Proc. Indian Acad. Sci. (Math. Sci.) Vol. 121, No. 3, August 2011, pp. 369–377.

An almost sure central limit theorem for the weight function sequences of NA random variables

QUNYING WU

College of Science, Guilin University of Technology, Guilin 541004,
People's Republic of China
E-mail: wqy666@glite.edu.cn

MS received 1 January 2010

Abstract. Consider the weight function sequences of NA random variables. This paper proves that the almost sure central limit theorem holds for the weight function sequences of NA random variables. Our results generalize and improve those on the almost sure central limit theorem previously obtained from the i.i.d. case to NA sequences.

Keywords. Weight function sequence; NA random variables; almost sure central limit theorem.

1. Introduction and main results

DEFINITION

Random variables $X_1, X_2, \ldots, X_n, n \geq 2$ are said to be negatively associated (NA) if for every pair of disjoint subsets A_1 and A_2 of $\{1, 2, \ldots, n\}$,

$$\operatorname{cov}(f_1(X_i; i \in A_1), f_2(X_j; j \in A_2)) \leq 0,$$

where f_1 and f_2 are increasing for every variable (or decreasing for every variable) such that this covariance exists. A sequence of random variables $\{X_i; i \geq 1\}$ is said to be NA if every finite subfamily is NA.

Obviously, if $\{X_i; i \geq 1\}$ is a sequence of NA random variables, and $\{f_i; i \geq 1\}$ is a sequence of nondecreasing (or non-increasing) functions, then $\{f_i(X_i); i \geq 1\}$ is also a sequence of NA random variables.

This definition was introduced by Joag-Dev and Proschan [6]. Statistical test depends greatly on sampling. The random sampling without replacement from a finite population is NA, but is not independent. NA sampling has wide applications such as in multivariate statistical analysis and reliability theory. Because of the wide applications of NA sampling, the limit behaviors of NA random variables have received more and more attention recently. One can refer to Joag-Dev and Proschan [6] for fundamental properties, Matula [8] for the three series theorem, Shao [12] for the moment inequalities, Wu [14] for the strong consistency, and Wu and Jiang [15,16] for the law of the iterated logarithm.

Starting with Brosamler [2] and Schatte [11], in the last two decades several authors investigated the almost sure central limit theorem (ASCLT) for i.i.d. random variables. We refer the reader to [2], [11], [7], [5], [4], [9] and [17]. The simplest form of the ASCLT [2,11,7] reads as follows: Let $\{X_n; n \geq 1\}$ be i.i.d. random variables with mean 0, variance $\sigma^2 > 0$ and partial sums S_n. Then

$$\lim_{n\to\infty} \frac{1}{\ln n} \sum_{k=1}^{n} \frac{1}{k} I\left\{\frac{S_k}{\sigma\sqrt{k}} < x\right\} = \Phi(x) \quad \text{a.s. for any } x \in R, \tag{1.1}$$

where I denotes indicator function, $\Phi(x)$ is the standard normal distribution function. Relation (1.1) is a logarithmic means of the sequence $I\{\frac{S_k}{\sigma\sqrt{k}} < x\}$; by Theorem 1 of [11], the arithmetic means of this sequence does not converge to $\Phi(x)$ with probabity one, i.e.

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} I\left\{\frac{S_k}{\sigma\sqrt{k}} < x\right\} = \Phi(x) \quad \text{a.s. for any } x \in R \tag{1.2}$$

fails.

It is natural to ask that if there exist other different averaging methods which also work in the ASCLT, i.e., whether there exists a weight sequence $\{d_k \neq 1/k; k \geq 1\}$ such that

$$\lim_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k I\left\{\frac{S_k}{\sigma\sqrt{k}} < x\right\} = \Phi(x) \quad \text{a.s. for any } x \in R \tag{1.3}$$

holds, where $D_n = \sum_{k=1}^{n} d_k$.

