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Bounds on Gromov hyperbolicity constant in graphs

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Abstract. If X is a geodesic metric space and $x_1, x_2, x_3 \in X$, a *geodesic triangle* $T = \{x_1, x_2, x_3\}$ is the union of the three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$ in X. The space X is δ-*hyperbolic* (in the Gromov sense) if any side of T is contained in a δ-neighborhood of the union of two other sides, for every geodesic triangle T in X. If X is hyperbolic, we denote by $\delta(X)$ the sharp hyperbolicity constant of X, i.e. $\delta(X) = \inf\{\delta \geq 0 : X \text{ is } \delta\text{-hyperbolic}\}$. In this paper we relate the hyperbolicity constant of a graph with some known parameters of the graph, as its independence number, its maximum and minimum degree and its domination number. Furthermore, we compute explicitly the hyperbolicity constant of some class of product graphs.

Keywords. Infinite graphs; Cartesian product graphs; independence number; domination number; geodesics; Gromov hyperbolicity.

1. Introduction

The study of mathematical properties of Gromov hyperbolic spaces and its applications is a topic of recent and increasing interest in graph theory; see, for instance [3–5, 8–10, 15–17, 18–20, 22, 23, 25–27].

The theory of Gromov's spaces was used initially for the study of finitely generated groups, where it was demonstrated to have an enormous practical importance. This theory was applied principally to the study of automatic groups (see [21]), that play an important role in the sciences of computation. Another important application of these spaces is the secure transmission of information through the internet (see [15, 16]). In particular, hyperbolicity also plays an important role in the spread of viruses through the network (see [15, 16]) and is also useful in the study of DNA data (see [8]).

In recent years several researchers have been interested in showing that metrics used in geometric function theory are Gromov hyperbolic. For instance, the Gehring–Osgood j-metric is Gromov hyperbolic; and the Vuorinen j-metric is not Gromov hyperbolic except in the punctured space (see [12]). The study of Gromov hyperbolicity of the quasi-hyperbolic and the Poincaré metrics is the subject of importance in [1, 6, 13, 14, 22, 24, 27]. The equivalence of the hyperbolicity of Riemannian manifolds and the hyperbolicity of a simple graph are proved in [22, 27]. Hence, it is useful to know hyperbolicity criteria for graphs.

In our study on hyperbolic graphs we use the notations of [11]. We give now the basic facts about Gromov's spaces. If γ is a continuous curve in a metric space (X, d), we say that γ is a *geodesic* if it is an isometry, i.e. $d(\gamma(t), \gamma(s)) = s - t$ for every $t < s$. We say that X is a *geodesic metric space* if for every $x, y \in X$ there exists a geodesic joining x and y; we denote by $[xy]$ any such geodesics (since we do not require uniqueness of geodesics, this notation is ambiguous, but it is convenient). It is clear that every geodesic metric space is path-connected. If X is a graph, we use the notation $[u, v]$ for the edge of a graph joining the vertices u and v.

In order to consider a graph G as a geodesic metric space, we must identify (by an isometry) any edge $[u, v] \in E(G)$ with a real interval with length $l := L([u, v])$; therefore, any point in the interior of the edge $[u, v]$ is a point of G. A connected graph G is naturally equipped with a distance or, more precisely, metric defined on its points, induced by taking shortest paths in G. Then, we see G as a metric graph.

Throughout the paper, by graph we mean a connected finite graph with edges of length 1; we do not allow loops or multiple edges in the graphs. These conditions guarantee that the graph is a geodesic space.

If X is a geodesic metric space and $J = \{J_1, J_2, \dots, J_n\}$ is a polygon, with sides $J_j \subseteq X$, we say that J is δ-*thin* if for every $x \in J_i$ we have that $d(x, \cup_{j \neq i} J_j) \leq \delta$. We denote by $\delta(J)$ the sharp thin constant of J, i.e. $\delta(J) := \inf\{\delta \geq 0 : \ J \text{ is } \delta\text{-thin}\}$. If $x_1, x_2, x_3 \in X$, a *geodesic triangle* $T = \{x_1, x_2, x_3\}$ is the union of the three geodesics $[x_1 x_2]$, $[x_2 x_3]$ and $[x_3 x_1]$. The space X is δ-*hyperbolic* (or satisfies the *Rips condition* with constant δ) if every geodesic triangle in X is δ-thin. We denote by $\delta(X)$ the sharp hyperbolicity constant of X, i.e. $\delta(X) := \sup\{\delta(T) : \ T \text{ is a geodesic triangle in } X\}$. We say that X is *hyperbolic* if X is δ-hyperbolic for some $\delta \geq 0$. If X is hyperbolic, then $\delta(X) = \inf\{\delta \geq 0 : \ X \text{ is } \delta\text{-hyperbolic}\}$.

