followings holds*:

(1) *$G = NH$ is a Frobenius group with a kernel N and a complement H, where N is the minimal normal subgroup of G and H is either a cyclic group or a product of Q_8 with a cyclic group of odd order.*
(2) *$Z(G) \neq 1$, G is a quasi-Frobenius group with a abelian complement, and for any nonabelian subgroup H of G, H is normal in G, or H is a product of Q_8 with a cyclic group of odd order.*

Proof. It is easy to see that $G/Z(G)$ is not nilpotent. Choose a maximal subgroup H of G such that $Z(G) \leq H$ and H is not normal in G. Then $H = N_G(H)$.

Suppose that H is nonabelian. By the hypotheses, H is a self-normalizing TI-subgroup of G. So G is a Frobenius group with a complement H, and every Sylow subgroup of H is either a cyclic group or a generalized quaternion group. Let N be the kernel of G and $H_1 \neq 1$ be any subgroup of H. Then NH_1 is a nonabelian subgroup of G. So, by the hypotheses, NH_1 is normal in G. It follows that $H_1 = NH_1 \cap H$ is normal in H. So H is nilpotent. Notice that Q_8 is the only generalized quaternion group all of whose subgroups are normal. Therefore H is either a cyclic group or a product of Q_8 with a cyclic group of odd order, and conclusion (1) of the theorem holds.

In what follows, assume that H is abelian. In this case, we have that $H^g \cap H \leq Z(G)$ for any $g \in G - H$. In fact, let $t \in H^g \cap H$, then t centralizes $\langle H^g, H \rangle = G$. So $t \in Z(G)$. If $Z(G) = 1$, then conclusion (1) holds. Assume that $Z(G) \neq 1$. Then, by Lemma 1.2, G is a quasi-Frobenius group and H is an abelian complement. Let T be any nonabelian subgroup of G. Suppose that T is not normal in G. Choose $K \leq G$ minimal such that $T \leq K$ and K is normal in G. Since T is not normal in G, by Proposition 2.3, T is not normal in K. Now let M be a maximal subgroup of K with $T \leq M$. If M is normal in K, then, again by Proposition 2.3, M is normal in G, which contradicts the choice of K. So M is a self-normalizing TI-subgroup of K. By the above arguments, we see that K is a Frobenius group with a complement M and M is either a cyclic group or a product of Q_8 with a cyclic group of odd order. This implies that conclusion (2) of the theorem holds. □

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