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On the L_p affine isoperimetric inequalities

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Abstract. We obtain an isoperimetric inequality which estimate the affine invariant p-surface area measure on convex bodies. We also establish the reverse version of L_p-Petty projection inequality and an affine isoperimetric inequality of $\Gamma_{-p}K$.

Keywords. Isotropic measure; L_p-Petty projection inequality; p-isoperimetric inequality.

1. Introduction

Projection bodies have a long and complicated history which goes back to Minkowski [3]. The classical Petty projection inequality plays a central role in the framework of the affine isoperimetric inequalities (see the survey article by Lutwak [10]). It states that [19]:

If K is a convex body in $\mathbb{R}^n$, *then*

$$V(\Pi^* K)V(K)^{n-1} \leq \omega_n^n, \tag{1.1}$$

where $\Pi^* K$ *is the polar body of projection body* ΠK *and equality holds if and only if K is an ellipsoid.*

The Petty projection inequality is the statement that the quantity $V(\Pi^* K)V(K)^{n-1}$ is maximized precisely when K is an ellipsoid.

Of all convex bodies of fixed (say, unit) volume, the inequality states that the ones whose polar projection bodies have minimal volume are precisely the simplices is known as the 'Zhang projection inequality' [24]. It states that:

If K is a convex body in $\mathbb{R}^n$, *then*

$$V(\Pi^* K)V(K)^{n-1} \geq \frac{(2n!)}{n^n (n!)^2} \tag{1.2}$$

with equality if and only if K is a simplex.

In recent years, the L_p-analogs of projection bodies have received considerable attentions [11–14,21,22]. Lutwak, Yang and Zhang [12] established the L_p-analog of the Petty projection inequality. It states that:

If K is a convex body in $\mathbb{R}^n$, then for $1 < p < \infty$,

$$V(\Pi_p^* K)V(K)^{(n-p)/p} \leq \omega_n^{n/p}, \tag{1.3}$$

with equality if and only if K is an ellipsoid.

Here $\Pi_p^* K$ denotes the polar L_p-projection body of K.

The main purpose of this paper is to establish the L_p-analog of Zhang projection inequality. In order to obtain this reverse L_p affine isoperimetric inequality, we should first establish the L_p-analog of isoperimetric inequality and its reverse form by combining a result of p-isotropic surface area measure obtained by Yu [23] with Ball's well-known reverse isoperimetric inequality [1]. The study of p-isotropic surface area measure was motivated by the paper of Giannopoulos and Papadimitrakis [7].

Our main results are as follows:

Theorem 1. *Let K be a convex body in $\mathbb{R}^n$ with $\int_{S^{n-1}} u \mathrm{d}S_p(K,u) = 0$, and $\triangle_n$ be a regular simplex circumscribed to B_n. Then*

$$\partial_p(B_n) \leq \partial_p(K) \leq \partial_p(\triangle_n). \tag{1.4}$$

Theorem 2. *Let K be a convex body in $\mathbb{R}^n$ with $\int_{S^{n-1}} u \mathrm{d}S_p(K,u) = 0$, then for $1 < p < \infty$,*

$$V(\Pi_p^* K)V(K)^{(n-p)/p} \geq \frac{\omega_n^{(n+p)/p} n!}{n^{n/2}(n+1)^{(n+1)/2}}, \tag{1.5}$$

with equality if and only if $p = 2$ and K is a regular simplex.

2. Notations and background materials

We shall work in $\mathbb{R}^n$ equipped with a fixed Euclidean structure and write $|\cdot|$ for the corresponding Euclidean norm. We denote the Euclidean unit ball and the unit sphere by B_n and S^{n-1} respectively. The volume of appropriate dimension will be denoted by $V(\cdot)$. We shall write ω_n for the volume of the Euclidean unit ball in $\mathbb{R}^n$. Note that $\omega_n = \pi^{n/2}/\Gamma(1+\frac{n}{2})$ defines ω_n for all non-negative real n (not just the positive integers). For real $p \geq 1$, define $c_{n,p} = \dfrac{(n+p)\omega_{n+p}}{\omega_2 \omega_{p-1}}$.