It is clear that every graph in this paper is hyperbolic since we just consider finite graphs.

A *bigon* is a geodesic triangle $\{x_1, x_2, x_3\}$ with $x_2 = x_3$. Therefore, every bigon in a δ-hyperbolic geodesic metric space is δ-thin.

There are several definitions of Gromov hyperbolicity (see e.g. [7, 11]). These different definitions are equivalent in the sense that if X is δ_A-hyperbolic with respect to the definition A, then it is δ_B-hyperbolic with respect to the definition B, and there exist universal constants c_1, c_2 such that $c_1 \delta_A \leq \delta_B \leq c_2 \, \delta_A$. However, for a fixed $\delta \geq 0$, the set of δ-hyperbolic graphs with respect to the definition A, is different, in general, from the set of δ-hyperbolic graphs with respect to the definition B. We have chosen this definition since it has a deep geometric meaning (see e.g. [11]).

The following are interesting examples of hyperbolic spaces. The real line $\mathbb{R}$ is 0-hyperbolic: in fact, any point of a geodesic triangle in the real line belongs to two sides of the triangle simultaneously, and therefore we can conclude that $\mathbb{R}$ is 0-hyperbolic. The Euclidean plane $\mathbb{R}^2$ is not hyperbolic: it is clear that equilateral triangles can be drawn with arbitrarily large diameter, so that $\mathbb{R}^2$ with the Euclidean metric is not hyperbolic. This argument can be generalized in a similar way to higher dimensions: a normed vector space E is hyperbolic if and only if $\dim \ E = 1$. Every arbitrary length metric tree is 0-hyperbolic: in fact, all point of a geodesic triangle in a tree belongs simultaneously to two sides of the triangle. Every bounded metric space X is $(\operatorname{diam} X)$-hyperbolic. Every simply connected complete Riemannian manifold with sectional curvature verifying $K \leq -k^2$, for some positive constant k, is hyperbolic. We refer to [7, 11] for more background and further results.

If D is a closed subset of X, we always consider in D the *inner metric* obtained by the restriction of the metric in X, that is

$$\begin{aligned} d_D(z, w) &:= \inf\{L_X(\gamma) : \gamma \subset D \text{ is a continuous curve joining } z \text{ and } w\} \\ &\geq d_X(z, w) . \end{aligned}$$

Consequently, $L_D(\gamma) = L_X(\gamma)$ for every curve $\gamma \subset D$.

We would like to point out that computing the hyperbolicity constant of a space is usually extraordinarily difficult: Notice that, first of all, we have to consider an arbitrary geodesic triangle T, and calculate the minimum distance from an arbitrary point P of T to the union of the other two sides of the triangle to which P does not belong to. And then we have to take supremum over all the possible choices for P and then over all the possible choices for T. Without disregarding the difficulty of solving this minimax problem, notice that in general the main obstacle is that we do not know the location of geodesics in the space. Therefore, it is interesting to obtain inequalities relating the hyperbolicity constant and other parameters of graphs.

In §2 of this paper we relate the hyperbolicity constant and some parameters in graphs; in particular, in Theorem 2.2 we find upper bounds for the hyperbolicity constant of a graph in terms of its independence number and in Theorems 2.8 and 2.9 and Corollary 2.12 we find upper bounds for the hyperbolicity constant of a graph in terms of its domination number.

Section 3 is devoted to study the relation between the hyperbolicity constant and the maximum (see Theorems 3.2 and 3.3) and the minimum (see Theorem 3.4) degree of a graph. See also Theorem 2.10.

In §4, we compute explicitly the hyperbolicity constant of some class of product graphs (see Theorem 4.1 and Corollaries 4.2–4.4).

2. Hyperbolicity constant and parameters in graphs

We say that a subset $A \subset V(G)$ is an *independent set* if $[v, w] \notin E(G)$ for every $v, w \in A$. We denote by $\beta(G)$ the *independence number* of G, i.e. the cardinal of the largest independent set in G.

In Theorem 8 of [25] we find the following result.

