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Optimal combinations bounds of root-square and arithmetic means for Toader mean

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Abstract. We find the greatest values α_1 and α_2, and the least values β_1 and β_2, such that the double inequalities $\alpha_1 S(a,b) + (1-\alpha_1)A(a,b) < T(a,b) < \beta_1 S(a,b) + (1-\beta_1)A(a,b)$ and $S^{\alpha_2}(a,b)A^{1-\alpha_2}(a,b) < T(a,b) < S^{\beta_2}(a,b)A^{1-\beta_2}(a,b)$ hold for all $a, b > 0$ with $a \neq b$. As applications, we get two new bounds for the complete elliptic integral of the second kind in terms of elementary functions. Here, $S(a,b) = [(a^2+b^2)/2]^{1/2}$, $A(a,b) = (a+b)/2$, and $T(a,b) = \frac{2}{\pi}\int_0^{\pi/2}\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\,\mathrm{d}\theta$ denote the root-square, arithmetic, and Toader means of two positive numbers a and b, respectively.

Keywords. Root-square mean; arithmetic mean; Toader mean; complete elliptic integrals.

1. Introduction

For $a, b > 0$ with $a \neq b$ the power mean $M_{\mathrm{p}}(a,b)$ is defined by

$$M_{\mathrm{p}}(a,b) = \begin{cases} \left(\frac{a^p+b^p}{2}\right)^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0. \end{cases}$$

It is well known that $M_{\mathrm{p}}(a,b)$ is continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$. Many means are special cases of $M_{\mathrm{p}}(a,b)$, for example, $M_{-1}(a,b) = H(a,b) = 2ab/(a+b)$, $M_0(a,b) = G(a,b) = \sqrt{ab}$,

$$M_1(a,b) = A(a,b) = (a+b)/2 \tag{1.1}$$

and

$$M_2(a,b) = S(a,b) = \sqrt{(a^2+b^2)/2} \tag{1.2}$$

are known in the literature as harmonic, geometric, arithmetic and root-square means, respectively.

Recently, the power mean M_p has been the subject of intensive research. In particular, many remarkable inequalities for M_p can be found in the literature [1, 2, 10, 12–15, 17, 19, 22].

In [18], Toader introduced the Toader mean $T(a,b)$ of two positive numbers a and b as follows:

$$T(a,b) = \frac{2}{\pi}\int_0^{\pi/2}\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\,\mathrm{d}\theta = \begin{cases} 2a\mathcal{E}\left(\sqrt{1-(b/a)^2}\right)/\pi, & a > b, \\ 2b\mathcal{E}\left(\sqrt{1-(a/b)^2}\right)/\pi, & a < b, \\ a, & a = b, \end{cases} \tag{1.3}$$

where $\mathcal{E}(r) = \int_0^{\pi/2}(1-r^2\sin^2 t)^{1/2}\mathrm{d}t$ ($r \in [0,1]$) is the complete elliptic integral of the second kind.

Vuorinen [20] conjectured that

$$M_{3/2}(a,b) < T(a,b) \tag{1.4}$$

for all $a, b > 0$ with $a \neq b$. This conjecture was proved by Qiu and Shen [16], and Barnard *et al.* [5], respectively.

In [3], Alzer and Qiu presented the best possible upper power mean bound for the Toader mean as follows:

$$T(a,b) < M_{\log 2/\log(\pi/2)}(a,b) \tag{1.5}$$

for all $a, b > 0$ with $a \neq b$.

From (1.1), (1.2), (1.4) and (1.5) we clearly see that

$$A(a,b) < T(a,b) < S(a,b) \tag{1.6}$$

for all $a, b > 0$ with $a \neq b$.

The main purpose of the paper is to find the greatest values α_1 and α_2, and the least values β_1 and β_2, such that the double inequalities $\alpha_1 S(a,b) + (1-\alpha_1)A(a,b) < T(a,b) < \beta_1 S(a,b) + (1-\beta_1)A(a,b)$ and $S^{\alpha_2}(a,b)A^{1-\alpha_2}(a,b) < T(a,b) < S^{\beta_2}(a,b)A^{1-\beta_2}(a,b)$ hold for all $a, b > 0$ with $a \neq b$. As applications, we get two new bounds for the complete elliptic integral of the second kind in terms of elementary functions.

