Proc. Indian Acad. Sci. (Math. Sci.) Vol. 121, No. 4, November 2011, pp. 457–468.

Periodic and subharmonic solutions for second order p-Laplacian difference equations

XIA LIU[1], YUANBIAO ZHANG[2], BO ZHENG[3] and HAIPING SHI[4]

[1]Oriental Science and Technology College, Hunan Agricultural University, Changsha 410128, People's Republic of China
[2]Packaging Engineering Institute, Jinan University, Zhuhai 519070, People's Republic of China
[3]College of Mathematics and Information Sciences, Guangzhou University, Guangzhou 510006, People's Republic of China
[4]Basic Courses Department, Guangdong Construction Vocational Technology Institute, Guangzhou 510450, China
E-mail: shp7971@163.com

MS received 30 October 2010; revised 20 January 2011

Abstract. In this paper, some sufficient conditions for the existence and multiplicity of periodic and subharmonic solutions to second order p-Laplacian difference equations are obtained by using the critical point theory. The proof is based on the Linking theorem in combination with variational technique.

Keywords. Periodic and subharmonic solutions; p-Laplacian; difference equations; discrete variational theory.

1. Introduction

Let $\mathbf{N}$, $\mathbf{Z}$ and $\mathbf{R}$ denote the sets of all natural numbers, integers and real numbers respectively. For $a, b \in \mathbf{Z}$, define $\mathbf{Z}(a) = \{a, a+1, \ldots\}$, $\mathbf{Z}(a, b) = \{a, a+1, \ldots, b\}$ when $a \leq b$. $*$ denotes the transpose of a vector.

In this paper, we consider the following forward and backward difference equation

$$\Delta(\varphi_p(\Delta u_{n-1})) + f(n, u_{n+1}, u_n, u_{n-1}) = 0, \quad n \in \mathbf{Z}, \tag{1.1}$$

where Δ is the forward difference operator $\Delta u_n = u_{n+1} - u_n$, $\Delta^2 u_n = \Delta(\Delta u_n)$, $\varphi_p(s)$ is the p-Laplacian operator $\varphi_p(s) = |s|^{p-2}s (1 < p < \infty)$, $f \in C(\mathbf{Z} \times \mathbf{R}^3, \mathbf{R})$, and $f(n, v_1, v_2, v_3)$ is T-periodic in n for a given positive integer T.

We may think of eq. (1.1) as being a discrete analogue of the second order functional differential equation

$$\left[\varphi_p(u')\right]' + f(t, u(t+1), u(t), u(t-1)) = 0, \quad t \in \mathbf{R} \tag{1.2}$$

which includes the following equation

$$c^2 u''(t) = V'(u(t+1) - u(t)) - V'(u(t) - u(t-1)), \quad t \in \mathbf{R} \tag{1.3}$$

where $f \in C(\mathbf{R}^4, \mathbf{R})$. Equation (1.3) has been studied extensively by many scholars. For example, Smets and Willem [25] have obtained the existence of solitary waves of lattice differential equations.

The motivation of this paper is as follows. When the functional $F(n, v_1, v_2) \geq a_1 \left(\sqrt{v_1^2 + v_2^2}\right)^{\mu} - a_2,\ \mu > p$, where $F(n, v_1, v_2)$ can be referred to as Theorem 1.1. Chen and Fang [8] have obtained a sufficient condition for the existence of periodic and subharmonic solutions of the second-order p-Laplacian difference equation (1.1). Since eq. (1.1) contains both advance and retardation, there are very few manuscripts dealing with this subject. Furthermore, we found that [8] is the only paper which deals with the problem of periodic and subharmonic solutions to p-Laplacian difference equation of (1.1). When $\mu = p$ we can still find the periodic and subharmonic solutions of eq. (1.1).

The theory of nonlinear difference equations has been widely used to study discrete models appearing in many fields such as computer science, economics, neural network, ecology, cybernetics, etc. For the general background of difference equations, one can refer to monographs [1,16,21]. Since the last decade, there has been much progress on the qualitative properties of difference equations, which included results on stability and attractivity [9,16,19,31,32] and results on oscillation and other topics, see [1–6, 11–15,17,18,26,27,29,30]. It is well known that critical point theory is a powerful tool that deals with the problems of differential equations [7,20,22,28]. Starting 2003, critical point theory has been employed to establish sufficient conditions on the existence of periodic solutions of difference equations. In particular, Guo and Yu [11–13] and Shi *et al* [24] considered second-order nonlinear difference equations. However, to the best of our knowledge, the results on periodic solutions of p-Laplacian difference equations are very scarce in the literature. The main purpose of this paper is to give some sufficient conditions for the existence and multiplicity of periodic and subharmonic solutions to second-order p-Laplacian difference equations. The main approach used in our paper is the variational technique and the Linking theorem. One of our results is more general than the result in the literature [8]. In fact, one can see the following Remark 1.2 for details.