The terminology of summation procedures (see e.g. [3]) shows that the larger the weight sequence $\{D_k; k \geq 1\}$ in (1.3) is, the stronger the relation becomes. By this argument, one should also expect to get stronger results if we use larger weights. And it would be of considerable interest to determine the optimal weights.

The main purpose of this paper is to study function sequences of NA random variables and try to find some larger weight sequences $\{d_k; k \geq 1\}$ such that (1.3) holds.

In the following, $a_n \sim b_n$ denotes $a_n/b_n \to 1, n \to \infty$, and $a_n \ll b_n$ denotes that there exists a constant $c > 0$ such that $a_n \leq cb_n$ for sufficiently large n. The symbol $\mathscr{C}_G$ denotes the set of continuity points of G, and c stands for a generic positive constant which may differ from one place to another.

Let $c_n > 0$ with

$$c_n \uparrow \infty, \qquad \lim_{n\to\infty} \frac{c_{n+1}}{c_n} = c < \infty. \tag{1.4}$$

$f_k : R^k \to R$ $(k = 1, 2, \ldots)$ and $f_{k,l} : R^{l-k} \to R$ $(k < l)$ are increasing for every variable (or decreasing for every variable).

Theorem 1. *Let $\{X_n; n \geq 1\}$ be a sequence of NA random variables satisfying*

$$E|f_l(X_1, \ldots, X_l) - f_{k,l}(X_{k+1}, \ldots, X_l)| \ll (c_l/c_k)^{-\delta} \tag{1.5}$$

and

$$|\text{Cov}(f_k(X_1, \ldots, X_k), f_{k,l}(X_{k+1}, \ldots, X_l))| \ll (c_l/c_k)^{-\delta} \tag{1.6}$$

for some constant $\delta > 0$. Put

$$d_k = \ln \frac{c_{k+1}}{c_k} \exp(\ln^\alpha c_k), \quad 0 \le \alpha < 1/2, \quad D_n = \sum_{k=1}^n d_k. \tag{1.7}$$

Then for any distribution function G the relations

$$\lim_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^n d_k I\{f_k(X_1, \ldots, X_k) < x\} = G(x) \quad a.s.\ for\ any\ x \in \mathscr{C}_G \tag{1.8}$$

and

$$\lim_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^n d_k P\{f_k(X_1, \ldots, X_k) < x\} = G(x) \quad for\ any\ x \in \mathscr{C}_G \tag{1.9}$$

are equivalent.

In particular, let $f_k(X_1, \ldots, X_k) = \frac{S_k}{\sigma\sqrt{k}}$. We have

Theorem 2. *Let $\{X_n; n \ge 1\}$ be a stationary sequence of NA random variables with $EX_1 = 0, 0 < EX_1^2 < \infty$ and $\sigma^2 \triangleq EX_1^2 + 2\sum_{k=2}^\infty EX_1X_k > 0$. Assume that there exists a constant $\gamma > 0$ such that*

$$(c_l/c_k)^\gamma \ll l/k, \quad k < l. \tag{1.10}$$

Then

$$\lim_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^n d_k I\left\{\frac{S_k}{\sigma\sqrt{k}} < x\right\} = \Phi(x) \quad a.s.\ for\ any\ x \in R, \tag{1.11}$$

*where d_k and D_n are defined by (*1.7).

Remark 1. By the terminology of summation procedures, Theorems 1 and 2 remain valid if we replace the weight sequence $\{d_k; k \ge 1\}$ by $\{d_k^*; k \ge 1\}$ such that $0 \le d_k^* \le d_k$, $\sum_{k=1}^\infty d_k^* = \infty$.

Remark 2. Assume that $\{X_n; n \ge 1\}$ is a sequence of i.i.d. random variables.

