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Fixed points of *IA*-endomorphisms of a free metabelian Lie algebra

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Abstract. Let L be a free metabelian Lie algebra of finite *rank* at least 2. We show the existence of non-trivial fixed points of an *IA*-endomorphism of L and give an algorithm detecting them. In particular, we prove that the fixed point subalgebra Fix φ of an *IA*-endomorphism φ of L is not finitely generated.

Keywords. Free metabelian Lie algebra; fixed point.

1. Introduction

One of the important problem in the theory of Lie algebras is to determine the non-trivial fixed points of endomorphisms of free Lie algebras.

The most important results about fixed points of a finite group acting on a free algebra were obtained by Formanek [5]. Similar results for free Lie algebras were proved by Bryant [2] and Drensky [4]. They showed that if F is a free Lie algebra of finite rank n ($n \geq 2$) and G is a non-trivial finite group of automorphisms of F then under some assumptions the fixed point subalgebra $F^G = \{u \in F : ug = u \quad \text{for all } g \in G\}$ of F is not finitely generated. In [3], Bryant and Papistas have extended these results. Some of the results about fixed points of a finite group of automorphisms of a free algebra can also apply to fixed points for a single endomorphism. In [8], Shpilrain has given a matrix characterization of the fixed points of an *IA*-endomorphism of a free metabelian group. Matrix methods have been used by a number of authors to get interesting results on endomorphisms of free metabelian Lie algebras and groups. In this paper we obtain a criterion for detecting non-trivial fixed points of an *IA*-endomorphism of a free metabelian Lie algebra. We use the method of [8] to obtain a matrix characterization of *IA*-endomorphisms of a free metabelian Lie algebra with non-trivial fixed points.

We also prove that the non-trivial fixed point subalgebra of an *IA*-endomorphism of a free metabelian Lie algebra is not finitely generated. In our proof we use the method developed in [3].

2. Preliminaries

Let K be a field and let F be a free Lie algebra (over K) freely generated by the set $X = \{x_1, \ldots, x_n\}$ with $n \geq 2$. We use commutator notation $[f, g]$ to denote the product of elements f and g of F, while $[f_1, \ldots, f_n]$ denotes the right-normed products of elements

$f_1, \ldots, f_n$ of F. We denote by F' and F'' the derived algebra $[F, F]$ and the second derived algebra $[[F, F], [F, F]]$ respectively. We identify a free metabelian Lie algebra L of rank n with F/F'' in the usual way.

An endomorphism of F which induces the identity on F/F' is called an *IA*-endomorphism of the free Lie algebra F.

We write $U(F)$ for the universal enveloping algebra of the free Lie algebra F. Let P be the polynomial algebra $K[x_1, \ldots, x_n]$ and let V be the subspace spanned by $\{x_1, \ldots, x_n\}$. We regard $P \otimes_K V$ (tensor product taken over K) as a left P-module in the obvious way. Clearly it is a free P-module with $\{1 \otimes x : x \in X\}$ as a free generating set. It is easy to verify that the derived subalgebra L' of the free metabelian Lie algebra L may be viewed as a left P-module in which the image of an element g of L' under the action of a monomial $y_1 \ldots y_m$ of P is the right-normed Lie product $[y_1, \ldots, y_m, g]$. Note that P may be regarded as $U(F/F')$. For $g \in L'$, $u \in U(F/F')$ we write $u \cdot g$ to denote the image of g under the module action of u.

We denote by $\frac{\partial}{\partial x_i}$, $1 \leq i \leq n$ the left Fox derivatives [6,7]. The operators $\frac{\partial}{\partial x_i} : U(F) \to U(F)$ are linear mappings such that $\frac{\partial x_j}{\partial x_i} = \delta_{ij}$ (Kronecker delta), $\frac{\partial (u+v)}{\partial x_i} = \frac{\partial u}{\partial x_i} + \frac{\partial v}{\partial x_i}$, $\frac{\partial (uv)}{\partial x_i} = \frac{\partial u}{\partial x_i} \varepsilon(v) + u \frac{\partial v}{\partial x_i}$, where $\varepsilon : U(F) \to K$ is the augmentation homomorphism defined as $\varepsilon(x_i) = 0$ for all $i = 1, ..., n$. By Δ we denote the kernel of the augmentation homomorphism ε. The ideal Δ is a free left $U(F)$-module with basis $\{x_1, x_2, ..., x_n\}$. Thus any element $u \in \Delta$ can be uniquely written in the form $u = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} x_i$.