Our main results are as follows.

Theorem 1.1. *Assume that the following hypotheses are satisfied*:

(F_1) *There exists a functional* $F(n, v_1, v_2) \in C^1(\mathbf{Z} \times \mathbf{R}^2, \mathbf{R})$ *with* $F(n, v_1, v_2) \geq 0$ *and it satisfies*

$$F(n + T, v_1, v_2) = F(n, v_1, v_2),$$

$$\frac{\partial F(n-1, v_2, v_3)}{\partial v_2} + \frac{\partial F(n, v_1, v_2)}{\partial v_2} = f(n, v_1, v_2, v_3);$$

(F_2) *There exist constants* $\delta > 0$, $\alpha \in \left(0, \frac{1}{2^{p/2}p}\left(\frac{c_1}{c_2}\right)^p \lambda_{\min}^{\frac{p}{2}}\right)$ *such that*

$$F(n, v_1, v_2) \leq \alpha \left(\sqrt{v_1^2 + v_2^2}\right)^p, \quad \text{for } n \in \mathbf{Z} \text{ and } v_1^2 + v_2^2 \leq \delta^2;$$

(F_3) *There exist constants* $\rho > 0, \gamma > 0, \beta \in \left(\frac{1}{2^{p/2}p}\left(\frac{c_2}{c_1}\right)^p \lambda_{\max}^{\frac{p}{2}}, +\infty\right)$ *such that*

$$F(n, v_1, v_2) \geq \beta\left(\sqrt{v_1^2+v_2^2}\right)^p - \gamma, \quad \text{for } n \in \mathbf{Z} \text{ and } v_1^2+v_2^2 \geq \rho^2,$$

where $c_1,\ c_2$ *are constants which can be referred to* (2.4), *and* $\lambda_{\min},\ \lambda_{\max}$ *are constants which can be referred to* (2.7).

Then for any given positive integer $q > 0$, *eq.* (1.1) *has at least three* qT*-periodic solutions.*

Remark 1.1 By (F_3) it is easy to see that there exists a constant $\gamma' > 0$ such that

$$(\mathrm{F}_3') \qquad F(n, v_1, v_2) \geq \beta\left(\sqrt{v_1^2+v_2^2}\right)^p - \gamma', \quad \forall (n, v_1, v_2) \in \mathbf{Z} \times \mathbf{R}^2.$$

As a matter of fact, let $\gamma_1 = \max\left\{\left|F(n, v_1, v_2) - \beta\left(\sqrt{v_1^2+v_2^2}\right)^p + \gamma\right| : n \in \mathbf{Z},\ v_1^2+v_2^2 \leq \rho^2\right\}$, $\gamma' = \gamma + \gamma_1$, and we can easily get the desired result.

COROLLARY 1.1

Assume that (F_1)–(F_3) *is satisfied. Then for any given positive integer* $q > 0$, *eq.* (1.1) *has at least two nontrivial* qT*-periodic solutions.*

Remark 1.2 Corollary 1.1 reduces to Theorem 3.1 in [8].

The rest of the paper is organized as follows. In §2 we shall transfer the solutions of eq. (1.1) into the critical points of some functional, which is called the variational framework. By applying critical point theory to the functional, we prove the existence of solutions of eq. (1.1). In §3 we shall complete the proof of the main results and give an example to illustrate the result.

About the basic knowledge of variational methods, the reader can refer to [10,20,23].

2. Variational structure and some lemmas

In order to apply the critical point theory, we shall establish the corresponding variational framework for eq. (1.1) and give some basic notations and useful lemmas.

Let S be the set of sequences $u = (\ldots, u_{-n}, \ldots, u_{-1}, u_0, u_1, \ldots, u_n, \ldots) = \{u_n\}_{n=-\infty}^{+\infty}$, that is

$$S = \{\{u_n\} | u_n \in \mathbf{R}, \quad n \in \mathbf{Z}\}.$$

For any $u, v \in S$, $a, b \in \mathbf{R}$, $au + bv$ is defined by

$$au + bv = \{au_n + bv_n\}_{n=-\infty}^{+\infty}.$$

Then S is a vector space.

