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On the existence of hydrodynamic instability in single diffusive bottom heavy systems with permeable boundaries

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Abstract. We utilize the reformulated equations of the classical theory, as derived by Banerjee *et al.* (*J. Math. Anal. Appl.* **175** (1993) 458), to establish mathematically, the existence of hydrodynamic instability in single diffusive bottom heavy systems, when considered in the more general framework of the boundary conditions of the type specified by Beavers and Joseph (*J. Fluid Mech.* **30** (1967) 197), in the parameter regime $T_0\alpha_2 > 1$, where T_0 and α_2 being some properly chosen mean temperature and coefficient of specific heat (at constant volume) variation due to temperature variation respectively.

Keywords. Convection; specific heat; permeable; hydrodynamic instability.

1. Introduction

Banerjee *et al.* [1] have reformulated the fundamental equations of Rayleigh's [8] classical theory, by taking into account the fact that the theoretical explanation of the linear problem of gravity-dominated thermal instability in a horizontal liquid layer heated underside (or overside), also known as simple Bénard instability [3, 4], should depend not only upon the Rayleigh number which is proportional to the uniform temperature difference maintained across the layer but also upon another parameter that arises due to the variation in the specific heat at constant volume on account of the variations in temperature.

In this paper, we prove the existence of hydrodynamic instability in a single diffusive bottom heavy system, with permeable boundaries at which the boundary condition of Beavers and Joseph [2] is applicable. The characteristic value problem is solved using the technique of Chandrasekhar [5], for situations with more general nature of the bounding surfaces. Further, it is observed that the limiting cases of the boundary parameters K_0 and K_1 characterizing the permeable nature of the lower and upper boundary respectively, give rise to the particular cases, namely, when both the bounding surfaces are either dynamically free ($K_0 \to 0$, $K_1 \to 0$) or rigid ($K_0 \to \infty$, $K_1 \to \infty$), and either one of them is dynamically free ($K_0 \to 0$ or $K_1 \to 0$) while the other is rigid ($K_0 \to \infty$ or $K_1 \to \infty$).

2. The eigenvalue problem in nondimensional form

A viscous finitely heat conducting modified Boussinesq liquid of infinite horizontal extension and finite vertical depth is statically confined between two horizontal boundaries at

$z = 0$ and $z = d$ which are respectively maintained at uniform temperatures T_0 and T_1. We choose a Cartesian coordinate system with the x and y axes in the plane of the lower boundary and the positive direction of the z-axis along the vertically upward direction. Further, both the bounding surfaces are assumed to be permeable at which the boundary condition of Beavers and Joseph [2] is applicable. We mathematically analyse the onset of hydrodynamic instability in the system under the force field of gravity. The nondimensional form of the modified governing linearized perturbation equations, which govern the initiation of thermal convection, are given by Banerjee *et al.* [1] as

$$(D^2 - a^2)\left(D^2 - a^2 - \frac{p}{\sigma}\right) w = \theta, \tag{1}$$

$$(D^2 - a^2 - p)\theta = -Ra^2\,(1 - \alpha_2 T_0)\,w, \tag{2}$$

where w is the z-component of the perturbation velocity, θ is the temperature perturbation, a is the wave number, σ is the Prandtl number, R is the Rayleigh number, $p = p_{\mathrm{r}} + ip_{\mathrm{i}}$ is a complex constant in general and represents the complex growth rate, p_{r} and p_{i} being real constants, and $D = \mathrm{d}/\mathrm{d}z$.

Since both the boundaries are fixed and maintained at constant temperatures, we must have

$$w = 0 \quad \text{and} \quad \theta = 0 \quad \text{at}\, z = 0\, \text{and}\, z = 1. \tag{3}$$

Further, the boundary condition of Beavers and Joseph [2] is that, at the permeable boundary the normal derivative of the tangential velocity is proportional to that velocity. If the direction of the normal is taken to be into the fluid then the constant of proportionality is positive. On making use of the equation of continuity, we have

$$D^2 w - K_0 D_w = 0 \quad \text{at}\, z = 0 \tag{4}$$

and

$$D^2 w + K_1 D w = 0 \quad \text{at}\, z = 1, \tag{5}$$

where K_0 and K_1 are positive constants, characterizing the permeable nature of the lower and upper boundary respectively, and the plus sign of the boundary parameter K_1 arises because at $z = 1$ the normal into the fluid is in the negative z-direction.

Equations (1) and (2) together with boundary conditions (3)–(5) pose a double eigenvalue problem for p for prescribed values of a^2, R, $\alpha_2 T_0$, K_0 and K_1 and a given normal mode is stable, neutral or unstable provided that the real part p_{r} of p is negative, zero or positive respectively. Further, the marginal state of the configuration is defined by $p_{\mathrm{r}} = 0$, and if $p_{\mathrm{r}} = 0$ implies $p_{\mathrm{i}} = 0$ for every wave number then the ensuing thermal convection is neutral and the 'principle of exchange of stabilities' is valid. Otherwise we will have overstability at least when instability sets in as a certain mode.