The Jacobian matrix J_φ of an endomorphism φ of F is defined as $J_\varphi = \left(\frac{\partial \varphi(x_i)}{\partial x_j}\right)_{1 \leq i,j \leq n}$, where $\frac{\partial}{\partial x_j}$ denotes partial Fox derivation with respect to x_j in the universal enveloping algebra $U(F)$ (see [7] for details). If $R \neq F$ is an ideal of F then we denote by Δ_R the ideal of $U(F)$ generated by R.

In [6], Fox has given a detailed account of the differential calculus in a free group ring. Since any associative algebra is naturally embedded in a free group algebra, most of the technical results remain valid for free associative algebras (see [7] for details).

Throughout this paper, we need the following lemmas.

Lemma 1. *Let J be an arbitrary ideal of F and let $u \in \Delta$. Then $u \in J\Delta$ if and only if $\frac{\partial u}{\partial x_j} \in J$ for each i, $1 \leq i \leq n$.*

Proof. Let $u = vw \in J\Delta$, *where* $v \in J$, $w \in \Delta$. The element w of Δ can be written in the form $w = \sum_{j=1}^{n} \frac{\partial w}{\partial x_j} x_j$. Therefore $\frac{\partial u}{\partial x_i} = v \frac{\partial w}{\partial x_i} \in J$ for each i, $1 \leq i \leq n$. The proof of the other part of the lemma is obvious. □

The proof of the next lemma can be found in [9].

Lemma 2. *Let R be an ideal of F and let $u \in F$. Then $u \in \Delta_R \Delta$ if and only if $u \in R'$.*

For any endomorphism φ of a Lie algebra G we write

$$\mathrm{Fix}\,\varphi = \{u \in G : \varphi(u) = u\}.$$

Thus $\mathrm{Fix}\,\varphi$ is the fixed point subalgebra of G. Let H be any subset of a vector space over K (K-space). We write $\langle H \rangle$ for the K-subspace spanned by H.

3. Matrix characterization of fixed points

Let K be a field and F be a free Lie algebra freely generated by the set $X = \{x_1, \ldots, x_n\}$, $n \geq 2$, over K and let $L = F/F''$. We denote the elements of the free Lie algebra F and their images in L by the same letters. When $u \in F$ or $u \in U(F)$ we denote by $\bar{u}$ the image of u in the free abelian Lie algebra F/F' or in the algebra $U(F/F')$ respectively.

Let φ be an *IA*-endomorphism of L defined as

$$\varphi : x_i \to x_i + u_i, \quad \text{where } u_i \in F', \quad 1 \leq i \leq n.$$

Then the Jacobian matrix J_φ of φ can be written as

$$J_\varphi = I + D_\varphi(u_1, \ldots, u_n),$$

where I is the identity matrix and

$$D_\varphi(u_1, \ldots, u_n) = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial u_n}{\partial x_1} & \cdots & \frac{\partial u_n}{\partial x_n} \end{pmatrix}.$$

The image of $D_\varphi(u_1, \ldots, u_n)$ over the abelianized algebra $U(F/F')$ will be denoted by $\bar{D}_\varphi(u_1, \ldots, u_n)$. By the *rank* of a matrix A over a commutative ring R we mean the maximal number of rows of A independent over R.

PROPOSITION 3

Let φ be an IA-endomorphism of L defined by $\varphi : x_i \to x_i + u_i$, $u_i \in F'$, $1 \leq i \leq n$. Then the columns of the matrix $\bar{D}_\varphi(u_1, \ldots, u_n)$ are dependent over $U(F/F')$.

Proof. Let $\varphi : L \to L$ defined by $\varphi : x_i \to x_i + u_i$, where $u_i \in F'$, $1 \leq i \leq n$. Now consider the image $\bar{D}_\varphi(u_1, \ldots, u_n)$ of the matrix $D_\varphi(u_1, \ldots, u_n)$ over $U(F/F')$. Then there is the following matrix equation:

$$\bar{D}_\varphi(u_1, \ldots, u_n) \begin{pmatrix} \overline{x_1} \\ \vdots \\ \overline{x_n} \end{pmatrix} = \begin{pmatrix} \overline{\frac{\partial u_1}{\partial x_1}} & \cdots & \overline{\frac{\partial u_1}{\partial x_n}} \\ \vdots & \ddots & \vdots \\ \overline{\frac{\partial u_n}{\partial x_1}} & \cdots & \overline{\frac{\partial u_n}{\partial x_n}} \end{pmatrix} \begin{pmatrix} \overline{x_1} \\ \vdots \\ \overline{x_n} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{i=1}^{n} \overline{\frac{\partial u_1}{\partial x_i}} \cdot \overline{x_i} \\ \vdots \\ \sum_{i=1}^{n} \overline{\frac{\partial u_n}{\partial x_i}} \cdot \overline{x_i} \end{pmatrix}.$$