Lemma 2.1. *In any graph G the inequality $\delta(G) \leq \frac{1}{2}\operatorname{diam} G$ holds and, furthermore, it is sharp.*

Theorem 2.2. *For every graph G with n vertices, we have*

$$\delta(G) \leq \min\left\{\beta(G),\ \frac{n-\beta(G)+2}{2}\right\}.$$

Proof. Let us consider a geodesic γ in G with $L(\gamma) = \operatorname{diam} V(G)$. Since γ is a geodesic, if $v, w \in \gamma \cap V(G)$ and $d(v, w) \geq 2$, then $[v, w] \notin E(G)$. Hence, it is possible to choose a set of vertices $v_1, v_2, \ldots, v_m \in \gamma \cap V(G)$ with $d(v_j, v_{j+1}) = 2$ and $m = \lfloor \operatorname{diam} V(G)/2 \rfloor + 1$. Thus, for every graph G we have

$$\beta(G) \geq \left\lfloor \frac{\operatorname{diam} V(G)}{2} \right\rfloor + 1.$$

This fact and Lemma 2.1 give

$$\delta(G) \le \frac{\operatorname{diam} G}{2} \le \frac{\operatorname{diam} V(G)}{2} + \frac{1}{2} \le \left\lfloor \frac{\operatorname{diam} V(G)}{2} \right\rfloor + 1 \le \beta(G).$$

In order to prove the other inequality, consider now a set $S \subset V(G)$ with $\beta(G) = |S|$. Note that since S is an independent set, we have $\operatorname{diam} V(G) \le \operatorname{diam} \bar{S} + 2 \le \left|\bar{S}\right| + 1 = n - \beta(G) + 1$. Thus $\operatorname{diam} G \le \operatorname{diam} V(G) + 1 \le n - \beta(G) + 2$ and Lemma 2.1 gives $\delta(G) \le (n - \beta(G) + 2)/2$. □

Notice that the above bound is attained, for instance, for the complete graph K_n with $n \ge 4$ vertices.

A subgraph Γ of G is said to be *isometric* if $d_\Gamma(x, y) = d_G(x, y)$ for every $x, y \in \Gamma$.

The following two results appear in Lemma 5 and Theorem 11 of [25].

Lemma 2.3. *If Γ is an isometric subgraph of G, then $\delta(\Gamma) \le \delta(G)$.*

Lemma 2.4. *The following graphs have these precise values of δ*:

- *The complete graphs with n vertices K_n verify $\delta(K_1) = \delta(K_2) = 0$, $\delta(K_3) = 3/4$, $\delta(K_n) = 1$ for every $n \ge 4$.*
- *The cycle graphs verify $\delta(C_n) = n/4$ for every $n \ge 3$.*
- *The wheel graphs with n vertices W_n verify $\delta(W_n) = 3/2$ if and only if $7 \le n \le 10$.*

The complement of a graph G is a graph $\bar{G}$ on the same vertices such that two vertices of $\bar{G}$ are adjacent if and only if they are not adjacent in G. We denote by $\bar{G}$ the complement of G.

PROPOSITION 2.5

We have the following results:

(a) *If a graph G verifies $\beta(G) \ge 4$, then there exists a connected component $\bar{G}_0$ of $\bar{G}$ with $\delta(\bar{G}_0) \ge 1$.*
(b) *If a graph G verifies $\beta(G) \ge 3$, then there exists a connected component $\bar{G}_0$ of $\bar{G}$ with $\delta(\bar{G}_0) \ge 3/4$.*

Proof.

(a) If G is an empty graph with n vertices, then $\bar{G}$ is isomorphic to the complete graph K_n. In particular, if $\beta(G) \ge 4$, then there exists an independent set of G with at least four vertices. Therefore, there exists a connected component $\bar{G}_0$ of $\bar{G}$ containing a subgraph Γ isomorphic to the complete graph K_4. Since Γ is an isometric subgraph of $\bar{G}_0$, we have $\delta(\bar{G}_0) \ge \delta(K_4)$ by Lemma 2.3. Since $\delta(K_4) = 1$ by Lemma 2.4, we conclude that $\delta(\bar{G}_0) \ge 1$.
(b) If G verifies $\beta(G) \ge 3$, then a similar argument gives $\delta(\bar{G}_0) \ge 3/4$. □

As usual, by *cycle* we mean a simple closed curve, i.e. a path with different vertices, unless the last vertex, which is equal to the first one.

Given any graph G, we denote by $\tau(G)$ the set of geodesic triangles in G which are cycles and such that each vertex is either a vertex in $V(G)$ or a midpoint of some edge in $E(G)$.

In [2], we found the following result.

Lemma 2.6. *In any graph* G,

$$\delta(G) = \sup\left\{\delta(T) : T \in \tau(G)\right\}.$$

In particular, $\delta(G)$ *is an integer multiple of* $1/4$.