For any given positive integers q and T, E_{qT} is defined as a subspace of S by

$$E_{qT} = \{u \in S | u_{n+qT} = u_n, \quad \forall n \in \mathbf{Z}\}.$$

Clearly, E_{qT} is isomorphic to $\mathbf{R}^{qT}$. E_{qT} can be equipped with the inner product

$$\langle u, v\rangle = \sum_{j=1}^{qT} u_j v_j, \quad \forall u, v \in E_{qT}, \tag{2.1}$$

by which the norm $\|\cdot\|$ can be induced by

$$\|u\| = \left(\sum_{j=1}^{qT} u_j^2\right)^{\frac{1}{2}}, \quad \forall u \in E_{qT}. \tag{2.2}$$

It is obvious that E_{qT} with the inner product (2.1) is a finite dimensional Hilbert space and linearly homeomorphic to $\mathbf{R}^{qT}$.

On the other hand, we define the norm $\|\cdot\|_r$ on E_{qT} as follows:

$$\|u\|_r = \left(\sum_{j=1}^{qT} |u_j|^r\right)^{\frac{1}{r}}, \tag{2.3}$$

for all $u \in E_{qT}$ and $r > 1$.

Since $\|u\|_r$ and $\|u\|_2$ are equivalent, there exist constants $c_1,\ c_2$ such that $c_2 \geq c_1 > 0$, and

$$c_1\|u\|_2 \leq \|u\|_r \leq c_2\|u\|_2, \quad \forall u \in E_{qT}. \tag{2.4}$$

Clearly, $\|u\| = \|u\|_2$. For all $u \in E_{qT}$, define the functional J on E_{qT} as follows:

$$J(u) = \sum_{n=1}^{qT}\left[\frac{1}{p}|\Delta u_n|^p - F(n, u_{n+1}, u_n)\right], \tag{2.5}$$

where

$$\frac{\partial F(n-1, v_2, v_3)}{\partial v_2} + \frac{\partial F(n, v_1, v_2)}{\partial v_2} = f(n, v_1, v_2, v_3).$$

Clearly, $J \in C^1(E_{qT}, \mathbf{R})$ and for any $u = \{u_n\}_{n\in\mathbf{Z}} \in E_{qT}$, by using $u_0 = u_{qT},\ u_1 = u_{qT+1}$, we can compute the partial derivative as

$$\frac{\partial J}{\partial u_n} = -(\Delta(\varphi_p(\Delta u_{n-1})) + f(n, u_{n+1}, u_n, u_{n-1})).$$

Thus, u is a critical point of J on E_{qT} if and only if

$$\Delta(\varphi_p(\Delta u_{n-1})) + f(n, u_{n+1}, u_n, u_{n-1}) = 0, \quad \forall n \in \mathbf{Z}(1, qT).$$

Due to the periodicity of $u = \{u_n\}_{n\in\mathbf{Z}} \in E_{qT}$ and $f(n, v_1, v_2, v_3)$ in the first variable n, we reduce the existence of periodic solutions of eq. (1.1) to the existence of critical points of J on E_{qT}. That is, the functional J is just the variational framework of eq. (1.1).

Let P be the $qT \times qT$ matrix defined by

$$P = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ -1 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}.$$

By matrix theory, we see that the eigenvalues of P are

$$\lambda_k = 2\left(1 - \cos\frac{2k}{qT}\pi\right), k = 0, 1, 2, \ldots, qT - 1. \tag{2.6}$$

Thus, $\lambda_0 = 0, \lambda_1 > 0, \lambda_2 > 0, \ldots, \lambda_{qT-1} > 0$. Therefore,

$$\left.\begin{aligned} &\lambda_{\min} = \min\{\lambda_1, \lambda_2, \ldots, \lambda_{qT-1}\} = 2\left(1 - \cos\tfrac{2}{qT}\pi\right), \\ &\lambda_{\max} = \max\{\lambda_1, \lambda_2, \ldots, \lambda_{qT-1}\} = \begin{cases} 4, & \text{when } qT \text{ is even,} \\ 2(1 + \cos\frac{1}{qT}\pi), & \text{when } qT \text{ is odd.} \end{cases} \end{aligned}\right\} \tag{2.7}$$

Let

$$W = \ker P = \{u \in E_{qT} | Pu = 0 \in \mathbf{R}^{qT}\}.$$

Then

$$W = \{u \in E_{qT} | u = \{c\},\ c \in \mathbf{R}\}.$$

Let V be the direct orthogonal complement of E_{qT} to W, i.e., $E_{qT} = V \oplus W$. For convenience, we identify $u \in E_{qT}$ with $u = (u_1, u_2, \ldots, u_{qT})^*$.