2. Basic knowledge and lemmas

In order to establish our main results we need some basic knowledge and lemmas, which we present in this section.

For real numbers a, b and c with $c \neq 0, -1, -2, \ldots$, the Gaussian hypergeometric function is defined by

$$F(a, b; c; x) = {}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)} \frac{x^n}{n!}, \quad |x| < 1,$$

where $(a, 0) = 1$ for $a \neq 0$ and (a, n) denotes the shifted factorial function

$$(a, n) = a(a + 1)(a + 2)(a + 3) \cdots (a + n - 1)$$

for $n = 1, 2, \ldots$, and it has the following derivative formula:

$$\frac{\mathrm{d}F(a, b; c; x)}{\mathrm{d}x} = \frac{ab}{c} F(a + 1, b + 1; c + 1; x).$$

For $r \in (0, 1)$ and $r' = \sqrt{1 - r^2}$, the well-known complete elliptic integrals of the first and second kinds are defined by

$$\begin{cases} \mathcal{K} = \mathcal{K}(r) = \int_0^{\pi/2} (1 - r^2 \sin^2 \theta)^{-1/2} \mathrm{d}\theta = \pi/2 F(1/2, 1/2; 1; r^2), \\ \mathcal{K}' = \mathcal{K}'(r) = \mathcal{K}(r'), \\ \mathcal{K}(0) = \pi/2, \quad \mathcal{K}(1) = +\infty \end{cases}$$

and

$$\begin{cases} \mathcal{E} = \mathcal{E}(r) = \int_0^{\pi/2} (1 - r^2 \sin^2 \theta)^{1/2} \mathrm{d}\theta = \pi/2 F(-1/2, 1/2; 1; r^2), \\ \mathcal{E}' = \mathcal{E}'(r) = \mathcal{E}(r'), \\ \mathcal{E}(0) = \pi/2, \quad \mathcal{E}(1) = 1, \end{cases}$$

respectively, and the following derivative formulas were presented in Appendix E, pp. 474–475 of [4]:

$$\frac{\mathrm{d}\mathcal{K}}{\mathrm{d}r} = \frac{\mathcal{E} - r'^2 \mathcal{K}}{r r'^2}, \qquad \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}r} = \frac{\mathcal{E} - \mathcal{K}}{r},$$

$$\frac{\mathrm{d}(\mathcal{E} - r'^2 \mathcal{K})}{\mathrm{d}r} = r\mathcal{K}, \qquad \frac{\mathrm{d}(\mathcal{K} - \mathcal{E})}{\mathrm{d}r} = \frac{r\mathcal{E}}{r'^2}.$$

Lemma 2.1. (*Theorem* 1.25 *of* [4]). *For* $-\infty < a < b < \infty$, *let* $f, g : [a, b] \to \mathbb{R}$ *be continuous on* $[a, b]$ *and differentiable on* (a, b), *and* $g'(x) \neq 0$ *on* (a, b). *If* $f'(x)/g'(x)$ *is increasing* (*decreasing*) *on* (a, b), *then so are*

$$\frac{f(x) - f(a)}{g(x) - g(a)} \quad \text{and} \quad \frac{f(x) - f(b)}{g(x) - g(b)}.$$

If $f'(x)/g'(x)$ *is strictly monotone, then the monotonicity in the conclusion is also strict.*

Lemma 2.2. *The function* $g(r) = (1 + r'^2)^{3/2}[(1 + r'^2)\mathcal{K} - 2\mathcal{E}]/r^4$ *is strictly increasing from* $(0, 1)$ *onto* $(\sqrt{2}\pi/8, \infty)$.