PROPOSITION 2.7

We have the following results:

(a) *For any graph G we have* $\delta(G) < 7/2$ *or there exists a connected component* $\bar{G}_0$ *of* $\bar{G}$ *with* $\delta\big(\bar{G}_0\big) \geq 1$.
(b) *For any graph G we have* $\delta(G) < 5/2$ *or there exists a connected component* $\bar{G}_0$ *of* $\bar{G}$ *with* $\delta\big(\bar{G}_0\big) \geq 3/4$.

Proof.

(a) Let us assume that $\delta(G) \geq 7/2$. By Lemma 2, there exist a geodesic triangle T in $\tau(G)$ with sides $\gamma_1, \gamma_2, \gamma_3$ and $p \in \gamma_1$ with $d(p, \gamma_2 \cup \gamma_3) = \delta(G) \geq 7/2$. Therefore, $L(\gamma_1) \geq 7$. Since $T \in \tau(G)$, each endpoint of γ_1 is either a vertex in $V(G)$ or a midpoint of some edge in $E(G)$ and there exists a geodesic $\gamma \subset \gamma_1$ with $L(\gamma) \geq 6$ and endpoints in $V(G)$; consequently, we have $\operatorname{diam} V(G) \geq 6$. Since $\beta(G) \geq \left\lfloor \frac{\operatorname{diam} V(G)}{2} \right\rfloor + 1$, we have $\beta(G) \geq 4$ and Proposition 2 gives that there exists a connected component $\bar{G}_0$ of $\bar{G}$ with $\delta\big(\bar{G}_0\big) \geq 1$.
(b) If G verifies $\delta(G) \geq 5/2$, then a similar argument gives $\delta\big(\bar{G}_0\big) \geq 3/4$. □

A set $S \subset V$ of a graph G, is a *dominating set* if every vetex not in S is adjacent to a vertex in S. The domination number of G, denoted by $\gamma(G)$ the *domination number* of G, is the minimum cardinality of a dominating set.

Theorem 2.8. *For every graph* G, *we have*

$$\delta(G) \leq \frac{3\gamma(G)}{2}.$$

Proof. It is well known that $\gamma(G) \geq (\operatorname{diam} V(G) + 1)/3$. Using this result, Lemma 2.1 gives

$$\delta(G) \leq \frac{\operatorname{diam} G}{2} \leq \frac{\operatorname{diam} V(G) + 1}{2} \leq \frac{3\gamma(G)}{2}.$$

□

Examples of equality in Theorem 2.8 are the wheel graphs with n vertices W_n, with $7 \leq n \leq 10$.

Let us define the *circumference* $c(G)$ of a graph G as the supremum of the lengths of their cycles. A cycle C is a *circumference* in G if $L(C) = c(G)$.

Theorem 2.9. *For any graph G with order n, if there exists a circumference C in G such that $V(G) \setminus V(C)$ is a dominating set, then*

$$\delta(G) \leq \frac{n - \gamma(G)}{4} .$$

Proof. Since $V(G) \setminus V(C)$ is a dominating set, we deduce that $\gamma(G) \leq n - L(C)$. Furthermore, by Proposition 4.8 of [26] we know that $\delta(G) \leq L(C)/4$. These inequalities give the result.

The above bound is attained, for instance, for the corona graphs $C_m \odot K_1$ (note that $\gamma(C_m \odot K_1) = m$ and $\delta(C_m \odot K_1) = m/4$).

Theorem 2.10. *For every graph G with minimal degree d_0 and maximum eigenvalue of the Laplace matrix μ_*, we have*

$$\delta(G) \leq \frac{n(\mu_* - d_0)}{\mu_*} .$$

Proof. By Theorem 2.2 we know that $\delta(G) \leq \beta(G)$. It is well known that $\gamma_r(G) \geq rn/\mu_*$, where $\gamma_r(G)$ is the r-domination number of G. Moreover, since G is a graph with order n, we have $\gamma_{d_0}(G) + \beta(G) \leq n$. Since $\delta(G) \leq \operatorname{diam} G/2$ by Lemma 2.1, the inequality follows. □

The equality in Theorem 2.10 is attained in the complete graph K_n.