Let E be a real Banach space, $J \in C^1(E, \mathbf{R})$, i.e., J is a continuously Fréchet-differentiable functional defined on E. J is said to be satisfying the Palais-Smale condition (PS condition, for short) if any sequence $\{u^{(k)}\} \subset E$ for which $\{J(u^{(k)})\}$ is bounded and $J'(u^{(k)}) \to 0 (k \to \infty)$ possesses a convergent subsequence in E.

Let B_ρ denote the open ball in E about 0 of radius ρ and let ∂B_ρ denote its boundary.

Lemma 2.1 (*Linking theorem* [23]). *Let E be a real Banach space, $E = E_1 \oplus E_2$, where E_1 is finite dimensional. Suppose that $J \in C^1(E, \mathbf{R})$ satisfies the PS condition and*

(J$_1$) *there exist constants $a > 0$ and $\rho > 0$ such that $J|_{\partial B_\rho \cap E_2} \geq a$;*
(J$_2$) *there exists an $e \in \partial B_1 \cap E_2$ and a constant $R_0 \geq \rho$ such that $J|_{\partial Q} \leq 0$, where $Q = (\bar{B}_{R_0} \cap E_1) \oplus \{re | 0 < r < R_0\}$.*

Then J possesses a critical value $c \geq a$, where

$$c = \inf_{h \in \Gamma} \sup_{u \in Q} J(h(u)),$$

and $\Gamma = \{h \in C(\bar{Q}, E) \mid h|_{\partial Q} = id\}$, where '$id$' denotes the identity operator.

Lemma 2.2. *Assume that* (F_1) *and* (F_3) *are satisfied. Then the functional* J *is bounded from above in* E_{qT}.

Proof. By (F_3') and (2.4), for any $u \in E_{qT}$,

$$\begin{aligned}
J(u) &= \sum_{n=1}^{qT}\left[\frac{1}{p}|\Delta u_n|^p - F(n,u_{n+1},u_n)\right] \\
&= \frac{1}{p}\left[\left(\sum_{n=1}^{qT}|\Delta u_n|^p\right)^{\frac{1}{p}}\right]^p - \sum_{n=1}^{qT}F(n,u_{n+1},u_n) \\
&\le \frac{1}{p}\left[c_2\left(\sum_{n=1}^{qT}|\Delta u_n|^2\right)^{\frac{1}{2}}\right]^p - \sum_{n=1}^{qT}\left[\beta\left(\sqrt{u_{n+1}^2+u_n^2}\right)^p - \gamma'\right] \\
&= \frac{1}{p}c_2^p\left[\sum_{n=1}^{qT}2\left(u_n^2 - u_n u_{n+1}\right)\right]^{\frac{p}{2}} - \beta\left[\left(\sum_{n=1}^{qT}\left(\sqrt{u_{n+1}^2+u_n^2}\right)^p\right)^{\frac{1}{p}}\right]^p + qT\gamma' \\
&\le \frac{1}{p}c_2^p\left[\sum_{n=1}^{qT}2\left(u_n^2 - u_n u_{n+1}\right)\right]^{\frac{p}{2}} - \beta c_1^p\left[\sum_{n=1}^{qT}\left(u_{n+1}^2+u_n^2\right)\right]^{\frac{p}{2}} + qT\gamma' \\
&= \frac{1}{p}c_2^p(u^*Pu)^{\frac{p}{2}} - \beta c_1^p(2\|u\|_2^2)^{\frac{p}{2}} + qT\gamma', \\
&\le \frac{1}{p}c_2^p\lambda_{\max}^{\frac{p}{2}}\|u\|_2^p - 2^{\frac{p}{2}}\beta c_1^p\|u\|_2^p + qT\gamma' \\
&= \left(\frac{1}{p}c_2^p\lambda_{\max}^{\frac{p}{2}} - 2^{\frac{p}{2}}\beta c_1^p\right)\|u\|_2^p + qT\gamma' \\
&\le qT\gamma'.
\end{aligned}$$

The proof of Lemma 2.2 is complete. □

Remark 2.1. The case $qT = 1$ is trivial. For the case $qT = 2$, P has a different form, namely,

$$P = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

However, in this special case, the argument need not be changed and we omit it.