Indeed $\sum_{i=1}^{n} \overline{\frac{\partial u_j}{\partial x_i}} \cdot \overline{x_i} = \overline{\sum_{i=1}^{n} \frac{\partial u_j}{\partial x_i} \cdot x_i} = \overline{u_j}$. Since $u_j \in F'$, $\overline{u_j} = 0$ in $U(F/F')$, $1 \leq j \leq n$. This yields the matrix equality

$$\bar{D}_\varphi (u_1, \ldots, u_n) \begin{pmatrix} \overline{x_1} \\ \vdots \\ \overline{x_n} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

in $U(F/F')$. Hence the columns of the matrix $\bar{D}_\varphi (u_1, \ldots, u_n)$ are dependent over $U(F/F')$. □

Our first main result guarantees the existence of the non-trivial fixed points of an *IA*-endomorphism of a free metabelian Lie algebra.

Theorem 4. *Let φ be an IA-endomorphism given by $\varphi : x_i \to x_i + u_i$, where $u_i \in L'$, $1 \leq i \leq n$. Then if rank $\bar{D}_\varphi (u_1, \ldots, u_n) \leq n - 2$ then φ has a non-trivial fixed point inside L'.*

Proof. Let $\varphi : L \to L$ be an endomorphism given by $\varphi : x_i \to x_i + u_i$, $u_i \in L'$, $1 \leq i \leq n$. For any element $f \in F$ we denote by $\bar{f}$ the image of f in the algebra F/F'.

If rank $\bar{D}_\varphi (u_1, \ldots, u_n) \leq n - 2$ then the rows of the matrix $\bar{D}_\varphi (u_1, \ldots, u_n)$ are dependent over the enveloping algebra $U(F/F')$. We write this dependence as

$$\sum_{i=1}^{n} a_i \overline{\frac{\partial u_i}{\partial x_j}} = 0, \qquad j = 1, \ldots, n \tag{1}$$

in $U(F/F')$ for some $a_s \in U(F/F')$, not all of them zero. Write system (1) as

$$\overline{\frac{\partial}{\partial x_j} \left(\sum_{i=1}^{n} a_i u_i \right)} = 0, \qquad j = 1, \ldots, n. \tag{2}$$

The system (2) is equivalent to the relation

$$\sum_{i=1}^{n} a_i u_i = 0$$

by Lemmas 1 and 2.

It is clear that the matrix $\bar{D}_\varphi (u_1, \ldots, u_n)$ is the image of the Jacobian matrix of the endomorphism $\varphi^* : x_i \to u_i$, $i = 1, \ldots, n$ of L in $U(F/F')$, i.e. $\overline{J_{\varphi^*}} = \bar{D}_\varphi(u_1, \ldots, u_n)$. If rank $\overline{J_{\varphi^*}} = k \leq n - 2$, then the maximal $U(F/F')$-independent subset of the set $\{u_1, \ldots, u_n\}$ has k elements. Let $\{u_1, \ldots, u_k\}$ be a maximal independent set. Therefore $u_1, \ldots, u_k$ generate a free submodule of the $U(F/F')$-module L'. Hence for any $j \geq k+1$ we have a relation of the form

$$a_j u_j = \sum_{i=1}^{k} b_{ji} u_i, \quad \text{where } b_{ji} \in U\left(F/F'\right). \tag{3}$$

Let $f \in L'$ be an element in the form

$$f = v_1 \cdot [x_1, x_2] + v_2 \cdot [x_2, x_3] + \cdots + v_{n-1} \cdot [x_{n-1}, x_n],$$

where $v_i \in U(F/F')$, $i = 1, \ldots, n-1$. We are going to construct elements $v_1, \ldots, v_{n-1}$ of $U(F/F')$ such that $f = v_1[x_1, x_2] + v_2[x_2, x_3] + \cdots + v_{n-1}[x_{n-1}, x_n]$ is a non-trivial fixed point of the endomorphism φ.