(1) If $c_k = k$ and $\alpha = 0$, (1.7) gives $d_k \sim \mathrm{e}k^{-1}$ and $D_n \sim \mathrm{e}\ln n$. Thus (1.11) becomes (1.1).
(2) Let $c_k = k$ in (1.7), then $d_k \sim k^{-1}\exp(\ln^\alpha k)$. For $\beta > -1, 0 < \alpha < 1/2$, since $d_k^* \triangleq k^{-1}\log^\beta k \le d_k$, $D_n^* = (\beta+1)^{-1}\log^{\beta+1} n \to \infty$. By Remark 1, (1.11) remains valid if we replace the weight sequence $\{d_k; k \ge 1\}$ by $\{d_k^*; k \ge 1\}$. Thus our Theorem 2 not only generalize and improve those on ASCLT previously obtained by Brosamler [2], Schatte [11] and Lacey and Philipp [7] from the i.i.d. case to NA sequences but also expand the scope of the weights $\{d_k; k \ge 1\}$.

Remark 3. In (1.7), let $\alpha = 1$, $c_k = k$, then $d_k \sim 1$, $D_n \sim n$. Thus, by (1.2) and Remark 1, Theorem 2 does not hold for $\alpha \geq 1$. Whether Theorem 2 holds also for some $1/2 \leq \alpha < 1$ remains open.

Remark 4. Equation (1.10) is essential, and if removed, Theorem 2 may not be set up. For example, let $c_k = 2^k$, we have $d_k \geq 1$. Hence, Theorem 2 does not hold for $c_k = 2^k$.

Remark 5. Let $\alpha > 0$ in (1.7). By the Stolz theorem, (1.4), and $\ln(1+x) \sim x$, $\mathrm{e}^x - 1 \sim x$, for $x \to 0$ we get

$$D_n \sim \frac{1}{\alpha} \ln^{1-\alpha} c_n \exp(\ln^{\alpha} c_n). \tag{1.12}$$

Remark 6. The weights in Theorems 1 and 2 have a very wide range. Taking different values of c_k and α in Theorem 2, we can get a series of results such as the following corollary.

COROLLARY

Let $\{X_n; n \geq 1\}$ be a stationary sequence of NA random variables with $EX_1 = 0, 0 < EX_1^2 < \infty$ and $\sigma^2 \,\widehat{=}\, EX_1^2 + 2\sum_{k=2}^{\infty} EX_1X_k > 0$.

(1) *If $c_k = k$, (1.7) gives $d_k \sim \mathrm{e}k^{-1}$, $D_n \sim \mathrm{e}\ln n$ for $\alpha = 0$ and $d_k \sim k^{-1}\exp(\ln^{\alpha} k)$, $D_n \sim \frac{1}{\alpha}\ln^{1-\alpha} n \exp(\ln^{\alpha} n)$ for $\alpha > 0$. Thus* (1.1) *and*

$$\lim_{n\to\infty} \frac{\alpha}{\ln^{1-\alpha} n \exp(\ln^{\alpha} n)} \sum_{k=1}^{n} \frac{\exp(\ln^{\alpha} k)}{k} I\left\{\frac{S_k}{\sigma\sqrt{k}} < x\right\} = \Phi(x) \quad a.s.\ for\ any\ x \in R$$

hold.

(2) *Set $\lg_0 x = x$ and denote $\lg_j x = \ln\{\max(\mathrm{e}, \lg_{j-1} x)\}$ for $j \geq 1$. We also define $\lg x = \lg_1 x$. If $c_k = \lg_j k$, we have $c_l/c_k = (\lg_j l)/(\lg_j k) \leq l/k$ by $(\lg_j k)/k \downarrow$. Equation* (1.7) *gives $d_k \sim \left(k\prod_{i=1}^{j}\lg_i k\right)^{-1}$, $D_n \sim \lg_{j+1} n$ for $\alpha = 0$ and $d_k \sim \frac{\exp((\lg_{j+1} k)^{\alpha})}{k\prod_{i=1}^{j}\lg_i k}$, $D_n \sim \frac{1}{\alpha}(\lg_{j+1} n)^{1-\alpha}\exp((\lg_{j+1} n)^{\alpha})$ for $\alpha > 0$. Thus*