3. The marginal state and solution of the problem

Case 1. $1 - \alpha_2 T_0 > 0$ *with* $R > 0$. When $1 - \alpha_2 T_0 > 0$ with $R > 0$, the governing equations and boundary conditions imply that we have a modified simple Bénard problem with permeable boundaries, wherein a liquid is heated underside, in the parameter regime

$\alpha_2 T_0 < 1$. Further, the technique of Pellew and Southwell [7] for the characterization of the marginal state is applicable to equations (1), (2) and boundary condition (3)–(5) with the following result:

Theorem 1. *If* $1 - \alpha_2 T_0 > 0$ *with* $R > 0$, *a necessary condition for the existence of nontrivial solutions for* w *and* θ *satisfying equations* (1), (2) *and boundary conditions* (3)–(5) *is that*

$$p_{\rm i} = 0. \tag{6}$$

Proof. Multiplying eq. (1) throughout by w^* (the complex conjugate of w), and integrating the resulting equation over the vertical range of z, and substituting for $\int_0^1 w^* \theta {\rm d}z$ in this equation from eq. (2), we then integrate each term of the equation so obtained, by parts, for a suitable number of times with the help of boundary conditions (3)–(5) and derive from the imaginary part of the integrated equation

$$p_{\rm i}\left[\frac{1}{\sigma}\int_0^1(|Dw|^2 + a^2|w|^2){\rm d}z + \frac{R^{-1}a^{-2}}{1-\alpha_2 T_0}\int_0^1|\theta|^2{\rm d}z\right] = 0. \tag{7}$$

From eq. (7), it follows that $p_{\rm i} = 0$.

This implies that the 'principle of exchange of stabilities' is valid for the problem under consideration and hence the marginal state is characterized by $p = 0$. In this case, equations (1), (2) and boundary conditions (3)–(5) can be treated as an eigenvalue problem in R for given values of a^2, $\alpha_2 T_0$, K_0 and K_1. Proceeding exactly along the same lines as given in Chandrasekhar [5], in the first approximation we get the following results: □

Theorem 2. *If* $1 - \alpha_2 T_0 > 0$ *with* $R > 0$ *and* $p = 0$, *a nontrivial solution for* w *and* θ *satisfying equations* (1), (2) *and boundary conditions* (3)–(5) *implies that the Rayleigh number* R *is given, in terms of* a, $\alpha_2 T_0$, K_0 *and* K_1, *by*

$$R = \frac{P^3}{a^2(1-\alpha_2 T_0)} \times \left[1 - \frac{8\pi^2 a\{K_0 K_1 (S_a - a)(1 + C_a) + (K_0 + K_1)aS_a^2\}}{P^2\{K_0 K_1 (S_a^2 - a^2) + 2(K_0 + K_1)(aS_aC_a - a^2) + 4a^2 S_a^2}\right]^{-1}, \tag{8}$$

where $P = \pi^2 + a^2$, $S_a = \sinh a$ *and* $C_a = \cosh a$.

For given values of K_0, K_1 and $\alpha_2 T_0$, eq. (8) gives R as a function of the wave number a. The minimum of R as a varies is the critical Rayleigh number $R_{\rm c}$ and the value of a at which R attains the minimum is the critical wave number $a_{\rm c}$.

Remark 1. It is easily seen from eqs (1), (2) and boundary conditions (3)–(5) that for the case when $K_0 \to 0$ and $K_1 \to 0$ we have governing equations for the modified simple Bénard problem with both boundaries free and in this case we find from eq. (8) that

$$R = \frac{(\pi^2 + a^2)^3}{a^2(1-\alpha_2 T_0)} \tag{9}$$

and this expression gives the critical Rayleigh number $R_c = 27\pi^4/4\,(1-\alpha_2 T_0)$ identical with that obtained by Banerjee *et al.* [1].

Remark 2. For the case when $K_0 \to \infty$, $K_1 \to \infty$, we find that eqs (1), (2) and boundary conditions (3)–(5) coincide with the governing equations for the modified simple Bénard problem with both boundaries rigid and in this case we find from eq. (8) that

$$R = \frac{(\pi^2+a^2)^3}{a^2(1-\alpha_2 T_0)}\left[1 - \frac{16\pi^2 a\cosh^2(a/2)}{(\pi^2+a^2)^2\,(S_a+a)}\right]^{-1} \tag{10}$$

and this expression gives $R_c = 1715.08/(1-\alpha_2 T_0)$. When $\alpha_2 T_0 = 0$, the value $R_c = 1715.08$ is the same as that obtained by Reid and Harris [9] in the first approximation. The value of $R_c = 1715.08/(1-\alpha_2 T_0)$ obtained by us, in the first approximation, is quite close to the exact value of $R_c = 1707.76/(1-\alpha_2 T_0)$ obtained by Banerjee *et al.* [1].