Assume that $\varphi(f) = f$. We have

$$\begin{aligned} &v_1 \cdot [x_1, x_2] + \cdots + v_{n-1} \cdot [x_{n-1}, x_n] \\ &\quad = v_1 \cdot [x_1 + u_1, x_2 + u_2] + \cdots + v_{n-1} \cdot [x_{n-1} + u_{n-1}, x_n + u_n]. \end{aligned}$$

From this equality we get

$$v_1 \cdot ([x_1, u_2] + [u_1, x_2]) + \cdots + v_{n-1} \cdot ([x_{n-1}, u_n] + [u_{n-1}, x_n]) = 0. \quad (4)$$

For $j \geq k+1$ multiply both sides of (4) by a_j, where a_j come from (3). Replace every $a_j u_j$ in (4) with $\sum_{i=1}^{k} b_{ji} u_i$. This gives a relation of the form

$$w_1 u_1 + \cdots + w_k u_k = 0, \quad (5)$$

where each w_i is a $U(F/F')$-linear combination of elements $v_1, v_2, \ldots, v_{n-1}$. Since $u_1, \ldots, u_k$ generate a free $U(F/F')$-submodule, the relation (5) leads a system of equations

$$w_j = 0, \quad 1 \leq j \leq k \quad (6)$$

with unknowns $v_1, v_2, \ldots, v_{n-1}$. Since $k \leq n-2$, this system has a non-trivial solution $(z_1, \ldots, z_{n-1})$ over $U(F/F')$. Hence

$$f = z_1 \cdot [x_1, x_2] + z_2 \cdot [x_2, x_3] + \cdots + z_{n-1} \cdot [x_{n-1}, x_n]$$

is a fixed point of the endomorphism φ which is contained in L'. □

Now we will consider the case rank $\bar{D}_\varphi(u_1, \ldots, u_n) = n-1$. In this case an *IA*-endomorphism may not have non trivial fixed points.

We are going to consider an element g of L', in the form

$$g = a_1 \cdot [x_1, x_2] + a_2 \cdot [x_2, x_3] + \cdots + a_{n-1} \cdot [x_{n-1}, x_n],$$

where $a_k \in U\left(F/F'\right)$, $1 \leq k \leq n-1$.

Lemma 5. *For an element $h \in L'$, there exist elements $z, a_1, \ldots, a_{n-1}$ of $U(F/F')$ such that $z \cdot h = a_1 \cdot [x_1, x_2] + a_2 \cdot [x_2, x_3] + \cdots + a_{n-1} \cdot [x_{n-1}, x_n]$.*

Proof. Right normed basic monomials $[x_{i_r}, \ldots, x_{i_2}, x_{i_1}]$, where $r \geq 2$ form a basis of the derived algebra L' (see [1] for details). Let P be the polynomial algebra $K[x_1, \ldots, x_n]$. We claim that

(i) the elements $[x_1, x_2], [x_2, x_3], \ldots, [x_{n-1}, x_n]$ generate a free P-submodule of rank $n-1$,

(ii) the P-module L' has no submodules of rank n.

Proof of (i). Let V be the subspace spanned by the set $\{x_1, \ldots, x_n\}$. Form the tensor product $P \otimes_K V$. It is shown in Lemma 4.1(i) of [3] that there is a P-module embedding

$$\theta : L' \to P \otimes_K V$$

in which $[v_r, \ldots, v_2, v_1] \to v_r \ldots v_3 \cdot v_2 \otimes v_1 - v_r \ldots v_3 \cdot v_1 \otimes v_2$ for all $r \geq 2$ and all $v_1, \ldots, v_r \in V$. Suppose that

$$f_1 \cdot [x_1, x_2] + f_2 \cdot [x_2, x_3] + \cdots + f_{n-1} \cdot [x_{n-1}, x_n] = 0$$

with $f_1, \ldots, f_{n-1} \in P$. Applying θ to the above equation, we get

$$\begin{aligned} f_1 x_1 \cdot (1 \otimes x_2) - f_1 x_2 \cdot (1 \otimes x_1) + f_2 x_2 \cdot (1 \otimes x_3) \\ - f_2 x_3 \cdot (1 \otimes x_2) + \cdots + f_{n-1} x_{n-1} \cdot (1 \otimes x_n) = 0. \end{aligned}$$