$$\lim_{n\to\infty} \frac{1}{\lg_{j+1} n} \sum_{k=1}^{n} \frac{1}{k\prod_{i=1}^{j}\lg_i k} I\left\{\frac{S_k}{\sigma\sqrt{k}} < x\right\} = \Phi(x) \quad a.s.\ for\ any\ x \in R,$$

and

$$\lim_{n\to\infty} \frac{\alpha}{(\lg_{j+1} n)^{1-\alpha}\exp((\lg_{j+1} n)^{\alpha})} \sum_{k=1}^{n} \frac{\exp((\lg_{j+1} k)^{\alpha})}{k\prod_{i=1}^{j}\lg_i k} I\left\{\frac{S_k}{\sigma\sqrt{k}} < x\right\} = \Phi(x) \quad a.s.\ for\ any\ x \in R.$$

(3) *If* $c_k = \exp(\ln^{\beta+1} k)$, $-1 < \beta < 0$, *we have* $c_k/c_l \geq k/l$ *by* $(\exp(\ln^{\beta+1} k))/k\downarrow$, *and* (1.7) *gives* $d_k \sim \frac{(\beta+1)\ln^{\beta} k}{k}$, $D_n \sim \frac{\ln^{\beta+1} n}{\beta+1}$ *for* $\alpha = 0$ *and* $d_k \sim \frac{(\beta+1)\ln^{\beta} k \ln^{\alpha(\beta+1)} k}{k}$, $D_n \sim \frac{1}{\alpha}(\ln n)^{(\beta+1)(1-\alpha)} \exp((\ln n)^{\alpha(\beta+1)})$ *for* $\alpha > 0$. *Thus*

$$\lim_{n\to\infty} \frac{(\beta+1)^2}{\ln^{\beta+1} n} \sum_{k=1}^{n} \frac{\ln^{\beta} k}{k} I\left\{\frac{S_k}{\sigma\sqrt{k}} < x\right\} = \Phi(x)$$

$$a.s.\ for\ any\ x \in R$$

and

$$\lim_{n\to\infty} \frac{\alpha(\beta+1)}{(\ln n)^{(\beta+1)(1-\alpha)} \exp((\ln n)^{\alpha(\beta+1)})} \sum_{k=1}^{n} \frac{\ln^{\beta} k \ln^{\alpha(\beta+1)} k}{k} I\left\{\frac{S_k}{\sigma\sqrt{k}} < x\right\} = \Phi(x)$$

$$a.s.\ for\ any\ x \in R.$$

2. Proofs

Proof of Theorem 1. Denote by $\mathscr{A}$ the class of bound functions with bounded continuous derivatives. By Theorem 7.1 of [1] and §2 of [10], (1.8) and (1.9) are equivalent to

$$\lim_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k g(f_k(X_1, \ldots, X_k)) = \int_{-\infty}^{\infty} g(x) \mathrm{d}G(x) \quad \text{a.s.}$$

and

$$\lim_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k E g(f_k(X_1, \ldots, X_k)) = \int_{-\infty}^{\infty} g(x) \mathrm{d}G(x)$$

for every $g \in \mathscr{A}$, respectively. Hence, it suffices to prove that for any $g \in \mathscr{A}$,

$$\lim_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k \left(g(f_k(X_1, \ldots, X_k)) - E g(f_k(X_1, \ldots, X_k))\right) = 0 \quad \text{a.s.} \tag{2.1}$$

Put

$$\xi_k = g(f_k(X_1, \ldots, X_k)) - E g(f_k(X_1, \ldots, X_k)).$$

Clearly, there is a constant $c > 0$ such that

$$|g(x)| \leq c, \quad |g'(x)| \leq c,$$

$$|g(x) - g(y)| \leq c|x - y| \quad \text{for any} \quad x, y \in R, \quad \text{and}$$