Proof. Making use of series expansion we have

$$
\begin{aligned}
&(1+r'^2)\mathcal{K}-2\mathcal{E}\\
&=\frac{\pi}{2}\left[(2-r^2)\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2},n\right)\left(\frac{1}{2},n\right)}{(n!)^2}r^{2n}-2\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2},n\right)\left(\frac{1}{2},n\right)}{(n!)^2}r^{2n}\right]\\
&=\frac{\pi}{2}\left[2\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2},n\right)\left(\frac{1}{2},n\right)}{(n!)^2}r^{2n}-\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2},n\right)\left(\frac{1}{2},n\right)}{(n!)^2}r^{2n+2}\right.\\
&\qquad\left.-2\sum_{n=1}^{\infty}\frac{\left(-\frac{1}{2},n\right)\left(\frac{1}{2},n\right)}{(n!)^2}r^{2n}\right]\\
&=\frac{\pi}{2}\left[2\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2},n+1\right)\left(\frac{1}{2},n+1\right)}{[(n+1)!]^2}r^{2n+2}-\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2},n\right)\left(\frac{1}{2},n\right)}{(n!)^2}r^{2n+2}\right.\\
&\qquad\left.+\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2},n\right)\left(\frac{1}{2},n+1\right)}{[(n+1)!]^2}r^{2n+2}\right]\\
&=\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{n(n+1)\left(\frac{1}{2},n\right)\left(\frac{1}{2},n\right)}{[(n+1)!]^2}r^{2n+2}=\frac{\pi}{2}\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2},n\right)\left(\frac{1}{2},n\right)}{(n-1)!(n+1)!}r^{2n+2}\\
&=\frac{\pi}{16}\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2},n\right)\left(\frac{3}{2},n\right)}{(3,n)n!}r^{2n+4}=\frac{\pi}{16}r^4F\left(\frac{3}{2},\frac{3}{2};3;r^2\right).
\end{aligned}
\tag{2.1}
$$

It follows from (2.1) that the function $g(r)$ can be rewritten as

$$
g(r)=\frac{\pi}{16}(1+r'^2)^{3/2}F\left(\frac{3}{2},\frac{3}{2};3;r^2\right).
\tag{2.2}
$$

Differentiating $g(r)$ in (2.1) one has

$$
\begin{aligned}
\frac{32}{3\pi r(1+r'^2)^{1/2}}g'(r)=&(2-r^2)F\left(\frac{5}{2},\frac{5}{2};4;r^2\right)-2F\left(\frac{3}{2},\frac{3}{2};3;r^2\right)\\
=&2\sum_{n=0}^{\infty}\frac{\left(\frac{5}{2},n\right)\left(\frac{5}{2},n\right)}{n!(4,n)}r^{2n}-\sum_{n=0}^{\infty}\frac{\left(\frac{5}{2},n\right)\left(\frac{5}{2},n\right)}{n!(4,n)}r^{2n+2}\\
&-2\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2},n\right)\left(\frac{3}{2},n\right)}{n!(3,n)}r^{2n}\\
=&\sum_{n=0}^{\infty}\frac{(2n+5)\left(\frac{5}{2},n\right)\left(\frac{5}{2},n\right)}{2n!(4,n+1)}r^{2n+2}>0.
\end{aligned}
\tag{2.3}
$$

Therefore, Lemma 2.2 follows from (2.2) and (2.3) together with the limiting values $g(0^+)=\sqrt{2}\pi/8$ and $g(1^-)=+\infty$.

Lemma 2.3.

(1) *The function* $G(r)=[(\mathcal{E}-r'^2\mathcal{K})^2+r'^2(\mathcal{K}-\mathcal{E})^2]/r^4$ *is strictly decreasing from* $(0,1)$ *onto* $(1,\pi^2/8)$;

(2) *The function* $H(r) = [r'(1+r'^2)\mathcal{K} - 2r'\mathcal{E}]/[(1-r')^2\mathcal{E}]$ *is strictly decreasing from* $(0, 1)$ *onto* $(0, 1/2)$.

Proof.

(1) Let $G_1(r) = (\mathcal{E} - r'^2\mathcal{K})^2 + r'^2(\mathcal{K} - \mathcal{E})^2$ and $G_2(r) = r^4$. Then $G_1(0) = G_2(0) = 0$, $G(r) = G_1(r)/G_2(r)$ and

$$\frac{{G_1}'(r)}{{G_2}'(r)} = \frac{r^2\mathcal{K}^2 - 2(\mathcal{K} - \mathcal{E})^2}{2r^2}. \tag{2.4}$$

Let

$$G_3(r) = \frac{r^2\mathcal{K}^2 - 2(\mathcal{K} - \mathcal{E})^2}{2r^2}. \tag{2.5}$$

Then simple computation leads to

$${G_3}'(r) = \frac{\mathcal{K}(\mathcal{E} - r'^2\mathcal{K})}{rr'^2}\left(1 - 2\frac{\mathcal{K} - \mathcal{E}}{r^2\mathcal{K}}\right). \tag{2.6}$$

It is well-known that the function $r \mapsto (\mathcal{K} - \mathcal{E})/(r^2\mathcal{K})$ is strictly increasing from $(0, 1)$ onto $(1/2, 1)$, then from (2.6) we clearly see that $G_3(r)$ is strictly decreasing in $(0, 1)$.