Since $\{1 \otimes x_1, \ldots, 1 \otimes x_n\}$ is a free generating set of $P \otimes_K V$ as P-module, we obtain

$$\begin{aligned} -f_1 x_2 &= 0, \\ f_1 x_1 - f_2 x_3 &= 0, \\ f_2 x_2 - f_3 x_4 &= 0, \\ &\vdots \\ f_{n-1} x_{n-1} &= 0. \end{aligned}$$

Since P is a principal ideal domain (PID), we have $f_1 = \cdots = f_{n-1} = 0$ and so, we have the required result.

Proof of (ii). To get a contradiction we assume that there exists a P-submodule M of L' with rank n. Since $P \otimes_K V$ is a free P-module and P is a PID, we obtain $\theta(M)$ is a free P-module and rank $(\theta(M)) \leq n$. Since θ is P-module embedding, we get rank $(\theta(M)) = n$. Therefore θ is onto which is a contradiction. (For example, for any $u \in L', \theta(u) \neq x_1 \otimes x_1$.)

We now proceed with the proof of the lemma. For any pair k, l we have

$$b_{k,l} \cdot [x_k, x_l] = c_1 \cdot [x_1, x_2] + c_2 \cdot [x_2, x_3] + \cdots + c_{n-1} \cdot [x_{n-1}, x_n] \tag{7}$$

for some $b_{k,l}, c_j \in U(F/F')$. Therefore any k-linear combination of elements of the form (7) is in the form

$$a_1 \cdot [x_1, x_2] + a_2 \cdot [x_2, x_3] + \cdots + a_{n-1} \cdot [x_{n-1}, x_n], \text{ where } a_k \in U(F/F').$$

This completes the proof. □

The following examples will show that the situation is very subtle in the case rank $\bar{D}_\varphi(u_1, \ldots, u_n) = n - 1$. In this case, anything can happen.

PROPOSITION 6

Let $n \geq 3$ and let φ be an endomorphism of L defined by

$$\begin{aligned} \varphi \ : \ & x_i \to x_i + u_i, \quad 1 \leq i \leq n-2 \\ & x_{n-1} \to x_{n-1} + u, \\ & x_n \to x_n + u, \end{aligned}$$

where $u_i, u \in L'$ and the elements u_i, u are $U(F/F')$-independent, $1 \leq i \leq n-2$. Then φ has no non-trivial fixed points inside L'.

Proof. For any element $f \in L'$ there exist an element $z \in U(F/F')$ such that $z \cdot f = v_1 \cdot [x_1, x_2] + \cdots + v_{n-1} \cdot [x_{n-1}, x_n]$, where $v_i \in U(F/F')$. Suppose $\varphi(f) = f$. Multiplying this equality with z from left, we get $\varphi(z \cdot f) = z \cdot f$. Then we have (cf. (4))

$$v_1 (x_1 u_2 + u_1 x_2) + \cdots + v_{n-2} (x_{n-2} u + u_{n-2} x_{n-1}) \\ + v_{n-1} (x_{n-1} u + u x_n) = 0.$$

Since $u, u_1, \ldots, u_{n-2}$ are independent, we get the system

$$\begin{aligned} -v_1 x_2 &= 0, \\ v_j x_j - v_{j+1} x_{j+2} &= 0, \quad j = 1, \ldots, n-3 \\ v_{n-2} x_{n-2} + v_{n-1} (x_{n-1} - x_n) &= 0 \end{aligned} \tag{8}$$

in $U(F/F')$. Since $U(F/F')$ is an integral domain, all of the elements $v_1, \ldots, v_{n-1}$ are equal to zero. Hence system (8) has no non-zero solutions. This completes the proof. □

PROPOSITION 7

Let M be a free metabelian Lie algebra freely generated by $\{x_1, x_2\}$ and φ be an endomorphism of M defined by

$$\begin{aligned} \varphi: \ & x_1 \to x_1 + [x_1, u], \\ & x_2 \to x_2 + u, \end{aligned}$$

for some non-zero $u \in M'$. Then φ has no non-trivial fixed points.