$$|\xi_k| \leq 2c \quad \text{for any} \quad k. \tag{2.2}$$

For any $1 \leq k < l$, note that $f_k(X_1, \ldots, X_k)$ and $f_{k,l}(X_{k+1}, \ldots, X_l)$ are increasing for every variable (or decreasing for every variable), and thus $f_k(X_1, \ldots, X_k)$ and $f_{k,l}(X_{k+1}, \ldots, X_l)$ are NA. We get, using (1.5), (1.6), (2.2) and Lemma 2.3 of [18],

$$\begin{aligned}
&|E\xi_k\xi_l| \\
&\quad = |\mathrm{Cov}(g(f_k(X_1, \ldots, X_k)), g(f_l(X_1, \ldots, X_l)))| \\
&\quad \leq |\mathrm{Cov}(g(f_k(X_1, \ldots, X_k)), g(f_l(X_1, \ldots, X_l)) \\
&\qquad - g(f_{k,l}(X_{k+1}, \ldots, X_l)))| \\
&\qquad + |\mathrm{Cov}(g(f_k(X_1, \ldots, X_k)), g(f_{k,l}(X_{k+1}, \ldots, X_l)))| \\
&\quad \ll E|f_l(X_1, \ldots, X_l) - f_{k,l}(X_{k+1}, \ldots, X_l)| \\
&\qquad + |\mathrm{Cov}(f_k(X_1, \ldots, X_k), f_{k,l}(X_{k+1}, \ldots, X_l))| \ll (c_l/c_k)^{-\delta} \qquad (2.3)
\end{aligned}$$

and

$$\begin{aligned}
E\left(\sum_{k=1}^{n} d_k\xi_k\right)^2 &\leq 2\sum_{1\leq k\leq l\leq n} d_k d_l |E\xi_k\xi_l| \\
&= 2\sum_{1\leq k\leq l\leq n;\, c_l/c_k\geq \ln^{2/\delta} D_n} d_k d_l |E\xi_k\xi_l| \\
&\quad +2\sum_{1\leq k\leq l\leq n;\, c_l/c_k< \ln^{2/\delta} D_n} d_k d_l |E\xi_k\xi_l| \\
&\widehat{=}\ 2(T_{n1} + T_{n2}). \qquad (2.4)
\end{aligned}$$

By (2.3),

$$T_{n1} \ll \sum_{1\leq k\leq l\leq n;\, c_l/c_k\geq \ln^{2/\delta} D_n} \frac{d_k d_l}{\ln^2 D_n} \leq \frac{D_n^2}{\ln^2 D_n}. \qquad (2.5)$$

On the other hand, by (1.12),

$$\exp(\ln^\alpha c_n) \sim \frac{\alpha D_n}{(\ln D_n)^{(1-\alpha)/\alpha}}.$$

Thus combining $|\xi_k| \leq 2c$ for any k,

$$\begin{aligned}
T_{n2} &\ll \sum_{k=1}^{n} d_k \sum_{1\leq k\leq l\leq n;\, c_l/c_k<(\ln D_n)^{2/\delta}} (\ln c_{l+1} - \ln c_l)\exp(\ln^\alpha c_l) \\
&\ll \exp(\ln^\alpha c_n)\ln\ln D_n \sum_{k=1}^{n} d_k \\
&\ll \frac{D_n^2 \ln\ln D_n}{(\ln D_n)^{(1-\alpha)/\alpha}}.
\end{aligned}$$

Since $\alpha < 1/2$ implies $(1-2\alpha)/(2\alpha) > 0$ and $\varepsilon \mathrel{\widehat{=}} 1/(2\alpha) - 1 > 0$. Thus, for sufficiently large n, we get

$$T_{n2} \ll \frac{D_n^2}{(\ln D_n)^{1/(2\alpha)}} \frac{\ln\ln D_n}{(\ln D_n)^{(1-2\alpha)/(2\alpha)}} \leq \frac{D_n^2}{(\ln D_n)^{1/(2\alpha)}} = \frac{D_n^2}{(\ln D_n)^{1+\varepsilon}}. \tag{2.6}$$

