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Generalizations of some zero sum theorems

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Abstract. Given an abelian group G of order n, and a finite non-empty subset A of integers, the *Davenport constant of G with weight A*, denoted by $D_A(G)$, is defined to be the least positive integer t such that, for every sequence $(x_1, \ldots, x_t)$ with $x_i \in G$, there exists a non-empty subsequence $(x_{j_1}, \ldots, x_{j_l})$ and $a_i \in A$ such that $\sum_{i=1}^{l} a_i x_{j_i} = 0$. Similarly, for an abelian group G of order n, $E_A(G)$ is defined to be the least positive integer t such that every sequence over G of length t contains a subsequence $(x_{j_1}, \ldots, x_{j_n})$ such that $\sum_{i=1}^{n} a_i x_{j_i} = 0$, for some $a_i \in A$. When G is of order n, one considers A to be a non-empty subset of $\{1, \ldots, n-1\}$. If G is the cyclic group $\mathbb{Z}/n\mathbb{Z}$, we denote $E_A(G)$ and $D_A(G)$ by $E_A(n)$ and $D_A(n)$ respectively.

In this note, we extend some results of Adhikari *et al* (*Integers* **8** (2008) Article A52) and determine bounds for $D_{R_n}(n)$ and $E_{R_n}(n)$, where $R_n = \{x^2 : x \in (\mathbb{Z}/n\mathbb{Z})^*\}$. We follow some lines of argument from Adhikari *et al* (*Integers* **8** (2008) Article A52) and use a recent result of Yuan and Zeng (*European J. Combinatorics* **31** (2010) 677–680), a theorem due to Chowla (*Proc. Indian Acad. Sci. (Math. Sci.)* **2** (1935) 242–243) and Kneser's theorem (*Math. Z.* **58** (1953) 459–484; **66** (1956) 88–110; **61** (1955) 429–434).

Keywords. Weighted zero sum problems; Davenport constant; EGZ theorem.

1. Introduction

Let G be an abelian group of order n, written additively. The *Davenport constant* $D(G)$ is defined to be the smallest natural number t such that any sequence of length t over G has a non-empty subsequence whose sum is zero. Another combinatorial invariant $E(G)$ (known as *the EGZ constant*) is the smallest natural number t such that any sequence of length t over G has a subsequence of length $|G|$ whose sum is zero. A classical theorem of Erdős *et al.* [6] says that $E(\mathbb{Z}/n\mathbb{Z}) = 2n - 1$. These two constants are related by a theorem of Gao [7], which states that $E(G) = D(G) + n - 1$.

Generalizations of the constants $E(G)$ and $D(G)$ with weights were considered in [2] and [4] for finite cyclic groups. Later in [1], generalizations for an arbitrary finite abelian group G were introduced. Given an abelian group G of order n, and a finite non-empty subset A of integers, the *Davenport constant of G with weight A*, denoted by $D_A(G)$, is defined to be the least positive integer t such that, for every sequence $(x_1, \ldots, x_t)$ with $x_i \in G$, there exists a non-empty subsequence $(x_{j_1}, \ldots, x_{j_l})$ and $a_i \in A$ such that $\sum_{i=1}^{l} a_i x_{j_i} = 0$. Similarly, for an abelian group G of order n, $E_A(G)$ is defined to be the least positive integer t such that every sequence over G of length t contains a subsequence

$(x_{j_1}, \ldots, x_{j_n})$ such that $\sum_{i=1}^{n} a_i x_{j_i} = 0$, for some $a_i \in A$. When G is of order n, one may consider A to be a non-empty subset of $\{0, 1, \ldots, n-1\}$ and for obvious reasons one assumes that $0 \notin A$. If G is the cyclic group $\mathbb{Z}/n\mathbb{Z}$, we denote $E_A(G)$ and $D_A(G)$ by $E_A(n)$ and $D_A(n)$ respectively.

Adhikari *et al.* [3] considered the problem of determining the values of $D_{R_n}(n)$ and $E_{R_n}(n)$, where $R_n = \{x^2 : x \in (\mathbb{Z}/n\mathbb{Z})^*\}$ and $(\mathbb{Z}/n\mathbb{Z})^*$ is the group of units modulo n. The case $n = p$, a prime has already been dealt with in [4]. In this note, we will be extending some results from [3].

In what follows, for a positive integer n, $\Omega(n)$ (resp. $\omega(n)$) denotes the number of prime factors of n counted with multiplicity (resp. without multiplicity).

We shall prove the following theorems.

Theorem 1. *Let $n = 3^\alpha$. Then we have*

(i) $D_{R_n}(n) = 2\Omega(n) + 1$, *and*
(ii) $E_{R_n}(n) = n + 2\Omega(n)$.