Proof. First we prove that φ has no non-trivial fixed points in M'. Let $h \in M'$ and $\varphi(h) = h$. Since h has the form $h = w[x_1, x_2]$ for some $w \in U(F/F')$, we have

$$w \cdot [x_1, x_2] = w \cdot [x_1 + [x_1, u], x_2 + u].$$

It follows that

$$w (x_1 - x_2 x_1) u = 0.$$

Hence $w(x_1 - x_2 x_1) = 0$ in $U(F/F')$. This is possible only if $w = 0$. Thus $h = 0$.

Now we will show that φ has no non-trivial fixed points outside M'. Let m be an arbitrary element of M. Then m has the form $ax_1 + bx_2 + w[x_1, x_2]$ for some $w \in U(F/F')$ and $a, b \in K$. Suppose $\varphi(m) = m$. Then we have

$$ax_1 + bx_2 + w[x_1, x_2] \\ = a(x_1 + [x_1, u]) + b(x_2 + u) + w[x_1 + [x_1, u], x_2 + u].$$

It follows that

$$(ax_1 + b + w(x_1 - x_2 x_1)) u = 0.$$

Hence

$$ax_1 + b + w(x_1 - x_2x_1) = 0 \tag{9}$$

in $U(F/F')$. This is possible only if $a = b = w = 0$. This completes the proof. □

Now we are going to prove that an *IA*-endomorphism of L has non-zero fixed points in a certain case. First we need some observations.

Let φ be an *IA*-endomorphism of L defined by $\varphi(x_i) = y_i = x_i + u_i$, where $u_i \in L'$, $1 \le i \le n$. If rank $\bar{D}_\varphi(u_1, \ldots, u_n) = n-1$ then the maximal $U(F/F')$-independent subset of $\{u_1, \ldots, u_n\}$ has $n-1$ elements. Let $\{u_1, \ldots, u_{n-1}\}$ be a maximal independent set. Therefore $u_1, \ldots, u_{n-1}$ generates a free submodule of the $U(F/F')$-module L'. Hence we have a relation of the form

$$a_n u_n = \sum_{i=1}^{n-1} a_i u_i, \tag{10}$$

for some $a_j \in U(F/F')$.

PROPOSITION 8

Let φ be an IA-endomorphism of L defined by $\varphi(x_i) = x_i + u_i$, where $u_i \in L'$, $1 \le i \le n$ and rank $\bar{D}_\varphi(u_1, \ldots, u_n) = n-1$. *Then φ has a non-trivial fixed point in L' if and only if the system*

$$\begin{aligned} -v_1x_2a_n + v_{n-1}x_{n-1}a_1 &= 0, \\ v_ix_ia_n - v_{i+1}x_{i+2}a_n + v_{n-1}x_{n-1}a_{i+1} &= 0, \quad 1 \le i \le n-2 \end{aligned}$$

with unknowns $v_1, \ldots, v_{n-1}$, has a non-zero solution in $U(F/F')$.

Proof. Let $\varphi(h) = h$ for any element h of L'. By Lemma 5 there exists an element $z \in U(F/F')$ such that

$$z \cdot h = v_1 \cdot [x_1, x_2] + v_2 \cdot [x_2, x_3] + \cdots + v_{n-1} \cdot [x_{n-1}, x_n]$$

for some elements $v_j \in U(F/F')$. Apply φ to this equality:

$$\varphi(z \cdot h) = v_1 \cdot [y_1, y_2] + v_2 \cdot [y_2, y_3] + \cdots + v_{n-1} \cdot [y_{n-1}, y_n]. \tag{11}$$

Multiplying the equality $\varphi(h) = h$ with z from left we get

$$\varphi(z \cdot h) = z \cdot h = v_1 \cdot [x_1, x_2] + v_2 \cdot [x_2, x_3] + \cdots + v_{n-1} \cdot [x_{n-1}, x_n]. \tag{12}$$

Combining (12) with (11) we obtain

$$v_1 \cdot [x_1, x_2] + \cdots + v_{n-1} \cdot [x_{n-1}, x_n] = v_1 \cdot [y_1, y_2] + \cdots + v_{n-1} \cdot [y_{n-1}, y_n]. \tag{13}$$

Equality (13) gives

$$-v_1x_2u_1 + \sum_{i=1}^{n-2} (v_ix_i - v_{i+1}x_{i+2})\, u_{i+1} + v_{n-1}x_{n-1}u_n = 0. \tag{14}$$