Therefore, Lemma 2.3(1) follows from (2.4), (2.5), Lemma 2.1 and the monotonicity of $G_3(r)$ together with the limiting values $G(0^+) = \pi^2/8$ and $G(1^-) = 1$.

(2) Let $H_1(r) = [(1+r'^2)\mathcal{K} - 2\mathcal{E}]/\mathcal{E}$ and $H_2(r) = (1-r')^2/r'$. Then $H_1(0) = H_2(0) = 0$, $H(r) = H_1(r)/H_2(r)$ and

$$\frac{{H_1}'(r)}{{H_2}'(r)} = \frac{[(\mathcal{E} - r'^2\mathcal{K})^2 + r'^2(\mathcal{K} - \mathcal{E})^2]/[rr'^2\mathcal{E}^2]}{r^3/r'^3} = G(r)\left(\frac{r'^{1/2}}{\mathcal{E}}\right)^2. \tag{2.7}$$

It is well-known that the function $r \mapsto r'^{1/2}/\mathcal{E}$ is strictly decreasing in $(0, 1)$. Then from Lemma 2.3(1) we conclude that ${H_1}'(r)/{H_2}'(r)$ is strictly decreasing in $(0, 1)$.

Therefore, Lemma 2.3(2) follows from (2.7), Lemma 2.1 and the monotonicity of ${H_1}'(r)/{H_2}'(r)$ together with the limiting values $H(0^+) = 1/2$ and $H(1^-) = 0$.

3. Main results

Theorem 3.1. *Inequality*

$$\alpha_1 S(a, b) + (1-\alpha_1)A(a, b) < T(a, b) < \beta_1 S(a, b) + (1-\beta_1)A(a, b) \tag{3.1}$$

holds for all $a, b > 0$ *with* $a \neq b$ *if and only if* $\alpha_1 \leq 1/2$ *and* $\beta_1 \geq (4-\pi)/[(\sqrt{2}-1)\pi] = 0.659\ldots$.

Proof. Without loss of generality, we assume that $a > b$. Let $r = b/a \in (0, 1)$, then from (1.1)–(1.3) we have

$$\frac{S(a, b) - T(a, b)}{S(a, b) - A(a, b)} = \frac{\left(\frac{1+r^2}{2}\right)^{1/2} - \frac{2}{\pi}\mathcal{E}'(r)}{\left(\frac{1+r^2}{2}\right)^{1/2} - \frac{1+r}{2}}. \tag{3.2}$$

Let $f_1(r)=[(1+r^2)/2]^{1/2}-2\mathcal{E}'(r)/\pi$, $f_2(r)=[(1+r^2)/2]^{1/2}-(1+r)/2$ and

$$f(r)=\frac{f_1(r)}{f_2(r)}=\frac{\left(\frac{1+r^2}{2}\right)^{1/2}-\frac{2}{\pi}\mathcal{E}'(r)}{\left(\frac{1+r^2}{2}\right)^{1/2}-\frac{1+r}{2}}. \tag{3.3}$$

Then simple computations lead to

$$f_1(1) = f_2(1)=0, \tag{3.4}$$

$$f_1{}'(r) = \frac{r}{\sqrt{2+2r^2}}-\frac{2r(\mathcal{K}'-\mathcal{E}')}{\pi r'^2}, \tag{3.5}$$

$$f_2{}'(r) = \frac{r}{\sqrt{2+2r^2}}-\frac{1}{2}, \tag{3.6}$$

$$f_1{}'(1) = f_2{}'(1)=0 \tag{3.7}$$

and

$$\frac{f_1{}''(r)}{f_2{}''(r)}=1-\frac{2\sqrt{2}}{\pi}(1+r^2)^{3/2}\frac{(1+r^2)\mathcal{K}'-2\mathcal{E}'}{r'^4}. \tag{3.8}$$

It follows from (3.8) and Lemma 2.2 that the function $f_1{}''(r)/f_2{}''(r)$ is strictly increasing from $(0,1)$ onto $(-\infty,1/2)$. Hence, $f(r)$ is strictly increasing directly from (3.3)–(3.7) and Lemma 2.1. Moreover

$$\lim_{r\to 0} f(r)=\frac{\sqrt{2}\pi-4}{(\sqrt{2}-1)\pi} \tag{3.9}$$

and

$$\lim_{r\to 1} f(r)=\frac{1}{2}. \tag{3.10}$$

Therefore, inequality (3.1) follows from (3.2), (3.9) and (3.10) together with the monotonicity of $f(r)$.