Lemma 2.3. *Assume that* (F_1) *and* (F_3) *are satisfied. Then the functional* J *satisfies the PS condition.*

Proof. Let $\{J(u^{(k)})\}$ be a bounded sequence from the lower bound, i.e. there exists a positive constant M_1 such that

$$-M_1 \le J(u^{(k)}), \quad \forall k \in \mathbf{N}.$$

By the proof of Lemma 2.2, it is easy to see that

$$-M_1 \leq J(u^{(k)}) \leq \left(\frac{1}{p}c_2^p\lambda_{\max}^{\frac{p}{2}} - 2^{\frac{p}{2}}\beta c_1^p\right)\|u^{(k)}\|_2^p + qT\gamma', \ \forall k \in \mathbf{N}.$$

Therefore,

$$\left(2^{\frac{p}{2}}\beta c_1^p - \frac{1}{p}c_2^p\lambda_{\max}^{\frac{p}{2}}\right)\|u^{(k)}\|_2^p \leq M_1 + qT\gamma'.$$

Since $\beta > \frac{1}{2^{p/2}p}\left(\frac{c_2}{c_1}\right)^p \lambda_{\max}^{\frac{p}{2}}$, it is not difficult to know that $\{u^{(k)}\}$ is a bounded sequence in E_{qT}. As a consequence, $\{u^{(k)}\}$ possesses a convergence subsequence in E_{qT}. And thus the PS condition is verified. □

3. Proof of the main results

3.1 *Proof of Theorem* 1.1

Assumptions (F_1) and (F_2) imply that $F(n, 0) = 0$ and $f(n, 0) = 0$ for $n \in \mathbf{Z}$. Then $u = 0$ is a trivial qT-periodic solution of eq. (1.1).

By Lemma 2.2, J is bounded from the upper bound on E_{qT}. We define $c_0 = \sup_{u\in E_{qT}} J(u)$. The proof of Lemma 2.2 implies $\lim_{\|u\|_2\to+\infty} J(u) = -\infty$. This means that $-J(u)$ is coercive. By the continuity of $J(u)$, there exists $\bar{u} \in E_{qT}$ such that $J(\bar{u}) = c_0$. Clearly, $\bar{u}$ is a critical point of J.

We claim that $c_0 > 0$. Indeed, by (F_2), for any $u \in V$, $\|u\|_2 \leq \delta$, we have

$$\begin{aligned}
J(u) &= \sum_{n=1}^{qT}\left[\frac{1}{p}|\Delta u_n|^p - F(n, u_{n+1}, u_n)\right] \\
&= \frac{1}{p}\left[\left(\sum_{n=1}^{qT}|\Delta u_n|^p\right)^{\frac{1}{p}}\right]^p - \sum_{n=1}^{qT}F(n, u_{n+1}, u_n) \\
&\geq \frac{1}{p}c_1^p\left[\left(\sum_{n=1}^{qT}|\Delta u_n|^2\right)^{\frac{1}{2}}\right]^p - \sum_{n=1}^{qT}F(n, u_{n+1}, u_n) \\
&= \frac{1}{p}c_1^p\left[\sum_{n=1}^{qT}2(u_n^2 - u_n u_{n+1})\right]^{\frac{p}{2}} - \sum_{n=1}^{qT}F(n, u_{n+1}, u_n) \\
&\geq \frac{1}{p}c_1^p(u^*Pu)^{\frac{p}{2}} - \sum_{n=1}^{qT}F(n, u_{n+1}, u_n) \\
&\geq \frac{1}{p}c_1^p\lambda_{\min}^{\frac{p}{2}}\|u\|_2^p - \alpha\sum_{n=1}^{qT}\left(\sqrt{u_{n+1}^2 + u_n^2}\right)^p
\end{aligned}$$

$$= \frac{1}{p}c_1^p\lambda_{\min}^{\frac{p}{2}}\|u\|_2^p - \alpha\left[\left(\sum_{n=1}^{qT}\left(\sqrt{u_{n+1}^2+u_n^2}\right)^p\right)^{\frac{1}{p}}\right]^p$$

$$\geq \frac{1}{p}c_1^p\lambda_{\min}^{\frac{p}{2}}\|u\|_2^p - \alpha c_2^p\left[\sum_{n=1}^{qT}(u_{n+1}^2+u_n^2)\right]^{\frac{p}{2}}$$

$$\geq \frac{1}{p}c_1^p\lambda_{\min}^{\frac{p}{2}}\|u\|_2^p - \alpha c_2^p(2\|u\|_2^2)^{\frac{p}{2}}$$

$$\geq \left(\frac{1}{p}c_1^p\lambda_{\min}^{\frac{p}{2}} - 2^{\frac{p}{2}}c_2^p\alpha\right)\|u\|_2^p.$$

Take $\sigma = (\frac{1}{p}c_1^p\lambda_{\min}^{\frac{p}{2}} - 2^{\frac{p}{2}}c_2^p\alpha)$. Then

$$J(u) \geq \sigma, \quad \forall u \in V \cap \partial B_\delta.$$

Therefore, $c_0 = \sup_{u\in E_{qT}} J(u) \geq \sigma > 0$. At the same time, we have also proved that there exist constants $\sigma > 0$ and $\delta > 0$ such that $J|_{\partial B_\delta \cap V} \geq \sigma$. That is to say, J satisfies the condition (J_1) of the Linking theorem.