Multiplying both sides of (14) by a_n and replacing $a_n u_n$ in (14) with $\sum_{i=1}^{n-1} a_i u_i$, we obtain a system of $n-1$ equations

$$\begin{aligned} -v_1 x_2 a_n + v_{n-1} x_{n-1} a_1 &= 0, \\ v_i x_i a_n - v_{i+1} x_{i+2} a_n + v_{n-1} x_{n-1} a_{i+1} &= 0, \qquad 1 \le i \le n-2 \end{aligned} \tag{15}$$

with unknowns $v_1, \ldots, v_{n-1}$. This completes the proof. The "if" part is obvious. □

Theorem 9. *There is an algorithm for detecting non-trivial fixed points of an arbitrary IA-endomorphism of the free metabelian Lie algebra L.*

Proof. Let φ be an *IA*-endomorphism of L defined by

$$\varphi : x_i \to x_i + u_i, \quad u_i \in F', \quad 1 \le i \le n.$$

First we compute the rank of the matrix $\bar{D}_\varphi(u_1, \ldots, u_n)$: Indeed by using elementary transformations of the rows of the matrix $\bar{D}_\varphi(u_1, \ldots, u_n)$, we can construct a basis of the free left $U(F/F')$-submodule of the free left $U(F/F')$-module $(U(L))^n$, and compute its rank.

We have two cases for rank $\bar{D}_\varphi(u_1, \ldots, u_n)$.

Case I. If rank $\bar{D}_\varphi(u_1, \ldots, u_n) \le n-2$, then we refer to Theorem 4.

Case II. If rank $\bar{D}_\varphi(u_1, \ldots, u_n) = n-1$, we will consider two cases:

(a) We find if there is a non-trivial fixed points inside L': We consider a system (15) as in the proof of Proposition 8. This is a system of $n-1$ homogeneous $U(F/F')$-linear equations in $n-1$ unknowns $v_1, \ldots, v_{n-1}$. Now we check the dependence of the rows of the coefficient matrix. If they are dependent then φ has a non-trivial fixed point inside L'. If these rows are independent then φ has no non-trivial fixed points inside L'. Then we consider the case (b).

(b) To find out if there is a non-trivial fixed point of φ outside L' we proceed as in the proof of Proposition 7, but instead of having just one equation of the form (9), we will have a system of $n-1$ $U(F/F')$-linear equations in $n-1$ unknowns. Since $U(F/F')$ is isomorphic to a polynomial algebra, using algorithms in polynomial algebras we can solve the system. This completes the proof. □

4. The fixed point subalgebra

Let K be a field and let L be the free metabelian Lie algebra generated by the set X over K. In this section we recall some definitions of [2]. Lie monomials of L are defined in the usual way as non-zero Lie products of elements of X. The degree of a monomial is the length of this product. For each positive integer n we write L_n for the K-subspace spanned by the monomials of degree n. Thus L is a K-space direct sum

$$L = L_1 \oplus L_2 \oplus \cdots .$$

The degree of any element f of L, denoted by deg f, is the smallest value of n such that $f \in L_1 \oplus L_2 \oplus \cdots \oplus L_n$. For each positive integer m we have

$$\gamma_m(L) = L_m \oplus L_{m+1} \oplus \cdots,$$

where $\gamma_m(L)$ is the m-th term of the lower central series of L. Thus, L is residually nilpotent. Let $x \in X$ and for $i = 1, \ldots, n$ let $L_{i,n}$ be the K-subspace spanned by all Lie monomials which have degree i in x. Then, for each $n \geq 0$, we can write

$$L_n = L_{0,n} \oplus L_{1,n} \oplus \cdots \oplus L_{n,n}.$$

Let $L(x)$ denote the subspace of L spanned by all monomials which have at least one factor from $X \setminus \{x\}$ .Thus

$$L(x) = L_{0,1} \oplus (L_{0,2} \oplus L_{1,2}) \oplus \cdots \oplus (L_{0,n} \oplus \cdots \oplus L_{n-1,n}) \oplus \cdots .$$

Note that for all $n \geq 2$, we have $L_{n,n} = \{0\}$, and $L(x) = L_{0,1} \oplus L'$.

Let q be a real number such that $0 \leq q \leq 1$. Define $L(x, q)$ to be the subspace of L, spanned by all subspaces $L_{i,n}$, with $n \geq 0$ and $i \leq qn$. We can write $L = L(x, 1)$ and $L(x) = \bigcup_{0 \leq q < 1} L(x, q)$.