Given a graph G and a natural number k, we say that a set of vertices $S \subseteq V(G)$ is *k-regular distance* if for every vertex not in S we have that $d(v, S) \leq k$. The subgraph induced by a set $S \subseteq V$ will be denoted by $\langle S \rangle$. We define the *k-regular distance number of G* as

$$\gamma_c^k(G) := \min \left\{ |S| : S \text{ is } k\text{-regular distance and } \langle S \rangle \text{ is connected} \right\} .$$

Note that if $d(v, S) = 1$ for every vertex not in S, S is a dominating set. Moreover, if $\langle S \rangle$ is a connected set, then S is a connected dominating set. We define the *connected domination number of G* as

$$\gamma_c(G) := \min \left\{ |S| : S \text{ is a dominating set and } \langle S \rangle \text{ is connected} \right\} .$$

It would be desirable to find functions f_k and/or g_k verifying, respectively, $\delta(G) \leq f_k(\delta(\langle S \rangle))$, $\delta(\langle S \rangle) \leq g_k(\delta(G))$, for every graph G and every k-regular distance set S with $\langle S \rangle$ connected. Unfortunately, this is not possible as the following examples show:

- If G is the cycle graph $G = C_n$, then there exists a k-regular distance set S with $\langle S \rangle$ isomorphic to the path graph P_{n-2k+1}. Since $\delta(G) = n/4$ (see Lemma 2.4) and $\delta(\langle S \rangle) = 0$, it is clear that we can not find a function f_k with the required property.

- Let us consider the graph G obtained from C_n by joining each vertex of C_n with a central vertex v_0 with a path of length k. Let us consider the set S of the vertices of G contained in C_n. We know by Lemma 2.4 that $\delta(\langle S\rangle) = \delta(C_n) = n/4$. We also have $\operatorname{diam} V(G) \le 2k$ and then $\operatorname{diam} G \le 2k+1$; therefore $\delta(G) \le k+1/2$ by Lemma 2.1. This shows that we can not find a function g_k with the required property.

These examples make the following result more valuable:

Theorem 2.11. *For any graph G and any natural number k the following inequality holds*:

$$\delta(G) \le k + \frac{1}{2}\,\gamma_c^k(G)\,.$$

Proof. For any k-regular distance set S with $\langle S\rangle$ connected, it is direct that

$$\operatorname{diam} V(G) \le \operatorname{diam} S + 2k \le |S| - 1 + 2k\,.$$

Therefore, we deduce

$$\operatorname{diam} G \le \operatorname{diam} V(G) + 1 \le |S| + 2k\,.$$

Since Lemma 2.1 gives that $\delta(G) \le \frac{1}{2}\operatorname{diam} G$, we obtain, by taking the minimum on $|S|$,

$$\delta(G) \le k + \frac{1}{2}\,\gamma_c^k(G)\,.$$

□

COROLLARY 2.12

For any graph G the following inequality holds:

$$\delta(G) \le 1 + \frac{1}{2}\,\gamma_c(G)\,.$$

3. Degree and hyperbolicity constant of a graph

From Proposition 5 and Theorem 7 of [19], we deduce the following result.

Lemma 3.1. *If G is any graph with a cycle g with length $L(g) \ge 3$, then $\delta(G) \ge 3/4$. If there exists a cycle g in G with length $L(g) \ge 4$, then $\delta(G) \ge 1$.*

Theorem 3.2. *Let G be any graph with n vertices and maximum degree $\Delta = n-1$ which is not a tree. Then*

$$\frac{3}{4} \le \delta(G) \le \frac{3}{2}$$

and both inequalities are sharp.

Proof. Since G is not a tree, there exists a cycle g in G with length $L(g) \ge 3$, and therefore, Lemma 3.1 gives $\delta(G) \ge 3/4$.

We now prove the upper bound of $\delta(G)$. Let v be a vertex in G with $\deg(v) = n - 1$. Then for every vertices $w_1, w_2 \in V(G)$ we have that $d(v, w_j) \le 1$ and $d(w_1, w_2) \le 2$. Therefore, $\operatorname{diam} G \le \operatorname{diam} V(G) + 1 \le 3$, and Lemma 2.1 gives $\delta(G) \le 3/2$.

The lower bound is attained in the cycle graph C_3; the upper bound is attained in the wheel graph with n vertices W_n with $7 \le n \le 10$ (see Lemma 2.4). □

We also have the following result.

Theorem 3.3. *Let G be any graph with m edges and maximum degree Δ. Then*

$$\delta(G) \le \frac{m + 2 - \Delta}{4} .$$

Furthermore, if $\Delta = 2$, then the inequality is attained if and only if G is isomorphic to C_m; if $\Delta = 3$, then the inequality is attained if and only if G is isomorphic to C_{m-1} with an edge attached joining two vertices of C_{m-1} at a distance of (in C_{m-1}) 2 or 3.

Proof. Let T be a geodesic triangle, and $\gamma_1, \gamma_2, \gamma_3$ be the geodesics joining the vertices of the triangle, and $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$ be the closed curve given by T. In order to compute $\delta(G)$, by Lemma 2.6, we can assume that γ is a cycle.