Noting that $\sum_{n=1}^{qT}|\Delta u_n|^p = 0$, for all $u \in W$, we have

$$J(u) = \frac{1}{p}\sum_{n=1}^{qT}|\Delta u_n|^p - \sum_{n=1}^{qT}F(n, u_{n+1}, u_n) = -\sum_{n=1}^{qT}F(n, u_{n+1}, u_n) \leq 0.$$

Thus, the critical point $\bar{u}$ of J corresponding to the critical value c_0 is a nontrivial qT-periodic solution of eq. (1.1).

In order to obtain another nontrivial qT-periodic solution of eq. (1.1) different from $\bar{u}$, we need to use the conclusion of Lemma 2.1. We have known that J satisfies the PS condition on E_{qT}. In the following, we shall verify the condition (J_2).

Take $e \in \partial B_1 \cap V$, for any $z \in W$ and $r \in \mathbf{R}$, and let $u = re + z$. Then

$$J(u) = \sum_{n=1}^{qT}\left[\frac{1}{p}|\Delta u_n|^p - F(n, u_{n+1}, u_n)\right]$$

$$= \sum_{n=1}^{qT}\left[\frac{1}{p}|r\Delta e_n|^p - F(n, re_{n+1}+z_{n+1}, re_n+z_n)\right]$$

$$\leq \frac{1}{p}r^p\left[\left(\sum_{n=1}^{qT}|\Delta e_n|^p\right)^{\frac{1}{p}}\right]^p$$

$$-\sum_{n=1}^{qT}\left\{\beta\left(\sqrt{(re_{n+1}+z_{n+1})^2+(re_n+z_n)^2}\right)^p - \gamma'\right\}$$

$$\begin{aligned}
&\leq \frac{1}{p} r^p c_2^p \left(\sum_{n=1}^{qT} |\Delta e_n|^2 \right)^{\frac{p}{2}} \\
&\quad -\beta c_1^p \left\{ \sum_{n=1}^{qT} [(re_{n+1} + z_{n+1})^2 + (re_n + z_n)^2] \right\}^{\frac{p}{2}} + qT\gamma' \\
&= \frac{1}{p} r^p c_2^p \left[\sum_{n=1}^{qT} 2(e_n^2 - e_{n+1} e_n) \right]^{\frac{p}{2}} \\
&\quad -\beta c_1^p \left\{ \sum_{n=1}^{qT} [(re_{n+1} + z_{n+1})^2 + (re_n + z_n)^2] \right\}^{\frac{p}{2}} + qT\gamma' \\
&\leq \frac{1}{p} r^p c_2^p \lambda_{\max}^{\frac{p}{2}} - \beta c_1^p \left[2 \sum_{n=1}^{qT} (re_n + z_n)^2 \right]^{\frac{p}{2}} + qT\gamma' \\
&= \frac{1}{p} r^p c_2^p \lambda_{\max}^{\frac{p}{2}} - \beta c_1^p r^p 2^{\frac{p}{2}} - \beta c_1^p 2^{\frac{p}{2}} \|z\|_2^p + qT\gamma' \\
&= \left(\frac{1}{p} c_2^p \lambda_{\max}^{\frac{p}{2}} - \beta c_1^p 2^{\frac{p}{2}} \right) r^p - \beta c_1^p 2^{\frac{p}{2}} \|z\|_2^p + qT\gamma' \\
&\leq -\beta c_1^p 2^{\frac{p}{2}} \|z\|_2^p + qT\gamma'.
\end{aligned}$$

Thus, there exists a positive constant $R_2 > \delta$ such that for any $u \in \partial Q$, $J(u) \leq 0$, where $Q = (\bar{B}_{R_2} \cap W) \oplus \{re | 0 < r < R_2\}$. By the Linking theorem, J possesses a critical value $c \geq \sigma > 0$, where

$$c = \inf_{h \in \Gamma} \sup_{u \in Q} J(h(u)),$$

and $\Gamma = \{h \in C(\bar{Q}, E_{qT}) \mid h|_{\partial Q} = id\}$.