Theorem 2. *Let $n = 2^\alpha$, $\alpha \geq 3$. Then we have $D_{R_n}(n) \leq 7\Omega(n) + 1$ and $E_{R_n}(n) \leq n + 7\Omega(n)$.*

Theorem 3. *Let $n = 5^l \prod_{i=2}^{k} p_i^{\alpha_i}$, where $l, \alpha_i \geq 0$ with $p_i \geq 7$ primes for each $i \in \{2, \ldots, k\}$. Let $m \geq 3\omega(n) + 1$ and $S = (x_1, x_2, \ldots, x_{m+2\Omega(n)+l})$ be a sequence of length $m + 2\Omega(n) + l$ over $\mathbb{Z}$. Then there exists a subsequence $(x_{i_1}, x_{i_2}, \ldots, x_{i_m})$ and $a_1, a_2, \ldots, a_m \in R_n$ such that $\sum_{j=1}^{m} a_j x_{i_j} \equiv 0 \pmod{n}$. In particular,*

$$E_{R_n}(n) \leq n + 2\Omega(n) + l.$$

As a consequence of the above theorem, we have the following:

Remark 1. Let n be an integer such that $\gcd(30, n) = 1$. Then, combining Theorem 3 with Theorem 1 from [3], we get $E_{R_n}(n) = n + 2\Omega(n)$. Then, using Theorem A (which will be stated in the next section), we get $D_{R_n}(n) = 2\Omega(n) + 1$.

2. Notations and preliminaries

First, we state some known results which we shall be using.

As conjectured in [4], a result similar to the result of Gao [7], the link between the constants $E_A(n)$ and $D_A(n)$ was established by Yuan and Zeng [15].

Theorem A [15]. *Let A be a finite non-empty subset of integers and n a positive integer. We have*

$$E_A(n) = D_A(n) + n - 1.$$

It should be remarked that the corresponding generalization of the above result for arbitrary finite abelian groups has been established by Grynkiewicz *et al.* [9].

We shall need the following theorem due to Chowla (see [5] and [14]).

Theorem B. *Let n be a natural number, and let A and B be two non-empty subsets of $\mathbb{Z}/n\mathbb{Z}$ such that $0 \in B$ and $A + B \neq \mathbb{Z}/n\mathbb{Z}$. If* $\gcd(x, n) = 1$ *for all* $x \in B \setminus \{0\}$, *then* $|A + B| \geq |A| + |B| - 1$.

For a non-empty subset A of an abelian group G, the *stabilizer* of A, denoted by $\mathrm{Stab}(A)$, is defined as follows:

$$\mathrm{Stab}(A) = \{x \in G : x + A = A\}.$$

Note that, if $\phi : G \to G/H$ is the canonical map with $H = \mathrm{Stab}(A)$, then $\phi(A)$ has a trivial stabilizer. We shall also need the following generalization of Theorem B due to Kneser [11–13] (for the statement in the following form, one may refer [14] or [8]).

Theorem C. *Let G be an abelian group, and let A and B be finite, non-empty subsets of G. Let* $H = \mathrm{Stab}(A + B)$. *Then*

$$|A + B| \geq |A + H| + |B + H| - |H|.$$

From Theorem C, the following can be easily deduced.

Theorem D. *Let G be an abelian group, and let $A_1, A_2, \ldots, A_k$ be k finite, non-empty subsets of G. Let* $H = \mathrm{Stab}(A_1 + A_2 + \cdots + A_k)$. *Then*

$$|A_1 + A_2 + \cdots + A_k| \geq |A_1 + H| + |A_2 + H| + \cdots + |A_k + H| - (k-1)|H|.$$

In what follows, for a positive integer n, we shall denote the set $\{1, 2, 3, \ldots, n\}$ by the symbol $[n]$.

If $n = p^a$, with p an odd prime and $a \in \mathbb{N}$, then $(\mathbb{Z}/n\mathbb{Z})^*$ is a (multiplicative) cyclic group (see [10]). Let x be a generator. Then clearly $R_n = \langle x^2 \rangle$. Hence $|R_n| = \mathrm{ord}(x^2) = \phi(n)/2$.

If $n = 2^a$, $a \in \mathbb{N}$ and $a \geq 3$, then $(\mathbb{Z}/n\mathbb{Z})^*$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2^{a-2}\mathbb{Z}$ (see [10]). Let x be a generator of $\mathbb{Z}/2^{a-2}\mathbb{Z}$. Then clearly $(0, 2x)$ is a generator of R_n. So $|R_n| = \mathrm{ord}((0, 2x)) = 2^{a-3}$.