If K is a convex body in $\mathbb{R}^n$, then its support function, $h_K(\cdot) = h(K,\cdot) : \mathbb{R}^n \to (0,\infty)$, is defined for $x \in \mathbb{R}^n$ by $h(K,x) = \max\{x \cdot y : y \in K\}$. If $T \in \mathrm{GL}(n)$, then $h(TK,x) = h(K,T^t x)$, where T^t denotes the transpose of T.

For $p \geq 1$, convex bodies K, L and $\varepsilon > 0$ the Firey L_p-combination $K +_p \varepsilon \cdot L$ [15] is defined as the convex body whose support function is given by

$$h(K +_p \varepsilon \cdot L, \cdot)^p = h(K,\cdot)^p + \varepsilon h(L,\cdot)^p. \tag{2.1}$$

For $p \geq 1$, the L_p-mixed volume, $V_p(K, L)$, of the convex bodies K, L was defined in [15] by

$$\frac{n}{p} V_p(K, L) = \lim_{\varepsilon \to 0^+} \frac{V(K +_p \varepsilon \cdot L) - V(K)}{\varepsilon}. \tag{2.2}$$

Similar to the definition of the surface area of K, we can define the p-surface area, $S_p(K)$ of K by

$$\frac{1}{p} S_p(K) = \lim_{\varepsilon \to 0^+} \frac{V(K +_p \varepsilon \cdot B_n) - V(K)}{\varepsilon}. \tag{2.3}$$

It was shown in [15], that corresponding to each convex body $K \in \mathbb{R}^n$ that contains the origin in its interior, there is a positive Borel measure, $S_p(K, \cdot)$ on S^{n-1} such that

$$V_p(K, Q) = \frac{1}{n} \int_{S^{n-1}} h_Q(u)^p \mathrm{d}S_p(K, u), \tag{2.4}$$

for each convex body Q.

The L_p analog of the classical Minkowski inequality states that for convex bodies K, L,

$$V_p(K, L) \geq V(K)^{\frac{n-p}{n}} V(L)^{\frac{p}{n}}, \tag{2.5}$$

with equality if and only if K and L are dilates [15]. The measure $S_1(K, \cdot)$ is just the classical surface area measure of K and is usually denoted by $S(K, \cdot)$, especially, $S(B_n, \cdot)$ is denoted by $S(\cdot)$. It was shown in [15] that the p-surface area measure $S_p(K, \cdot)$ is absolutely continuous with respect to the classical surface area measure $S(K, \cdot)$ and that the Radon–Nikodym derivative

$$\frac{\mathrm{d}S_p(K, \cdot)}{\mathrm{d}S(K, \cdot)} = h_K(\cdot)^{1-p}. \tag{2.6}$$

From (2.2), (2.3) and (2.4) we get

$$S_p(K) = nV_p(K, B_n) = \int_{S^{n-1}} \mathrm{d}S_p(K, u). \tag{2.7}$$

If the body K contains the origin in its interior and $\lambda > 0$, then for the dilate λK, we know $h_{\lambda K} = \lambda h_K$ and $S(\lambda K, \cdot) = \lambda^{n-1} S(K, \cdot)$. It follows immediately from (2.6) and (2.7) that $S_p(\lambda K) = \lambda^{n-p} S_p(K)$.

DEFINITION 1

If K is a convex body in $\mathbb{R}^n$, then define an affine invariant quantity $\partial_p(K)$ as follows:

$$\partial_p(K) = \min\{S_p(TK)/V(K)^{\frac{n-p}{n}} : T \in \mathrm{SL}(n)\}. \tag{2.8}$$

The isotropic measure was defined in [7] as follows.

DEFINITION 2

A Borel measure μ on S^{n-1} will be called isotropic if

$$\int_{S^{n-1}} |u \cdot \theta|^2 \mathrm{d}\mu(u) = \frac{\mu(S^{n-1})}{n} \tag{2.9}$$

for every $\theta \in S^{n-1}$.

The notation of *isotropic* is closely related to the study of $(n-1)$-dimensional sections of convex bodies and in particular to the classical slicing problem. Since its introduction in the end of 80's, it has attracted increased interest (see [2,4,5,8,9 and 18]). For general reference of the Brunn-Minkowski theory the reader may wish to consult the books of Gardner [6] and Schneider [20].