Let $\tilde{u} \in E_{qT}$ be a critical point associated to the critical value c of J, i.e. $J(\tilde{u}) = c$. If $\tilde{u} \neq \bar{u}$, then the conclusion of Theorem 1.1 holds. Otherwise, $\tilde{u} = \bar{u}$. Then $c_0 = J(\bar{u}) = J(\tilde{u}) = c$, that is $\sup_{u \in E_{qT}} J(u) = \inf_{h \in \Gamma} \sup_{u \in Q} J(h(u))$. Choosing $h = id$, we have $\sup_{u \in Q} J(u) = c_0$. Since the choice of $e \in \partial B_1 \cap V$ is arbitrary, we can take $-e \in \partial B_1 \cap V$. Similarly, there exists a positive number $R_3 > \sigma$, for any $u \in \partial Q_1$, $J(u) \leq 0$, where $Q_1 = (\bar{B}_{R_3} \cap W) \oplus \{-re | 0 < r < R_3\}$.

Again, by the Linking theorem, J possesses a critical value $c' \geq \sigma > 0$, where

$$c' = \inf_{h \in \Gamma_1} \sup_{u \in Q_1} J(h(u)),$$

and $\Gamma_1 = \{h \in C(\bar{Q}_1, E_{qT}) \mid h|_{\partial Q_1} = id\}$.

If $c' \neq c_0$, then the proof is finished. If $c' = c_0$, then $\sup_{u \in Q_1} J(u) = c_0$. Due to the fact that $J|_{\partial Q} \leq 0$ and $J|_{\partial Q_1} \leq 0$, J attains its maximum at some points in the interior of sets Q and Q_1. However, $Q \cap Q_1 \subset W$ and $J(u) \leq 0$ for any $u \in W$. Therefore, there must be a point $u' \in E_{qT}$, $u' \neq \tilde{u}$ and $J(u') = c' = c_0$. The proof of Theorem 1.1 is complete. □

Remark 3.1 Due to Theorem 1.1, the conclusion of Corollary 1.1 is obviously true.

4. Example

As an application of Theorem 1.1, we give an example to illustrate our result.

Example 4.1. For all $n \in \mathbf{Z}$, assume that

$$\Delta(\varphi_p(\Delta u_{n-1})) = \mu u_n \left[\left(1 + \cos^2\left(\frac{\pi n}{T}\right)\right) (u_{n+1}^2 + u_n^2)^{\frac{\mu}{2}-1} + \left(1 + \cos^2\left(\frac{\pi(n-1)}{T}\right)\right) (u_n^2 + u_{n-1}^2)^{\frac{\mu}{2}-1} \right], \quad (4.1)$$

where $1 < p < +\infty$, $\mu > p$, T is a given positive integer.

We have

$$f(n, v_1, v_2, v_3) = \mu v_2 \left[\left(1 + \cos^2\left(\frac{\pi n}{T}\right)\right) (v_1^2 + v_2^2)^{\frac{\mu}{2}-1} + \left(1 + \cos^2\left(\frac{\pi(n-1)}{T}\right)\right) (v_2^2 + v_3^2)^{\frac{\mu}{2}-1} \right]$$

and

$$F(n, v_1, v_2) = \left[1 + \cos^2\left(\frac{\pi n}{T}\right)\right] (v_1^2 + v_2^2)^{\frac{\mu}{2}}.$$

Then

$$\frac{\partial F(n-1, v_2, v_3)}{\partial v_2} + \frac{\partial F(n, v_1, v_2)}{\partial v_2} = \mu v_2 \left[\left(1+\cos^2\left(\frac{\pi n}{T}\right)\right) (v_1^2+v_2^2)^{\frac{\mu}{2}-1} + \left(1+\cos^2\left(\frac{\pi(n-1)}{T}\right)\right) (v_2^2+v_3^2)^{\frac{\mu}{2}-1} \right].$$

It is easy to verify that all the assumptions of Theorem 1.1 are satisfied. Consequently, for any given positive integer $q > 0$, eq. (4.1) has at least three qT-periodic solutions.

Acknowledgements

The present investigation was supported by specialized fund for the doctoral program of Higher Eduction under Grant 20071078001 of People's Republic of China, National Natural Science Foundation under Grant 11026059 of People's Republic of China and Foundation for Distinguished Young Talents in Higher Education under Grant LYM09105 of Guangdong, People's Republic of China.

References

[1] Agarwal R P, Difference Equations and Inequalities: Theory, Methods and Applications (New York: Marcel Dekker) (1992)

[2] Agarwal R P, Perera K and O'regan D, Multiple positive solutions of singular and nonsingular discrete problems via variational methods, *Nonlinear Anal.* **58** (2004) 69–73

[3] Agarwal R P, Perera K and O'regan D, Multiple positive solutions of singular discrete p-Laplacian problems via variational methods, *Adv. Differ. Equ.* **2005** (2005) 93–99

[4] Anderson D R, Avery R I and Henderson J, Existence of solutions for a one dimensional p-Laplacian on time-scales, *J. Differ. Equ. Appl.* **10** (2006) 889–896