We will be using following remark several times in this note.

Remark 2. Let $m, n \in \mathbb{N}$ with $m|n$. Let $\phi : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ be a surjective ring homomorphism. Then $(\mathbb{Z}/m\mathbb{Z})^* = \phi((\mathbb{Z}/n\mathbb{Z})^*)$ and $R_m = \phi(R_n)$.

The above remark can be observed as follows. Since ϕ is an abelian group homomorphism, it is completely determined by where it maps an additive generator, say the identity 1_n. On the other hand, ϕ being a surjective ring homomorphism, it must take the multiplicative identity to the multiplicative identity. Thus we have $\phi(1_n) = 1_m$, where 1_m is the multiplicative identity of $\mathbb{Z}/m\mathbb{Z}$. Hence for any $a \cdot 1_n \in \mathbb{Z}/n\mathbb{Z}$, we have $\phi(a \cdot 1_n) = a \cdot 1_m$. Therefore, ϕ is just reduction modulo m. Since $(\mathbb{Z}/n\mathbb{Z})^*$ is precisely the subset of all the generators of $\mathbb{Z}/n\mathbb{Z}$ (as an abelian group), we have $\phi((\mathbb{Z}/n\mathbb{Z})^*) \subset (\mathbb{Z}/m\mathbb{Z})^*$. Now it remains to show that for any $\alpha \cdot 1_m \in (\mathbb{Z}/m\mathbb{Z})^*$ there exists some $\beta \cdot 1_n \in (\mathbb{Z}/n\mathbb{Z})^*$ with $\alpha \cdot 1_m = \phi(\beta \cdot 1_n) = \beta \cdot 1_m$. This is equivalent to showing that, for any number $\alpha \in \mathbb{Z}$ that is relatively prime to m, there must be some $\alpha + mx$, where $x \in \mathbb{Z}$, that is relatively prime to n. If the prime divisors of n are a subset of those of m, then $\gcd(m, \alpha) = 1$ implies $\gcd(n, \alpha) = 1$, as needed. Let p be a prime divisor of n not dividing m. If $\alpha + xm \equiv 0 \pmod{p}$, for all $x \in \mathbb{Z}$, then $\alpha + xm \equiv \alpha + (x+1)m \pmod{p}$, which implies $m \equiv 0 \pmod{p}$, contradicting that p does not divide m. Thus we can find $x \in \mathbb{Z}$ such that $\alpha + xm$ is not divisible by p. Iterating this procedure for the remaining prime divisors of n but not m (replacing α by $\alpha + xm$ and m by pm each time) yields the integer with the needed properties. Thus $\phi((\mathbb{Z}/n\mathbb{Z})^*) = (\mathbb{Z}/m\mathbb{Z})^*$. Now one can easily observe that $\phi(R_n) = R_m$.

3. Proof of our theorems

Lemma 1. *Let $p \geq 7$ be any prime and $n = p^{\alpha}$. Then, given $x_1, x_2, x_3 \in (\mathbb{Z}/n\mathbb{Z})^*$, we have*

$$R_n x_1 + R_n x_2 + R_n x_3 = \mathbb{Z}/n\mathbb{Z}.$$

Proof. Let $H = \mathrm{Stab}\,(R_n x_1 + R_n x_2 + R_n x_3)$. Clearly, the quotient group $(\mathbb{Z}/n\mathbb{Z})/H$ is cyclic, say $\mathbb{Z}/m\mathbb{Z}$, where $m = p^{\beta},\ \beta \leq \alpha$. Consider $\phi : (\mathbb{Z}/n\mathbb{Z}) \to (\mathbb{Z}/m\mathbb{Z})$ to be the natural homomorphism with kernel H. Since $\phi(R_n) = R_m$, we have

$$\phi\left(\sum_{i=1}^{3} R_n x_i\right) = \sum_{i=1}^{3} R_m \phi(x_i).$$

As observed in § 2, we have $\mathrm{Stab}\left(\sum_{i=1}^{3} R_m \phi(x_i)\right) = \{\phi(0)\}$.