Finally, we prove that $S(a,b)/2+A(a,b)/2$ and $(4-\pi)/[(\sqrt{2}-1)\pi]S(a,b)+(\sqrt{2}\pi-4)/[(\sqrt{2}-1)\pi]A(a,b)$ are the best possible lower and upper convex combination bounds of root-square and arithmetic means for the Toader mean $T(a,b)$.

Let $p\in\mathbb{R}$ and $r\in(0,1)$. Then from (1.1)–(1.3) we get

$$\begin{aligned} &pS(1,r)+(1-p)A(1,r)-T(1,r)\\ &\quad=[S(1,r)-T(1,r)]-(1-p)[S(1,r)-A(1,r)]\\ &\quad=[S(1,r)-A(1,r)][f(r)+p-1], \end{aligned} \tag{3.11}$$

where $f(r)$ is defined as in (3.3).

We divide the proof into cases.

Case 1. $p>1/2$. From (3.10) we know that

$$\lim_{r\to 1}[f(r)+p-1]=p-\frac{1}{2}>0. \tag{3.12}$$

From (1.6), (3.11) and (3.12) we know that for any $p > 1/2$ there exists $0 < \delta_1 = \delta_1(r) < 1$, such that $pS(1,r) + (1-p)A(1,r) > T(1,r)$ for $r \in (\delta_1(r), 1)$.

Case 2. $p < (4-\pi)/[(\sqrt{2}-1)\pi]$. From (3.9) we get

$$\lim_{r\to 0}[f(r) + p - 1] = p - \frac{4-\pi}{(\sqrt{2}-1)\pi} < 0. \tag{3.13}$$

From (1.6), (3.11) and (3.13) we clearly see that for any $p < (4-\pi)/[(\sqrt{2}-1)\pi]$ there exists $0 < \delta_2 = \delta_2(r) < 1$, such that $pS(1,r) + (1-p)A(1,r) < T(1,r)$ for $r \in (0, \delta_2(r))$.

Theorem 3.2. *Inequality*

$$S^{\alpha_2}(a,b)A^{1-\alpha_2}(a,b) < T(a,b) < S^{\beta_2}(a,b)A^{1-\beta_2}(a,b) \tag{3.14}$$

holds for all $a, b > 0$ *with* $a \neq b$ *if and only if* $\alpha_2 \le 1/2$ *and* $\beta_2 \ge 4 - 2\log\pi/\log 2 = 0.697\ldots$.

Proof. Without loss of generality, we assume that $a > b$. Let $r = b/a \in (0,1)$. Then from (1.1)–(1.3) we have

$$\frac{\log S(a,b) - \log T(a,b)}{\log S(a,b) - \log A(a,b)} = \frac{\frac{1}{2}\log\frac{1+r^2}{2} - \log\frac{2}{\pi}\mathcal{E}'(r)}{\frac{1}{2}\log\frac{1+r^2}{2} - \log\frac{1+r}{2}}. \tag{3.15}$$

Let $F_1(r) = 1/2\log[(1+r^2)/2] - \log[2\mathcal{E}'(r)/\pi]$, $F_2(r) = 1/2\log[(1+r^2)/2] - \log[(1+r)/2]$ and

$$F(r) = \frac{F_1(r)}{F_2(r)} = \frac{\frac{1}{2}\log\frac{1+r^2}{2} - \log\frac{2}{\pi}\mathcal{E}'(r)}{\frac{1}{2}\log\frac{1+r^2}{2} - \log\frac{1+r}{2}}. \tag{3.16}$$