[5] Avery R I and Henderson J, Existence of three positive pseudo-symmetric solutions for a one dimensional discrete p-Laplacian, *J. Differ. Equ. Appl.* **10** (2004) 529–539

[6] Avery R I and Pererson A C, Three positive fixed points of nonlinear operators on ordered Banach space, *Comput. Math. Appl.* **42** (2001) 313–322

[7] Chang K C, Infinite Dimensional Morse Theory and Multiple Solution Problems (Boston: Birkhäuser) (1993)

[8] Chen P and Fang H, Existence of periodic and subharmonic solutions for second-order p-Laplacian difference equations, *Adv. Differ. Equ.* **2007** (2007) 1–9

[9] Erbe L H, Xia H and Yu J S, Global stability of a linear nonautonomous delay difference equations, *J. Differ. Equ. Appl.* **1** (1995) 151–161

[10] Guo D J, Nonlinear Functional Analysis (Jinan: Shandong Scientific Press) (1985)

[11] Guo Z M and Yu J S, Applications of critical point theory to difference equations, *Fields Inst. Commun.* **42** (2004) 187–200

[12] Guo Z M and Yu J S, The existence of periodic and subharmonic solutions for second-order superlinear difference equations, *Sci. China* **A46** (2003) 506–515

[13] Guo Z M and Yu J S, The existence of periodic and subharmonic solutions of subquadratic second order difference equations, *J. London Math. Soc.* **68** (2003) 419–430

[14] He Z, On the existence of positive solutions of p-Laplacian difference equations, *J. Comput. Appl. Math.* **161** (2003) 193–201

[15] Jiang D, Chu J, O'regan D and Agarwal R P, Positive solutions for continuous and discrete boundary value problems to the one-dimension p-Laplacian, *Math. Inequal. Appl.* **7** (2004) 523–534

[16] Kocic V L and Ladas G, Global Behavior of Nonlinear Difference Equations of High Order with Applications (Boston: Kluwer Academic Publishers) (1993)

[17] Li Y and Lu L, Existence of positive solutions of p-Laplacian difference equations, *Appl. Math. Lett.* **19** (2006) 1019–1023

[18] Liu Y and Ge W G, Twin positive solutions of boundary value problems for finite difference equations with p-Laplacian operator, *J. Math. Appl.* **278** (2003) 551–561

[19] Matsunaga H, Hara T and Sakata S, Global attractivity for a nonlinear difference equation with variable delay, *Comput. Math. Appl.* **41** (2001) 543–551

[20] Mawhin J and Willem M, Critical Point Theory and Hamiltonian Systems (New York: Springer) (1989)

[21] Moser J, Stable and Radom Motions in Dynamical Systems (Princeton: Princeton University Press) (1973)

[22] Pankov A and Zakhrchenko N, On some discrete variational problems, *Acta Appl. Math.* **65** (2001) 295–303

[23] Rabinowitz P H, Minimax Methods in Critical Point Theory with Applications to Differential Equations (Providence, RI, New York: Amer. Math. Soc.) (1986)

[24] Shi H P, Ling W P, Long Y H and Zhang H Q, Periodic and subharmonic solutions for second order nonlinear functional difference equations, *Commun. Math. Anal.* **5(2)** (2008) 50–59

[25] Smets D and Willem M, Solitary waves with prescribed speed on infinite lattices, *J. Funct. Anal.* **149** (1997) 266–275

[26] Tian Y and Ge W G, The existence of solutions for a second-order discrete Neumann problem with a p-Laplacian, *J. Appl. Math. Comput.* **26** (2008) 333–340

[27] Tian Y and Ge W G, Existence results for discrete Sturm-Liuville problem via variational methods, *J. Differ. Equ. Appl.* **13** (2007) 467–478

[28] Xu Y T and Guo Z M, Applications of a Z_p index theory to periodic solutions for a class of functional differential equations, *J. Math. Anal. Appl.* **257** (2001) 189–205

[29] Yu J S, Long Y H and Guo Z M, Subharmonic solutions with prescribed minimal period of a discrete forced pendulum equation, *J. Dyn. Differ. Equ.* **16** (2004) 575–586

[30] Yu J S and Guo Z M, On boundary value problems for a discrete generalized Emden-Fowler equation, *J. Differ. Equ.* **231** (2006) 18–31

[31] Zhou Z, Yu J S and Guo Z M, Periodic solutions of higher-dimensional discrete systems, *Proc. R. Soc. Edinb.* **A134** (2004) 1013–1022

[32] Zhou Z and Zhang Q, Uniform stability of nonlinear difference systems, *J. Math. Anal. Appl.* **225** (1998) 486–500