3. The proofs of theorems

Lemma 1 [23]. *Let K be a convex body in $\mathbb{R}^n$ containing the origin in its interior. Then $S_p(K) = \min\{S_p(TK) : T \in \mathrm{SL}(n)\}$ if and only if $S_p(K, \cdot)$ is isotropic. The minimal p-surface area position is unique up to orthogonal transformations.*

We shall say that a convex body K is *p-surface isotropic* if its p-surface area measure $S_p(K, \cdot)$ is isotropic.

Lemma 2. *Every p-surface isotropic convex body is the limit of a sequence of p-surface isotropic polytopes in the Hausdorff metric.*

Proof. Suppose K is a p-surface isotropic convex body. By the Blaschke selection theorem we assume that K_i is a sequence of polytopes converging to K in the Hausdorff metric. By Lemma 1, assume that T_i is a sequence of volume preserving transformations such that $\tilde{K_i} = T_i K_i$ is p-surface isotropic. Again the Blaschke selection theorem guarantees that the existence of a subsequence of the $T_i K_i$, which will also be denoted by $T_i K_i$, and a convex body L, such that $T_i K_i \to L = TK$ for some T with $|\det T| = 1$. Now by Proposition 1.4 in [16] we know that $S_p(\tilde{K_i}, \cdot) \to S_p(L, \cdot)$ and one can easily check that $S_p(L, \cdot)$ is isotropic. From Lemma 1 it follows that $T \in O(n)$, and the proof is complete. □

Let K be a convex body in $\mathbb{R}^n$. By the well-known fact that the space of affine equivalence classes of convex bodies in $\mathbb{R}^n$ is compact, we know that the affine equivalence continuous functional $\partial_p(K)$ attains its maximum and minimum. Using Ball's volume ratio inequality [1] we can get a sharp inequality of this affine invariant quantity.

Proof of Theorem 1. Assume that K is a polytope with facets F_i and normals u_i with $h_K(u_i) = a_i$, for $i = 1, \ldots, m$, which has isotropic p-surface area measure. And assume that K satisfies $\sum_{i=1}^{m} |F_i| a_i^{1-p} u_i = 0$. Then from the definition of $\partial_p(K)$ and Lemma 1, we know that

$$\partial_p(K) = S_p(K)/V(K)^{\frac{n-p}{n}}.$$

From (2.5) and (2.7) we get

$$\partial_p(K) = \frac{S_p(K)}{V(K)^{\frac{n-p}{n}}} = \frac{nV_p(K, B_n)}{V(K)^{\frac{n-p}{n}}} \geq nV(B_n)^{\frac{p}{n}} = \partial_p(B_n). \tag{3.1}$$

On the other hand, from (2.6) and (2.9) we know '$S_p(K, \cdot)$ is isotropic' means that

$$\sum_{i=1}^{m} \frac{n|F_i|a_i^{1-p}}{S_p(K)} u_i \otimes u_i = I, \tag{3.2}$$

where $|F_i|$ denotes the area of the facet F_i. From $\sum_{i=1}^{m} |F_i|a_i^{1-p}u_i = 0$ and (3.2) we know that λK is in John position (the unique ellipsoid of maximal volume contained in K is the unit ball) for some $\lambda > 0$. Then Ball's volume ratio result [1] states that

$$V(\lambda K) \leq V(\triangle_n). \tag{3.3}$$

We have $S_p(\lambda K) = nV_p(\lambda K, B_n) \leq nV(\lambda K)$, since $B_n \subseteq \lambda K$, and $S_p(\triangle_n) = S(\triangle_n) = nV(\triangle_n)$. Together with (3.3), this gives

$$\begin{aligned} \partial_p(K) &= \partial_p(\lambda K) \leq \frac{S_p(\lambda K)}{V(\lambda K)^{\frac{n-p}{n}}} \leq nV(\lambda K)^{\frac{p}{n}} \leq nV(\triangle_n)^{\frac{p}{n}} \\ &= \frac{S_p(\triangle_n)}{V(\triangle_n)^{\frac{n-p}{n}}} = \partial_p(\triangle_n) \end{aligned} \tag{3.4}$$

where the last equality is classical since $S_p(\triangle_n) = S(\triangle_n)$. Using Lemma 2 and by approximating we get the same estimate for a general p-surface isotropic convex body. Since $\partial_p(K)$ is an affinely invariant quantity, we know that (1.4) is true for arbitrary convex body K with $\int_{S^{n-1}} u \mathrm{d}S_p(K, u) = 0$. □