Then simple computations yield

$$F_1(1) = F_2(1) = 0, \tag{3.17}$$

$$\frac{F_1{}'(r)}{F_2{}'(r)} = \frac{r/(1+r^2) - r(\mathcal{K}' - \mathcal{E}')/(r'^2\mathcal{E}')}{r/(1+r^2) - 1/(1+r)} = \frac{r(1+r^2)\mathcal{K}' - 2r\mathcal{E}'}{(1-r)^2\mathcal{E}'}. \tag{3.18}$$

It follows from (3.18) and Lemma 2.3(2) that $F_1{}'(r)/F_2{}'(r)$ is strictly increasing from (0, 1) onto (0, 1/2). Then from (3.16) and (3.17) together with Lemma 2.1 we know that $F(r)$ is strictly increasing in (0, 1). Moreover

$$\lim_{r\to 0} F(r) = -3 + \frac{2\log\pi}{\log 2} \tag{3.19}$$

and

$$\lim_{r\to 1} F(r) = \frac{1}{2}. \tag{3.20}$$

Therefore, inequality (3.14) follows from (3.15), (3.16), (3.19) and (3.20) together with the monotonicity of $F(r)$.

Finally, we prove that $[S(a,b)]^{4-2\log\pi/\log 2}[A(a,b)]^{2\log\pi/\log 2-3}$ and $S^{1/2}(a,b)\times A^{1/2}(a,b)$ are the best possible upper and lower geometric combination bounds of root-square and arithmetic means for the Toader mean $T(a,b)$.

Let $q\in\mathbb{R}$ and $r\in(0,1)$. Then (1.1)–(1.3) lead to

$$\begin{aligned}
&\log[S^q(1,r)A^{1-q}(1,r)]-\log T(1,r)\\
&\quad=q\log S(1,r)+(1-q)\log A(1,r)-\log T(1,r)\\
&\quad=[\log S(1,r)-\log T(1,r)]-(1-q)[\log S(1,r)-\log A(1,r)]\\
&\quad=[\log S(1,r)-\log A(1,r)][F(r)+q-1], \qquad (3.21)
\end{aligned}$$

where $F(r)$ is defined as in (3.16).

We divide the proof into cases.

Case 1. $q>1/2$. From (3.20) we have

$$\lim_{r\to 1}[F(r)+q-1]=q-\frac{1}{2}>0. \qquad (3.22)$$

From (1.6), (3.21) and (3.22) we know that for any $q>1/2$ there exists $0<\delta_3=\delta_3(r)<1$ such that $S^q(1,r)A^{1-q}(1,r)>T(1,r)$ for $r\in(\delta_3(r),1)$.

Case 2. $q<4-2\log\pi/\log 2$. From (3.19) we get

$$\lim_{r\to 0}[F(r)+q-1]=q-\left(4-\frac{2\log\pi}{\log 2}\right)<0. \qquad (3.23)$$

From (1.6), (3.21) and (3.23) we know that for any $q<4-2\log\pi/\log 2$ there exists $0<\delta_4=\delta_4(r)<1$ such that $S^q(1,r)A^{1-q}(1,r)<T(1,r)$ for $r\in(0,\delta_4(r))$.

From Theorems 3.1 and 3.2 we get two new bounds for the complete elliptic integral $\mathcal{E}(r)$ of the second kind in terms of elementary functions as follows.

COROLLARY 3.3

For $r\in(0,1)$ and $r'=\sqrt{1-r^2}$, we have

$$\begin{aligned}
&\frac{\pi}{2}\left[\frac{1}{2}\sqrt{\frac{1+r'^2}{2}}+\frac{1+r'}{4}\right]<\mathcal{E}(r)\\
&\quad<\frac{\pi}{2}\left[\frac{4-\pi}{(\sqrt{2}-1)\pi}\sqrt{\frac{1+r'^2}{2}}+\frac{(\sqrt{2}\pi-4)(1+r')}{2(\sqrt{2}-1)\pi}\right], \qquad (3.24)
\end{aligned}$$

$$\begin{aligned}
&2^{-7/4}\pi(1+r'^2)^{1/4}(1+r')^{1/2}<\mathcal{E}(r)\\
&\quad<(1+r'^2)^{2-\log\pi/\log 2}(1+r')^{2\log\pi/\log 2-3}. \qquad (3.25)
\end{aligned}$$

Remark 3.4. In recent past, the complete elliptic integrals have been a subject of intensive research [6–9, 11, 21]. In [6], the authors established that

$$\mathcal{E}(r) \le \frac{\pi}{2}\left(\frac{2-r^2}{2}\right)^{1/2} \tag{3.26}$$

for all $r \in (0, 1)$.