Observe that, as each of the x_i's generates $\mathbb{Z}/n\mathbb{Z}$, we have $\langle \phi(x_i) \rangle = \mathbb{Z}/m\mathbb{Z}$, for each $i = 1, 2, 3$. Applying Kneser's theorem (Theorem D), we get

$$\begin{aligned}\left|\sum_{i=1}^{3} R_m \phi(x_i)\right| &\geq 3|R_m| - 2\\ &= \frac{3(p^{\beta} - p^{\beta-1})}{2} - 2\\ &\geq p^{\beta}.\end{aligned}$$

Thus, $\sum_{i=1}^{3} R_{p^{\beta}} \phi(x_i) = \mathbb{Z}/p^{\beta}\mathbb{Z}$. Therefore, $\mathrm{Stab}\,(\sum_{i=1}^{3} R_{p^{\beta}} \phi(x_i)) = \mathbb{Z}/p^{\beta}\mathbb{Z}$. Since $H = \mathrm{Stab}\,(R_n x_1 + R_n x_2 + R_n x_3)$, we get $\mathrm{Stab}\,(\phi(R_n x_1 + R_n x_2 + R_n x_3)) = \{0\}$. That is, $\mathbb{Z}/p^{\beta}\mathbb{Z} = \{0\}$. Therefore, $H = \mathbb{Z}/p^{\alpha}\mathbb{Z}$. Hence $R_n x_1 + R_n x_2 + R_n x_3 = \mathbb{Z}/n\mathbb{Z}$. □

The proof of the following lemma is similar to that of the above.

Lemma 2. *Let $n = 5^{\alpha}$. Given $x_1, x_2, x_3, x_4 \in (\mathbb{Z}/n\mathbb{Z})^*$, we have*

$$R_n x_1 + R_n x_2 + R_n x_3 + R_n x_4 = \mathbb{Z}/n\mathbb{Z}.$$

Proof of Theorem 1. In view of Theorem 1 of [3], for part (i) of the theorem we have only to prove that $D_{R_n}(n) \leq 2\Omega(n) + 1$.

Consider a sequence $x_1, x_2, \ldots, x_{2\Omega(n)+1}$ of elements of $(\mathbb{Z}/n\mathbb{Z}) \setminus \{0\}$. Observe that there are three elements, say $x_1,\ x_2$ and x_3, such that $3^r | x_i$ but 3^{r+1} does not divide x_i for $i = 1, 2, 3$ and some $r \in \{0, 1, \ldots, \alpha - 1\}$. Put $y_i = x_i/3^r$. By repeated application of Theorem B of Chowla, we see that

$$\begin{aligned}
&|(R_n y_1 + R_n y_2 \cup \{0\}) + R_n y_3 \cup \{0\}| \\
&\quad \geq \min\{n, |R_n y_1 + R_n y_2 \cup \{0\}| + |R_n y_3 \cup \{0\}| - 1\} \\
&\quad \geq \min\{n, \min\{n, 2|R_n|\} + |R_n y_3 \cup \{0\}| - 1\} \\
&\quad = \min\{n, 3|R_n|\} \\
&\quad = \min\left\{n, \frac{3(3^\alpha - 3^{\alpha-1})}{2}\right\} \\
&\quad = n.
\end{aligned}$$

Looking at the set $(R_n y_1 + R_n y_2 \cup \{0\}) + R_n y_3 \cup \{0\}$ and Theorem B of Chowla, one sees that the reason behind including 0 in $R_n y_2$ and $R_n y_3$ is just to get the setting in which we can apply Theorem B. Since y_2 and y_3 are coprime to n, $0 \notin R_n y_2 \cup R_n y_3$. So including 0 increases the cardinality of both the sets $R_n y_2$ and $R_n y_3$ by 1.

From the above inequality, it follows that $0 \in (R_n y_1 + R_n y_2 \cup \{0\}) + R_n y_3 \cup \{0\}$. Hence, $D_{R_n}(n) \leq 2\Omega(n) + 1$, as desired. Part (ii) follows from part (i) and Theorem A. □

Proof of Theorem 2. Observe that, by the structure of $(\mathbb{Z}/n\mathbb{Z})^*$, we get $|R_n| = 2^{\alpha-3}$. Observe that any sequence of length $7\Omega(n) + 1$ over $(\mathbb{Z}/n\mathbb{Z}) \setminus \{0\}$ contains 8 terms such that k is the exact power of 2 dividing these 8 terms for some $k \in \{0, 1, 2, \ldots, \alpha - 1\}$. Now we get the result by arguments similar to that employed in the above theorem. □

Proof of Theorem 3. We shall prove the theorem by induction on $\Omega(n)$. Suppose $\Omega(n) = 1$. Therefore, $n = p$, a prime.

First, suppose that $p = 5$. In this case, $m \geq 3(1) + 1 = 4$ and $S = (x_1, x_2, \ldots, x_{m+3})$. If there are at least four non-zero terms modulo 5 in the given sequence, then by Lemma 2 we shall get an R_n-weighted zero sum subsequence modulo 5 of length m. If the sequence does not have more than three non-zero terms modulo 5, then the sequence has at least m terms which are zero modulo 5 and so we are through.