We have $L(\gamma) \le m + 2 - \Delta$, and hence $L(\gamma_j) \le (m + 2 - \Delta)/2$, for every j. Let us consider $w \in \gamma$; without loss of generality we can assume that $w \in \gamma_1 =: [yz]$; then $d(w, \gamma_2 \cup \gamma_3) \le d(w, \{y, z\}) \le L(\gamma_j)/2 \le (m + 2 - \Delta)/4$ and consequently $\delta(T) \le (m + 2 - \Delta)/4$. Hence, we have $\delta(G) \le (m + 2 - \Delta)/4$.

If $\Delta = 2$, then the equality is attained if and only if $\delta(G) = m/4$; by Corollary 20 of [25], this happens if and only if G is isomorphic to C_m.

If $\Delta = 3$, then the equality is attained if and only if $\delta(G) = (m - 1)/4$; by Theorem 22 of [25], this happens if and only if G is isomorphic to C_{m-1} with an edge attached joining two vertices of C_{m-1} at a distance of (in C_{m-1}) 2 or 3. □

Theorem 3.4. *Let G be any graph with n vertices and minimum degree d_0. Then*

$$\delta(G) \le \max\left\{\frac{3}{2}, \frac{n + 2 - d_0}{4}\right\},$$

and the inequality is sharp.

Proof. Let T be a geodesic triangle with sides $\gamma_1, \gamma_2, \gamma_3$. In order to compute $\delta(G)$, by Lemma 2.6, we can assume that T is a cycle. Let us consider $p \in T$; without loss of generality we can assume that $p \in \gamma_1 := [xy]$.

If $L(T) \le n+2-d_0$, then $L(\gamma_1) \le (n+2-d_0)/2$ and $d(p, \gamma_2 \cup \gamma_3) \le d(p, \{x, y\}) \le L(\gamma_1)/2 \le (n + 2 - d_0)/4$.

If $L(T) \ge n + 3 - d_0$, then $d_0 \ge 3$ and $V(G) \setminus V(T)$ has at most $d_0 - 3$ vertices. Therefore, every vertex in $V(T)$ has at least 3 neighbors in $V(T)$ and, therefore, at least one neighbor in a different side of T. Now, there exists a vertex $v \in V(T)$ with $d(p, v) \le 1/2$. If $v \in \gamma_2 \cup \gamma_3$, then $d(p, \gamma_2 \cup \gamma_3) \le 1/2$. If $v \in \gamma_1$, then $d(p, \gamma_2 \cup \gamma_3) \le d(p, v) + d(v, \gamma_2 \cup \gamma_3) \le 3/2$. This finishes the proof of the inequality.

The equality is attained in the cycle graphs C_n with $n \ge 6$. □

Theorem 3.5. *Let G be any graph verifying the following conditions*:

(a) *There exists a cycle g with $L(g) = 6$ and* $\operatorname{diam} V(g) = 3$.
(b) *For every cycle C with $L(C) > 6$ we have* $\deg_C(v) \geq 3$ *for every vertex $v \in V(C)$.*

Then $\delta(G) = 3/2$.

Proof.

(a) Since there exists a cycle g in G with $L(g) = 6$ and $\operatorname{diam} V(g) = 3$, we can consider the vertices $v, w \in V(g)$ with $d(v, w) = 3$. Note that if $L(g) = 6$, there exist two geodesics g_1, g_2, joining v, w such that $g_1 \cap g_2 = \{v, w\}$ and $g_1 \cup g_2 = g$. If p is the midpoint of g_1, then $d(p, g_2) = d(p, \{v, w\}) = 3/2$. Therefore, we have that $\delta(G) \geq 3/2$.
(b) Let T be any fixed geodesic triangle with sides $\gamma_1, \gamma_2, \gamma_3$. In order to compute $\delta(G)$, by Lemma 2.6, we can assume that T is a cycle. Let us consider $p \in T$; without loss of generality we can assume that $p \in \gamma_1 := [xy]$.

If $L(T) > 6$, then every vertex in $V(T)$ has at least 3 neighbors in $V(T)$ and, therefore, at least one neighbor in a different side of T. Now, there exists a vertex $v \in V(T)$ with $d(p, v) \leq 1/2$. If $v \in \gamma_2 \cup \gamma_3$, then $d(p, \gamma_2 \cup \gamma_3) \leq 1/2$. If $v \in \gamma_1$, then $d(p, \gamma_2 \cup \gamma_3) \leq d(p, v) + d(v, \gamma_2 \cup \gamma_3) \leq 3/2$. If $L(T) \leq 6$, then $L(\gamma_1) \leq 3$ and $d(p, \gamma_2 \cup \gamma_3) \leq d(p, \{x, y\}) \leq L(\gamma_1)/2 \leq 3/2$. Therefore, $\delta(G) \leq 3/2$.