By Theorem 1, we obtain the p-isoperimetric inequality and its reverse form as follows:

COROLLARY 1

Let K be a convex body in $\mathbb{R}^n$. Then we have

$$S_p(B_n) \leq S_p(K), \quad \textit{if } V(K) = V(B_n). \tag{3.5}$$

COROLLARY 2

Let K be a convex body in $\mathbb{R}^n$ with $\int_{S^{n-1}} u \mathrm{d}S_p(K, u) = 0$, and $\triangle_n$ be a regular simplex circumscribed to B_n. Then there is an affine image of $\tilde{K}$ of K satisfying

$$S_p(\tilde{K}) \leq S_p(\triangle_n) \quad \textit{and} \quad V(\tilde{K}) = V(\triangle_n). \tag{3.6}$$

Theorem 1 is useful for questions related to L_p-projection bodies, $\Pi_p K$, of K. The L_p-projection bodies was first introduced in [12], and can be defined by

$$h(\Pi_p K, u) = \left(\frac{1}{c_{n,p}} \int_{S^{n-1}} |u \cdot v|^p \mathrm{d}S_p(K, v) \right)^{\frac{1}{p}}, \quad u \in S^{n-1}. \tag{3.7}$$

Note that the normalization is chosen so that for the standard unit ball B_n in R^n, we have $\Pi_p B_n = B_n$. So we have $c_{n,p} = \int_{S^{n-1}} |u \cdot v|^p \mathrm{d}S(v)$.

Proof of Theorem 2. To establish the inequality, observe that for $p < \infty$, from the definition of L_p-projection bodies, and the polar coordinate formula, together with the Hölder's inequality, and Fubini's theorem we get

$$\begin{aligned}
\left(\frac{V(\Pi_p^* K)}{\omega_n} \right)^{-\frac{p}{n}} &= \left[\frac{1}{n\omega_n} \int_{S^{n-1}} h(\Pi_p K, \theta)^{-n} \mathrm{d}S(\theta) \right]^{-\frac{p}{n}} \\
&\leq \frac{1}{n\omega_n} \int_{S^{n-1}} h(\Pi_p K, \theta)^p \mathrm{d}S(\theta) \\
&= \frac{1}{n\omega_n} \int_{S^{n-1}} \frac{1}{c_{n,p}} \int_{S^{n-1}} |u \cdot \theta|^p \mathrm{d}S_p(K, u) \mathrm{d}S(\theta) \\
&= \frac{1}{n\omega_n} \int_{S^{n-1}} \frac{1}{c_{n,p}} \int_{S^{n-1}} |u \cdot \theta|^p \mathrm{d}S(\theta) \mathrm{d}S_p(K, u) \\
&= \frac{1}{n\omega_n} \int_{S^{n-1}} \mathrm{d}S_p(K, u) \\
&= \frac{S_p(K)}{n\omega_n}.
\end{aligned} \tag{3.8}$$

Since $V(\Pi_p^* K) V(K)^{(n-p)/p}$ is an affine invariant quantity, from Corollary 2, we know that there is an affine image $\tilde{K}$ of K satisfying (3.6). So we get

$$\begin{aligned}
\left(\frac{V(\Pi_p^* K) V(K)^{(n-p)/p}}{\omega_n} \right)^{-\frac{p}{n}} &= \left(\frac{V(\Pi_p^* \tilde{K}) V(\tilde{K})^{(n-p)/p}}{\omega_n} \right)^{-\frac{p}{n}} \\
&\leq \frac{S_p(\tilde{K})}{n\omega_n V(\tilde{K})^{(n-p)/n}} \\
&\leq \frac{S_p(\triangle_n)}{n\omega_n V(\triangle_n)^{(n-p)/n}} \\
&= \frac{V(\triangle_n)^{\frac{p}{n}}}{\omega_n}.
\end{aligned} \tag{3.9}$$