Suppose $p \neq 5$. If the sequence contains at least three non-zero terms modulo p, then by Lemma 1 we shall get an R_n-weighted zero sum subsequence modulo p of length m. Otherwise, at most two terms of the sequence are units modulo p, which implies that at least m terms are divisible by p and we are through.

Thus the result is established for the case $\Omega(n) = 1$.

Now, assume that $\Omega(n) > 1$ and that the result holds for all N with $\Omega(N) < \Omega(n)$.

Case 1. Suppose there exists a prime divisor $p_t \neq 5$ of n such that number of terms in S which are coprime to p_t is at most 2. Let S_1 be the subsequence of S obtained by removing these terms. Clearly, the length of S_1 is at least $m + 2\Omega(n/p_t) + l$. Since $m \geq 3\omega(n/p_t) + 1$, by the induction hypothesis, we get a subsequence $(x_{i_1}, x_{i_2}, \ldots, x_{i_m})$ of S_1 such that

$$\sum_{j=1}^{m} a_j \frac{x_{i_j}}{p_t} \equiv 0 \pmod{n/p_t}, \quad \text{with } a_j \in R_{n/p_t}.$$

Since $m = n/p_t$ divides n, define $\phi : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ to be the reduction modulo m. By Remark 2, we can see that $\phi(R_n) = R_{n/p_t}$. Therefore, for each $j \in \{1, 2, \ldots, m\}$, there exists $a'_j \in R_n$ such that $\phi(a'_j) = a_j$. That is, $a'_j \equiv a_j$ modulo n/p_t. So

$$\sum_{j=1}^{m} a'_j \frac{x_{i_j}}{p_t} \equiv 0 \pmod{n/p_t}, \quad \text{with } a'_j \in R_n.$$

Therefore,

$$\sum_{j=1}^{m} a'_j x_{i_j} \equiv 0 \pmod{n}.$$

Case 2. Suppose the sequence contains at most three units modulo 5. Let S_1 be the sequence obtained by removing these terms from S. Clearly, the length of S_1 is at least $m + 2\Omega(n/5) + l - 1$. Since $m \geq 3\omega(n/5) + 1$, by applying the induction hypothesis, we get $\sum_{j=1}^{m} a_j \frac{x_{i_j}}{5} \equiv 0 \pmod{n/5}$, where $a_j \in R_{n/5}$. As in Case 1, using Remark 2, we get $a'_j \in R_n$ such that $\sum_{j=1}^{m} a'_j x_{i_j} \equiv 0 \pmod{n}$.

Case 3. Suppose the sequence contains at least four units modulo 5 and at least three units modulo p_t for each t. Then by Lemmas 1 and 2, we get a subsequence $(x_{i_1}, x_{i_2}, \ldots, x_{i_m})$ of S such that $\sum_{j=1}^{m} a_j^{(1)} x_{i_j} \equiv 0 \pmod{5^l}$ and $\sum_{j=1}^{m} a_j^{(i)} x_{i_j} \equiv 0 \pmod{p_i^{\alpha_i}}$, for some $a_j^{(i)} \in R_{p_i^{\alpha_i}}$ and $a_j^{(1)} \in R_{5^l}$. Since $a_j^{(1)} \in R_{5^l}$, there exists $b_j^{(1)} \in (\mathbb{Z}/5^l\mathbb{Z})^*$ such that $(b_j^{(1)})^2 \equiv a_j^{(1)} \pmod{5^l}$, for each $j \in \{1, 2, \ldots, m\}$. Similarly, there exists $b_j^{(i)} \in (\mathbb{Z}/p_i^{\alpha_i}\mathbb{Z})^*$ such that $(b_j^{(i)})^2 \equiv a_j^{(i)} \pmod{p_i^{\alpha_i}}$, for each $i \in \{2, 3, \ldots, k\}$ and $j \in \{1, 2, \ldots, m\}$. Now, for each $j \in \{1, 2, \ldots, m\}$ consider the following system of equations:

$$\begin{aligned} Y_j &\equiv b_j^{(1)} \pmod{5^l}, \\ Y_j &\equiv b_j^{(i)} \pmod{p_i^{\alpha_i}}, \ i \in \{2, 3 \cdots, k\}. \end{aligned}$$

By the Chinese Remainder theorem, for each $j \in \{1, 2, \ldots, m\}$, we get the unique solution Y_j modulo n. Thus, $\sum_{j=1}^{m} (Y_j)^2 x_{i_j} \equiv 0 \pmod{n}$. Hence we are through. □

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