Since $\delta(G) \geq 3/2$ and $\delta(G) \leq 3/2$ we conclude that $\delta(G) = 3/2$. □

4. Hyperbolicity constant of cartesian product graphs

For any graph G we define $\operatorname{diam}' G := \sup\{d_G(u, v) / u \in G, v \in V(G)\}$. It is clear that we have either $\operatorname{diam}' G = \operatorname{diam} V(G)$ or $\operatorname{diam}' G = \operatorname{diam} V(G) + 1/2$. The concept of $\operatorname{diam}' G$ plays an important role in the following result.

Theorem 4.1. *For every tree T and every graph G with* $\operatorname{diam} T = \operatorname{diam} V(G)$ *we have*

$$\delta(T \times G) = \frac{1}{2} (\operatorname{diam} T + \operatorname{diam}' G).$$

Proof. We distinguish three cases:

(1) $\operatorname{diam} G = \operatorname{diam} V(G) + 1$,
(2) $\operatorname{diam} G = \operatorname{diam} V(G) + 1/2$, and
(3) $\operatorname{diam} G = \operatorname{diam} V(G)$.

We first deal with case (1).

Let T_0 be a geodesic triangle in $T \times G$ with sides $\gamma_1, \gamma_2, \gamma_3$ and $w \in T_0$. Without loss of generality we can assume that $w \in \gamma_1 := [\alpha\beta]$. By Lemma 2.6 we know that in order to study the hyperbolicity constant, it suffices to consider triangles such that each vertex of T_0 is either a vertex of $T \times G$ or the midpoint of an edge of $T \times G$. Hence, we have that $L(\gamma_1)$ is either an integer or an integer plus one half. Note that $\operatorname{diam}(T \times G) = \operatorname{diam} T + \operatorname{diam} G$ by Corollary 10 of [20].

If $L(\gamma_1) < \operatorname{diam} T + \operatorname{diam} G$, then $L(\gamma_1) \le \operatorname{diam} T + \operatorname{diam} G - 1/2 = \operatorname{diam} T + \operatorname{diam}' G$. Hence,

$$d_{T\times G}(w, \gamma_2 \cup \gamma_3) \le d_{T\times G}(w, \{\alpha, \beta\}) \le \frac{1}{2} L(\gamma_1) \le \frac{1}{2} (\operatorname{diam} T + \operatorname{diam}' G).$$

Consider now the case $L(\gamma_1) = \operatorname{diam} T + \operatorname{diam} G$. Since $\operatorname{diam} G = \operatorname{diam} V(G) + 1$, there exists u, v which are midpoints of some edge of G with $d_G(u, v) = \operatorname{diam} G$ and $a, b \in V(T)$ with $d_T(a, b) = \operatorname{diam} T$, and $\alpha = (a, u), \beta = (b, v)$, i.e. $\gamma_1 = [(a, u)(b, v)]$.

Let $x := (x_0, y_0)$ be the midpoint of γ_1; since $\operatorname{diam} T = \operatorname{diam} V(G) = \operatorname{diam} G - 1$, we deduce that $x \in V(T\times G)$. Since $T\times G$ is path-connected, the set $(\gamma_2\cup\gamma_3)\cap(\{x_0\}\times V(G))$ is non-empty; thus there exists $y_1 \in V(G)$ with $(x_0, y_1) \in \gamma_2 \cup \gamma_3$. Consequently,

$$d_{T\times G}(x, \gamma_2 \cup \gamma_3) \le d_{T\times G}((x_0, y_0), (x_0, y_1)) = d_G(y_0, y_1) \le \operatorname{diam} V(G).$$

If $w \in \gamma_1 \cap \overline{B_{T\times G}(x, 1/4)}$, then

$$\begin{aligned} d_{T\times G}(w, \gamma_2 \cup \gamma_3) &\le d_{T\times G}(w, x) + d_{T\times G}(x, \gamma_2 \cup \gamma_3) \\ &\le \frac{1}{4} + \operatorname{diam} V(G) = \frac{1}{2} (\operatorname{diam} T + \operatorname{diam}' G). \end{aligned}$$

If $w \in \gamma_1 \setminus \overline{B_{T\times G}(x, 1/4)}$, then

$$\begin{aligned} d_{T\times G}(w, \gamma_2 \cup \gamma_3) &\le d_{T\times G}(w, \{\alpha, \beta\}) \le \frac{1}{2} L(\gamma_1) - \frac{1}{4} = \frac{1}{4} + \operatorname{diam} V(G) \\ &= \frac{1}{2} (\operatorname{diam} T + \operatorname{diam}' G). \end{aligned}$$