The following lemma which was proved by Bryant and Papistas gives a useful necessary condition for a subalgebra of a free Lie algebra to be finitely generated.

Lemma 10 [3].

(i) *For each* $0 \leq q \leq 1$, $L(x, q)$ *and* $L(x)$ *is a subalgebra of* L.
(ii) *Let* S *be a finitely generated subalgebra of* L *such that* $S \subseteq L(x)$. *Then* $S \subseteq L(x, q)$ *for some* q *with* $0 \leq q < 1$.

The following result is proved in [3] for residually nilpotent Lie algebras.

Lemma 11 [3]. *Let* φ *be a non trivial IA-endomorphism of a residually nilpotent Lie algebra* G. *Then* $\operatorname{Fix}\varphi + G' \neq G$.

Lemma 12 [3]. *If* u *is a non-zero element of* L' *and* v *is a non-zero element of* $U(L/L')$ *then* $v \cdot u \neq 0$.

Now we are going to show that the fixed point subalgebra of an *IA*-endomorphism of a free metabelian Lie algebra is not finitely generated. Our proof is a minor adaptation of the arguments given in [3].

Theorem 13. *Let* φ *be a non trivial IA-endomorphism of* L *such that* $\operatorname{Fix}\varphi \cap L' \neq \{0\}$. *Then* $\operatorname{Fix}\varphi$ *is not finitely generated.*

Proof. Since L is residually nilpotent by Lemma 11, $\operatorname{Fix}\varphi + L' \neq L$ and $L = \langle X \rangle \oplus L'$. Hence we have $\operatorname{Fix}\varphi \subseteq < X \setminus \{x\} > \oplus L' = L(x)$. To get a contradiction assume that $\operatorname{Fix}\varphi$ is finitely generated. By Lemma 10(ii), there exists a real number q with $0 \leq q \leq 1$

such that $\operatorname{Fix}\varphi \subseteq \mathrm{L}(x, q)$. Since $\operatorname{Fix}\varphi \cap \mathrm{L}' \neq \{0\}$ there exists a non-zero element g of $\operatorname{Fix}\varphi \cap \mathrm{L}'$. Assume that $\deg g = n$. Write $g = g_2 + \cdots + g_n$, where $g_i \in L_i$ for $i = 2, \ldots, n$. Since $\varphi(g_j) \in L_j$ for $j = 2, \ldots, n$, from $\varphi(g) = g$ we obtain $\varphi(g_j) = g_j$, where $j = 2, \ldots, n$. Hence without loss of generality, we can replace g by g_n. Write $g_n = \sum_{i=0}^{n} w_{i,n}$, where each $w_{i,n} \in L_{i,n}$ for $i = 0, \ldots, n$.

Now consider the element $h = [\underbrace{x, \ldots, x}_{m\text{-times}}, g_n]$ of L. We claim that $h \notin L(x, q)$. Since $\varphi(g_n) = g_n$, this implies

$$\varphi(h) = [x, \ldots, x, \varphi(g_n)] = [x, \ldots, x, g_n] = h.$$

Thus $h \in \operatorname{Fix}\varphi$. Consider the element

$$h = \sum_{i=0}^{n} [x, \ldots x, w_{i,n}],$$

where $w_{i,n} \in L_{i,n}$ for $i = 1, \ldots, n$. Then $\deg([x, \ldots x, w_{i,n}]) = n + m$. Now choose k maximal such that $w_{k,n} \neq 0$ and choose m such that $m > q(n + m)$. Then $[x, \ldots x, w_{i,n}] \neq 0$ and $h \neq 0$ by Lemma 12. Hence by the choice of m we have $k + m > m > q(n + m)$ and $h \in \bigoplus_{i>q(n+m)} L_{i,n+m}$. Thus $h \notin L(x, q)$. But $h \in \operatorname{Fix}\varphi \subseteq L(x, q)$. This contradiction completes the proof. □

COROLLARY 14

Let φ be a non-trivial IA-endomorphism of L. If $\operatorname{Fix}\varphi \cap L' \neq \{0\}$ *then it is not finitely generated.*

Proof. Let the element h and q be as in the proof of the Theorem 13. Since $\operatorname{Fix}\varphi \cap L' \subset \operatorname{Fix}\varphi \subseteq L(x) \subseteq L(x, q)$ the result follows. □

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