Let $T_n \mathrel{\widehat{=}} \frac{1}{D_n}\sum_{k=1}^{n} d_k\xi_k$. Combining (2.4)–(2.6), for sufficiently large n, we get

$$ET_n^2 \ll \frac{1}{(\ln D_n)^{1+\varepsilon}}.$$

By (1.4) and (1.7), we have $D_{n+1} \sim D_n$. Let $0 < \eta < \frac{\varepsilon}{1+\varepsilon}$, $n_k = \inf\{n;\, D_n \geq \exp(k^{1-\eta})\}$, then $D_{n_k} \geq \exp(k^{1-\eta})$, $D_{n_k-1} < \exp(k^{1-\eta})$. Therefore

$$1 \leq \frac{D_{n_k}}{\exp(k^{1-\eta})} \sim \frac{D_{n_k-1}}{\exp(k^{1-\eta})} < 1 \to 1,$$

i.e.,

$$D_{n_k} \sim \exp(k^{1-\eta}).$$

Since $(1-\eta)(1+\varepsilon) > 1$ from the definition of η, thus for any $\varepsilon_1 > 0$, we have

$$\sum_{k=1}^{\infty} P(|T_{n_k}| > \varepsilon_1) \ll \sum_{k=1}^{\infty} ET_{n_k}^2 \ll \sum_{k=1}^{\infty} \frac{1}{k^{(1-\eta)(1+\varepsilon)}} < \infty.$$

By the Borel–Cantelli lemma,

$$T_{n_k} \to 0 \quad \text{a.s.}$$

Now for $n_k < n \leq n_{k+1}$, by $|\xi_k| \leq 2c$ for any k,

$$|T_n| \leq |T_{n_k}| + \frac{2c}{D_n}\sum_{i=n_k+1}^{n} d_i \leq |T_{n_k}| + 2c\left(1 - \frac{D_{n_k}}{D_{n_{k+1}}}\right) \to 0 \quad \text{a.s.}$$

from $\frac{D_{n_k}}{D_{n_{k+1}}} \sim \frac{\exp((k+1)^{1-\eta})}{\exp(k^{1-\eta})} = \exp(k^{1-\eta}((1+1/k)^{1-\eta} - 1)) \sim \exp((1-\eta)k^{-\eta}) \to 1$, i.e. (2.1) holds. This completes the proof of Theorem 1. □

Proof of Theorem 2. Under the assumption of Theorem 2, by Corollary 2.2 of [13] we get

$$\frac{S_n}{\sigma\sqrt{n}} \xrightarrow{\mathscr{D}} N(0,1),$$

i.e.,

$$\lim_{n\to\infty} P\left\{\frac{S_n}{\sigma\sqrt{n}} < x\right\} = \Phi(x) \quad \text{for any } x \in R.$$

Hence

$$\lim_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k P\left\{\frac{S_k}{\sigma\sqrt{k}} < x\right\} = \Phi(x) \quad \text{for any } x \in R.$$

By Theorem 1, to prove (1.11) it suffices to prove that (1.5) and (1.6) hold for $f_l(X_1, \ldots, X_l) = \frac{S_l}{\sigma\sqrt{l}}$ and $f_{k,l}(X_{k+1}, \ldots, X_l) = \frac{S_l - S_k}{\sigma\sqrt{l}}$. By (1.10),

$$E\left|\frac{S_l}{\sigma\sqrt{l}} - \frac{S_l - S_k}{\sigma\sqrt{l}}\right| = \frac{E|S_k|}{\sigma\sqrt{l}} \le \frac{(ES_k^2)^{1/2}}{\sigma\sqrt{l}}$$

$$\ll \frac{\sqrt{kEX_1^2}}{\sqrt{l}} \ll \left(\frac{l}{k}\right)^{-1/2} \ll \left(\frac{c_l}{c_k}\right)^{-\gamma/2}.$$