Remark 3. For the case when either $K_0 \to 0$ and $K_1 \to \infty$ or $K_0 \to \infty$ and $K_1 \to 0$, we find that eqs (1), (2) and boundary conditions (5) coincide with the governing equations for the modified simple Bénard problem when either one of them is dynamically free while the other is rigid and in this case we find from eq. (8) that

$$R = \frac{(\pi^2+a^2)^3}{a^2\,(1-\alpha_2 T_0)}\left[1 - \frac{4\pi^2 a\,S_a^2}{(\pi^2+a^2)^2\,(S_a\,C_a - a)}\right]^{-1} \tag{11}$$

and this expression gives $R_c = 1112.07/(1-\alpha_2 T_0)$, in the first approximation, which is also quite close to the exact value of $R_c = 1100.65/(1-\alpha_2 T_0)$ obtained by Banerjee *et al.* [1].

Remark 4. For the case when $\alpha_2 T_0 = 0$ and $K_0 = K_1 = K$, we find that eqs (1), (2) and boundary conditions (3)–(5) coincide with the classical equations for the simple Bénard problem with identical permeable boundaries, and in this case we find from eq. (8) that

$$R = \frac{P^3}{a^2}\left[1 - \frac{8\pi^2 aK\{K(S_a-a)(1+C_a)+2aS_a^2\}}{P^2\{K^2(S_a^2-a^2)+4K(aS_a\,C_a-a^2)+4a^2S_a^2\}}\right]^{-1} \tag{12}$$

and this expression for the Rayleigh number is identical with that obtained by Nield [6].

Case 2. $1-\alpha_2 T_0 > 0$ *with* $R < 0$. When $1-\alpha_2 T_0 > 0$ with $R < 0$, the governing equations and boundary conditions imply that we have a modified simple Bénard problem with permeable boundaries, wherein a liquid is heated overside, in the parameter regime $\alpha_2 T_0 < 1$. Further, the stability of the system can be established along the classical lines as given in Chandrasekhar [5] so that any oscillation which exists in the system must decay. We have the following theorem:

Theorem 3. *If* $1-\alpha_2 T_0 > 0$ *with* $R < 0$, *a necessary condition for the existence of nontrivial solutions for w and* θ *satisfying eqs* (1), (2) *and boundary conditions* (3)–(5) *is that*

$$p_r < 0. \tag{13}$$

Proof. Multiplying eq. (1) throughout by w^* (the complex conjugate of w), and integrating the resulting equation over the vertical range of z, and substituting for $\int_0^1 w^*\theta \mathrm{d}z$ in this equation from eq. (2), we then integrate each term of the equation so obtained by parts for a suitable number of times with the help of boundary conditions (3)–(5) and derive from the imaginary part of the integrated equation

$$K_1(|Dw|^2)_1 + K_0(|Dw|^2)_0 + \int_0^1 (|D^2w|^2 + 2a^2|Dw|^2 + a^4|w|^2)\mathrm{d}z$$
$$+ p_\mathrm{r}\left[\frac{1}{\sigma}\int_0^1 (|Dw|^2 + a^2|w|^2)\mathrm{d}z\right] = \frac{R^{-1}a^{-2}}{1-a_2T_0}\int_0^1 (|D\theta|^2 + a^2|\theta|^2 + p_\mathrm{r}|\theta|^2)\mathrm{d}z, \tag{14}$$

where $(|Dw|^2)_0$ and $(|Dw|^2)_1$ are values of $|Dw|^2$ at the lower and upper boundary surfaces respectively. From eq. (14), it follows that $p_\mathrm{r} < 0$.

Theorem 3 establishes the stability of the system when the liquid layer under consideration is heated from above. It may, however, be remarked that the usual physical circumstances are characterized by parameter regime $1 - \alpha_2 T_0 > 0$ and it is only in this parameter regime that the above results are valid. □

Case 3. $1 - \alpha_2 T_0 < 0$ *with* $R < 0$. When $1 - \alpha_2 T_0 < 0$ with $R < 0$, the governing equations and boundary conditions imply that we have a simple Bénard problem with permeable boundaries, wherein a liquid is heated overside, in the parameter regime $\alpha_2 T_0 > 1$. Since R is negative the initial distribution of density is bottom heavy and therefore statically gravitationally stable. This stabilizing effect together with the joint stabilizing effect of viscosity and conduction is expected to impart, in the usual parameter regime characterized by $1 - \alpha_2 T_0 > 0$ an overall stabilizing effect to the system. That this is really the case is borne out by Theorem 3 wherein the stability of the system is proved in such situations. The nature of the problem, however, is completely different in the regime $1 - \alpha_2 T_0 < 0$ in which case we have with $R < 0$, $R(1 - \alpha_2 T_0) > 0$. This in turn introduces a new instability into the system as $|R|$ goes beyond a critical value with the 'principle of exchange of stabilities' being valid at the marginal state.