Hence, we have

$$\delta(T \times G) \le \frac{1}{2} (\operatorname{diam} T + \operatorname{diam}' G).$$

In order to obtain the reverse inequality, we will construct an appropriate geodesic bigon. Since $\operatorname{diam} G = \operatorname{diam} V(G)+1$, there exists u, v which are midpoints, respectively, of the edges $[u_1, u_2], [v_1, v_2] \in E(G)$ with $d_G(u, v) = \operatorname{diam} G$, and two geodesics g_1, g_2 in G joining u and v with $[u_1u]\cup[v_1v] \subset g_1$ and $[u_2u]\cup[v_2v] \subset g_2$. Let g be a geodesic in T joining a and b with $d_T(a, b) = \operatorname{diam} T$. Consider the geodesics

$$\begin{aligned} \gamma_1 &:= (\{a\} \times [uu_1]) \cup ([ab] \times \{u_1\}) \cup (\{b\} \times (g_1 \setminus [uu_1])), \\ \gamma_2 &:= (\{a\} \times (g_2 \setminus [v_2v])) \cup ([ab] \times \{v_2\}) \cup (\{b\} \times [v_2v])), \end{aligned}$$

joining $z_1 := (a, u)$ and (b, v), and the geodesic bigon in $T \times G$ with sides γ_1 and γ_2. Let $z := (b, u_1)$ be the midpoint of γ_1 and z_0 be the point in the geodesic $[z_1z]$ with $d_{T\times G}(z_0, z) = 1/4$. It is not difficult to check that

$$\delta(T \times G) \ge d_{T\times G}(z_0, \gamma_2) = \frac{1}{4} + \operatorname{diam} T = \frac{1}{2} (\operatorname{diam} T + \operatorname{diam}' G).$$

We now deal with cases (2) and (3). By Theorem 24 of [20], we know that in these cases the following equality holds:

$$\delta(T \times G) = \frac{1}{2}\,(\operatorname{diam} T + \operatorname{diam} G),$$

since $\operatorname{diam} T = \operatorname{diam} V(G) > \operatorname{diam} G - 1$. In order to finish the proof, it suffices to note that in both cases (2) and (3) we have that $\operatorname{diam} G = \operatorname{diam}' G$. □

COROLLARY 4.2

Let T be any tree and G be any graph. Then

$$\delta(T \times G) = \begin{cases} (\operatorname{diam} T + \operatorname{diam}' G)/2, & \textit{if } \operatorname{diam} T = \operatorname{diam} V(G), \\ \operatorname{diam} V(G) + 1/2, & \textit{if } \operatorname{diam} T > \operatorname{diam} V(G). \end{cases}$$

Proof. Theorem 4.1 proves the equality if $\operatorname{diam} T = \operatorname{diam} V(G)$. Theorem 24 of [20] gives the equality if $\operatorname{diam} T > \operatorname{diam} V(G)$. □

We denote by P_2 the path graph with 2 vertices.

COROLLARY 4.3

For every $n \geq 3$ we have $\delta(P_2 \times K_n) = 5/4$.

Proof. Since $\operatorname{diam} P_2 = 1 = \operatorname{diam} V(K_n)$, Theorem 4.1 gives

$$\delta(P_2 \times K_n) = \frac{1}{2}\,(\operatorname{diam} P_2 + \operatorname{diam}' K_n) = \frac{1}{2}\left(1 + \frac{3}{2}\right) = \frac{5}{4}\,.$$

COROLLARY 4.4

Let us consider any $n \geq 4$ and any tree T. Then

$$\delta(T \times K_n) = \begin{cases} 1, & \textit{if } |V(T)| = 1, \\ 5/4, & \textit{if } |V(T)| = 2, \\ 3/2, & \textit{if } |V(T)| \geq 3. \end{cases}$$

Proof. If T has just one vertex, then $T \times K_n$ is isometric to K_n, and Lemma 2.4 gives that $\delta(K_n) = 1$.

If T has just two vertices, then $T \times K_n$ is isometric to $P_2 \times K_n$ and Corollary 4.3 implies that $\delta(P_2 \times K_n) = 5/4$.

If T has at least three vertices, then the result is a consequence of Corollary 4.2, since $\operatorname{diam} T \geq 2 > 1 = \operatorname{diam} V(K_n)$. □

The reader is referred to [20] for more details on hyperbolicity constant of cartesian product graphs.

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