And by $0 < -\sum_{m=2}^{\infty} EX_1X_m = \frac{EX_1^2 - \sigma^2}{2} < \infty$ and (1.10),

$$\begin{aligned}\left|\mathrm{Cov}\left(\frac{S_k}{\sigma\sqrt{k}}, \frac{S_l - S_k}{\sigma\sqrt{l}}\right)\right| &= -\frac{1}{\sigma^2\sqrt{kl}}\mathrm{Cov}\left(\sum_{i=1}^{k} X_i, \sum_{j=k+1}^{l} X_j\right) \\ &= -\frac{1}{\sigma^2\sqrt{kl}} \sum_{i=1}^{k} \sum_{m=k-i+2}^{l-i+1} \mathrm{Cov}(X_1, X_m) \\ &\le -\frac{k}{\sigma^2\sqrt{kl}} \sum_{m=2}^{\infty} EX_1X_m \ll \left(\frac{l}{k}\right)^{-1/2} \\ &\ll \left(\frac{c_l}{c_k}\right)^{-\gamma/2}.\end{aligned}$$

This completes the proof of Theorem 2. □

Acknowledgements

This work was supported by the National Natural Science Foundation of China (11061012), the support program of the New Century Guangxi, China, Ten-hundred-thousand Talents Project (2005214), and the support program of the Guangxi, China Science Foundation (2010GXNSFA013120).

References

[1] Billingsley P, Convergence of probability measures (New York: Wiley) (1968)

[2] Brosamler G A, An almost everywhere central limit theorem, *Math. Proc. Cambridge Philos. Soc.* **104** (1988) 561–574

[3] Chandrasekharan K and Minakshisundaram S, Typical means (Oxford: Oxford University Press) (1952)

[4] Hörmann S, Optimal averaging procedures in almost sure cenrtal limit theory, *Metodološki zvezki.* **2**(**2**) (2005) 271–282

[5] Ibragimov I A and Lifshits M, On the convergence of generalized moments in almost sure central limit theorem, *Statist. Probab. Lett.* **40** (1998) 343–351

[6] Joag-Dev K and Proschan F, Negative association of random variables with applications, *Ann. Statist.* **11**(**1**) (1983) 286–295

[7] Lacey M T and Philipp W, A note on the almost sure central limit theorem, *Statist. Probab. Lett.* **9** (1990) 201–205

[8] Matula P A, A note on the almost sure convergence of sums of negatively dependent random variables, *Statist. Probab. Lett.* **15** (1992) 209–213

[9] Miao Y, Central limit theorem and almost sure central limit theorem for the product of some partial sums, *Proc. Indian Acad. Sci. (Math. Sci.)* **118**(**2**) (2008) 289–294

[10] Peligrad M and Shao Q M, A note on the almost sure central limit theorem for weakly dependent random variables, *Statist. Probab. Lett.* **22** (1995) 131–136

[11] Schatte P, On strong versions of the central limit theorem, *Math. Nachr.* **137** (1988) 249–256

[12] Shao Q M, A comparison theorem on moment inequalities between negatively associated and independent random variables, *J. Theor. Probab.* **13**(**2**) (2000) 343–356

[13] Su C and Chi X, Some results on CLT for non-stationary NA sequences, *Acta Math. Appl. Sin.* **21**(**1**) (1988) 9–21 (in Chinese)

[14] Wu Q Y, Strong consistency of M estimator in linear model for negatively associated samples, *J. Syst. Sci. Complex.* **19**(**4**) (2006) 592–600

[15] Wu Q Y and Jiang Y Y, A law of the iterated logarithm of partial sums for NA random variables, *J. Korean Stat. Soc.* **39**(**2**) (2010a) 199–206

[16] Wu Q Y and Jiang Y Y, Chover's law of the iterated logarithm for NA sequences, *J. Syst. Sci. Complex.* **23**(**2**) (2010b) 293–302

[17] Wu Q Y, Almost sure limit theorems for stable distribution, *Stat. Probab. Lett.* **81**(**6**) (2011) 662–672

[18] Zhang L X, The weak convergence for functions of negatively associated random variables, *J. Multivariate Anal.* **78** (2001) 272–298