Since $V(\triangle_n) = \dfrac{n^{n/2}(n+1)^{(n+1)/2}}{n!}$, we get reverse L_p-Petty projection inequality (1.5) for convex body K with $\int_{S^{n-1}} u \mathrm{d}S_p(K, u) = 0$. Now let us check the equality condition. Since the equality in (3.9) holds if and only if K is a regular simplex, and (3.8) with equality if and only if $\Pi_p K$ is a ball. Observe the fact that was pointed out by Lutwak, Yang and Zhang in [11] that when $p = 2$, the projection body of the regular simplex (e.g. $\Pi_2 \triangle_n$) is a ball. So we get (1.5) with equality if and only if $p = 2$ and K is a regular simplex. □

COROLLARY 3

Let K be a convex body in $\mathbb{R}^n$ with $\int_{S^{n-1}} u \mathrm{d}S_p(K, u) = 0$. Then

$$V(\Pi_2^* K) V(K)^{(n-2)/2} \geq \frac{\omega_n^{(n+2)/2} n!}{n^{n/2}(n+1)^{(n+1)/2}}, \tag{3.10}$$

with equality if and only if K is a regular simplex.

Lutwak, Yang and Zhang [11] recently introduced a new ellipsoid $\Gamma_{-2}K$ associated with each convex body K that contains the origin in its interior. One of their results is as follows:

If $K \subset \mathbb{R}^n$ is a convex body positioned so that its John point is at the origin, then

$$V(\Gamma_{-2}K) \geq \frac{n! \omega_n}{n^{n/2}(n+1)^{(n+1)/2}} V(K), \tag{3.11}$$

with equality if and only if K is a simplex.

The normalized L_p polar projection body of K, $\Gamma_{-p}K$, for $p > 0$ is defined as the body whose radial function, for $u \in S^{n-1}$ is given by

$$\rho_{\Gamma_{-p}K}^{-p}(u) = \frac{1}{V(K)} \int_{S^{n-1}} |u \cdot v|^p \, \mathrm{d}S_p(K, v).$$

For more details on $\Gamma_{-p}K$, see [17].

Using Theorem 2, we can obtain an L_p version of inequality (3.11) as follows:

COROLLARY 4

Let K be a convex body in $\mathbb{R}^n$ with $\int_{S^{n-1}} u \mathrm{d}S_p(K, u) = 0$. Then for $1 < p < \infty$,

$$V(\Gamma_{-p}K) \geq \frac{\omega_n^{(n+p)/p} n!}{(c_{n,p})^{n/p} n^{n/2}(n+1)^{(n+1)/2}} V(K), \tag{3.12}$$

with equality if and only if $p = 2$ and K is a regular simplex.

Proof. From the definitions of $\Pi_p K$ and $\Gamma_{-p} K$, we have

$$
\begin{aligned}
V(\Pi_p^* K) &= \frac{1}{n} \int_{S^{n-1}} \rho(\Pi_p^* K, \theta)^n \mathrm{d}S(\theta) \\
&= \frac{1}{n} \int_{S^{n-1}} \left(\frac{1}{c_{n,p}} \int_{S^{n-1}} |\theta \cdot v|^p \, \mathrm{d}S_p(K, v) \right)^{-\frac{n}{p}} \mathrm{d}S(\theta) \\
&= \frac{1}{n} \int_{S^{n-1}} \left(\frac{V(K)}{c_{n,p}} \right)^{-\frac{n}{p}} \\
&\quad \times \left(\tfrac{1}{V(K)} \int_{S^{n-1}} |\theta \cdot v|^p \, \mathrm{d}S_p(K, v) \right)^{-\frac{n}{p}} \mathrm{d}S(\theta) \\
&= \frac{1}{n} \int_{S^{n-1}} \left(\frac{V(K)}{c_{n,p}} \right)^{-\frac{n}{p}} \rho(\Gamma_{-p} K, \theta)^n \mathrm{d}S(\theta) \\
&= \left(\frac{V(K)}{c_{n,p}} \right)^{-\frac{n}{p}} V(\Gamma_{-p} K).
\end{aligned} \tag{3.13}
$$

Combining (3.13) with inequality (1.5) gives the desired inequality. □

Note that when $p = 2$, from the definition of isotropic p-surface area measure we can compute that $c_{n,2} = \omega_n$. So (3.12) is equivalent to (3.11) for $p = 2$.

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