We now prove the following two theorems to show the existence of this new instability.

Theorem 4. *If* $1 - \alpha_2 T_0 < 0$ *with* $R < 0$, *a necessary condition for the existence of nontrivial solutions for w and θ satisfying eqs* (1), (2) *and boundary conditions* (3)–(5) *is that*

$$p_\mathrm{i} = 0. \tag{15}$$

Proof. Proceeding exactly as in the proof of Theorem 1 we have in place of eq. (7),

$$p_\mathrm{i}\left[\frac{1}{\sigma}\int_0^1 (|Dw|^2 + a^2|w|^2)\mathrm{d}z + \frac{|R^{-1}|a^{-2}}{|1-\alpha_2 T_0|}\int_0^1 |\theta|^2\mathrm{d}z\right] = 0. \tag{16}$$

From eq. (16), it follows that $p_\mathrm{i} = 0$. It implies that the 'principle of exchange of stabilities' is valid in the present case and hence the marginal state is characterized by $p = 0$. In this case, eqs (1), (2) and boundary conditions (3)–(5) can be treated as an eigenvalue problem in R for given values of a^2, $\alpha_2 T_0$, K_l and K_u. Proceeding exactly along the same lines as given in Chandrasekhar [5], we get the following result: □

Theorem 5. *If* $1 - \alpha_2 T_0 < 0$ *with* $R < 0$ *and* $p = 0$, *a nontrivial solution for* w *and* θ *satisfying equations* (1), (2), *and boundary conditions* (3)–(5) *implies that the Rayleigh number* R *is given, in terms of* a, $\alpha_2 T_0$, K_0 *and* K_1, *by*

$$|R| = \frac{P^3}{a^2|(1-\alpha_2 T_0)|} \times \left[1 - \frac{8\pi^2 a\{K_0 K_1 (S_a - a)(1 + C_a) + (K_0 + K_1) a S_a^2\}}{P^2\{K_0 K_1 (S_a^2 - a^2) + 2(K_0 + K_1)(a S_a C_a - a^2) + 4a^2 S_a^2\}}\right]^{-1}. \tag{17}$$

Remark 5. It is easily seen from eq. (17) that the results analogous to those given in Remarks 1–4 are also valid, in the present case.

Case 4. $1 - \alpha_2 T_0 < 0$ *with* $R > 0$. When $1 - \alpha_2 T_0 < 0$ with $R > 0$, the governing equations and boundary conditions imply that we have a simple Bénard problem with permeable boundaries, wherein a liquid is heated underside, in the parameter regime $1 - \alpha_2 T_0 < 0$. Since R is positive the initial distribution of density is top heavy and therefore potentially gravitationally unstable. This destabilizing effect together with the joint stabilizing effects of viscosity and conduction is expected to impart, in the usual parameter regime characterized by $1 - \alpha_2 T_0 > 0$, the character of a simple Bénard problem with permeable boundaries, in which the liquid layer is heated from below; that is, instability must set in the system when R goes beyond a critical value with the 'principle of exchange of stabilities' being valid at the marginal state. The nature of the problem, however, is completely different in the regime $1 - \alpha_2 T_0 < 0$ in which case we have with $R > 0$, $R(1 - \alpha_2 T_0) < 0$ and this in turn forces all the perturbations to decay, thus making the system stable.

We now prove the following theorem to show the existence of this new stabilizing mechanism.

Theorem 6. *If* $1 - \alpha_2 T_0 < 0$ *with* $R > 0$, *a necessary condition for the existence of nontrivial solutions for* w *and* θ *satisfying equations* (1), (2) *and boundary conditons* (3)–(5) *is that*

$$p_r < 0. \tag{18}$$

Proof. Proceeding exactly as in the proof of Theorem 3 we have in place of eq. (14),

$$K_1(|Dw|^2)_1 + K_0(|Dw|^2)_0 + \int_0^1 (|D^2 w|^2 + 2a^2|Dw|^2 + a^4|w|^2)\mathrm{d}z + p_r\left[\frac{1}{\sigma}\int_0^1 (|Dw|^2 + a^2|w|^2)\mathrm{d}z\right] = -\frac{R^{-1}a^{-2}}{|1 - \alpha_2 T_0|}\int_0^1 (|D\theta|^2 + a^2|\theta|^2 + p_r|\theta|^2)\mathrm{d}z. \tag{19}$$

From equation (19), it follows that $p_r < 0$. □

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