Guo and Qi [11] proved that

$$\frac{\pi}{2} - \frac{1}{2}\log\frac{(1+r)^{1-r}}{(1-r)^{1+r}} < \mathcal{E}(r) < \frac{\pi-1}{2} + \frac{1-r^2}{4r}\log\frac{1+r}{1-r}, \tag{3.27}$$

for all $r \in (0, 1)$.

Computational and numerical experiments show that the bounds in (3.24) and (3.25) for $\mathcal{E}(r)$ are better than that in (3.26) and (3.27) for some $r \in (0, 1)$, respectively. In fact, if we let

$$\begin{aligned}
J_1(r) &= \frac{\pi}{2}\left[\frac{1}{2}\sqrt{\frac{1+r'^2}{2}} + \frac{1+r'}{4}\right],\\
J_2(r) &= \frac{\pi}{2}\left[\frac{4-\pi}{(\sqrt{2}-1)\pi}\sqrt{\frac{1+r'^2}{2}} + \frac{(\sqrt{2}\pi-4)(1+r')}{2(\sqrt{2}-1)\pi}\right],\\
J_3(r) &= \frac{\pi}{2}\left(\frac{2-r^2}{2}\right)^{1/2},\\
J_4(r) &= \frac{\pi}{2} - \frac{1}{2}\log\frac{(1+r)^{1-r}}{(1-r)^{1+r}},\\
J_5(r) &= \frac{\pi-1}{2} + \frac{1-r^2}{4r}\log\frac{1+r}{1-r},\\
I_1(r) &= 2^{-7/4}\pi(1+r'^2)^{1/4}(1+r')^{1/2}
\end{aligned}$$

Table 1. Comparisons of $J_1(r)$ with $J_4(r)$, and $J_2(r)$ with $J_3(r)$ and $J_5(r)$ for some $r \in (0, 1)$.

r	$J_1(r)$	$J_4(r)$	$J_2(r)$	$J_3(r)$	$J_5(r)$
0.1	1.5668···	1.4699···	1.566862···	1.566864···	1.5674···
0.2	1.5549···	1.3639···	1.5549···	1.5550···	1.5573···
0.3	1.5348···	1.2471···	1.5349···	1.5350···	1.5402···
0.4	1.5059···	1.1122···	1.5061···	1.5066···	1.5156···
0.5	1.4674···	0.9495···	1.4680···	1.4693···	1.4827···
0.6	1.4180···	0.7437···	1.4194···	1.4224···	1.4404···
0.7	1.3555···	0.4678···	1.3585···	1.3648···	1.3867···

Table 2. Comparisons of $I_1(r)$ with $J_4(r)$, and $I_2(r)$ with $J_3(r)$ and $J_5(r)$ for some $r \in (0, 1)$.

r	$I_1(r)$	$J_4(r)$	$I_2(r)$	$J_3(r)$	$J_5(r)$
0.1	$1.5668\cdots$	$1.4699\cdots$	$1.566862\cdots$	$1.566864\cdots$	$1.5674\cdots$
0.2	$1.5549\cdots$	$1.3639\cdots$	$1.5549\cdots$	$1.5550\cdots$	$1.5573\cdots$
0.3	$1.5348\cdots$	$1.2471\cdots$	$1.5349\cdots$	$1.5350\cdots$	$1.5402\cdots$
0.4	$1.5059\cdots$	$1.1122\cdots$	$1.5062\cdots$	$1.5066\cdots$	$1.5156\cdots$
0.5	$1.4674\cdots$	$0.9495\cdots$	$1.4682\cdots$	$1.4693\cdots$	$1.4827\cdots$
0.6	$1.4180\cdots$	$0.7437\cdots$	$1.4197\cdots$	$1.4224\cdots$	$1.4404\cdots$
0.7	$1.3555\cdots$	$0.4678\cdots$	$1.3592\cdots$	$1.3648\cdots$	$1.3867\cdots$

and

$$I_2(r) = (1 + r'^2)^{2-\log\pi/\log 2}(1 + r')^{2\log\pi/\log 2-3},$$

then we have tables 1 and 2 via elementary computation.

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