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Value functions for certain class of Hamilton Jacobi equations

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Abstract. We consider a class of Hamilton Jacobi equations (in short, HJE) of type

$$u_t + \frac{1}{2}\left(|u_{x_1}|^2 + \cdots + |u_{x_{n-1}}|^2\right) + \frac{e^u}{m}|u_{x_n}|^m = 0,$$

in $\mathbb{R}^n \times \mathbb{R}_+$ and $m > 1$, with bounded, Lipschitz continuous initial data. We give a Hopf-Lax type representation for the value function and also characterize the set of minimizing paths. It is shown that the minimizing paths in the representation of value function need not be straight lines. Then we consider HJE with Hamiltonian decreasing in u of type

$$u_t + H_1\left(u_{x_1}, \ldots, u_{x_i}\right) + e^{-u} H_2\left(u_{x_{i+1}}, \ldots, u_{x_n}\right) = 0$$

where H_1, H_2 are convex, homogeneous of degree $n, m > 1$ respectively and the initial data is bounded, Lipschitz continuous. We prove that there exists a unique viscosity solution for this HJE in Lipschitz continuous class. We also give a representation formula for the value function.

Keywords. Hamilton Jacobi equation; viscosity solution; minimizing paths; value function.

1. Introduction

Let $H : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a continuous function and $g : \mathbb{R}^n \to \mathbb{R}$ be a bounded, Lipschitz continuous function. Consider the Hamilton Jacobi equation (HJE)

$$\begin{aligned} u_t + H(u, Du) &= 0 \ \text{ for } (x,t) \in \mathbb{R}^n \times \mathbb{R}_+, \\ u(x,0) &= g(x), \end{aligned} \tag{1.1}$$

where $Du = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right)$. This problem has been studied extensively in [2,4,5] and the references therein. Some of the open issues related to problem (1.1) are the following:

(i) Does there exist a viscosity solution for HJE (1.1) and is the solution unique?
(ii) Is it possible to give a Hopf-Lax type representation for the value function (solution)? Are the minimizing paths always straight lines?

The question (ii) was also asked in [3]. It has been shown in [5] that apart from the usual hypothesis in H if the H is increasing in u then (1.1) has a unique viscosity solution. But when the Hamiltonian is decreasing in u then a few results are known. Even in the general set up when $H(u, p)$ depends on u, getting a Hopf-Lax type representation for u is quite difficult. For $n = 1$, an explicit Hopf-Lax type representation is given in [1]. But the dimensionality was crucially used there and its implementation for higher dimension is quite difficult. Here we consider a certain class of HJE

$$u_t + \frac{1}{2}\left(|u_{x_1}|^2 + \cdots + |u_{x_{n-1}}|^2\right) + \frac{\mathrm{e}^u}{m}|u_{x_n}|^m = 0$$

and give a Hopf-Lax type representation for $u(x, t)$. We also characterize the set of minimizers for the representation and show that the minimizing paths could be of exponential types. Apart from minimizer characterizations, it is also possible to give the Hopf-Lax type formula for solution $u(x, t)$ for a more general Hamiltonian (see Remark 2.2). Even though it is not known, in general, whether the viscosity solution exists or not when the Hamiltonian is decreasing in u, we give a class of Hamiltonian decreasing in u and show that there exists a unique viscosity solution for the corresponding HJE. For the convenience of the reader we recall the definition of viscosity solution,

DEFINITION 1.1

Let U be a bounded, continuous function in $\mathbb{R}^n \times \mathbb{R}_+$.

1. U is said to be a subsolution of (1.1) if for any $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}_+$, $v \in C^1(\mathbb{R}^n \times \mathbb{R}_+)$ such that (x_0, t_0) is a local maximum for $U - v$, with $U(x_0, t_0) = v(x_0, t_0)$, then at (x_0, t_0), $v_t + H(v, Dv) \leq 0$ and $\lim_{t \to 0} U(x, t) \leq g(x)$.
2. U is said to be a supersolution of (1.1) if for any $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}_+$, $v \in C^1(\mathbb{R}^n \times \mathbb{R}_+)$ such that (x_0, t_0) is a local minimum for $U - v$, with $U(x_0, t_0) = v(x_0, t_0)$, then at (x_0, t_0), $v_t + H(v, Dv) \geq 0$ and $\lim_{t \to 0} U(x, t) \geq g(x)$.

U is said to be a viscosity solution if it is both sub and supersolution.

The paper is organized as follows: In §2 we consider HJE with Hamiltonian increasing in u. We prove that there exists a unique viscosity solution for the HJE. Then we characterize the minimizing paths in the representation of $u(x, t)$ in §3. In §4 we consider HJE with Hamiltonian decreasing in u and prove that there exists a unique viscosity solution for the corresponding HJE. In §5 we provide an example to strengthen the arguments of §3.

2. Value function for HJE increasing in u

Let $g : \mathbb{R}^n \to \mathbb{R}$ be a bounded, Lipschitz continuous function. Consider the HJE

$$\begin{aligned} u_t + H(u, Du) &= 0 \text{ for } (x, t) \in \mathbb{R}^n \times \mathbb{R}_+, \\ u(x, 0) &= g(x), \end{aligned} \tag{2.1}$$

where $Du = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right)$ and

$$H(u, p) = \frac{1}{2}\left(\sum_{i=1}^{n-1} |p_i|^2\right) + \frac{e^u}{m}|p_n|^m \text{ for } (u, p) \in \mathbb{R} \times \mathbb{R}^n.$$

We assume that $m > 1$ so that $H(u, \cdot)$ becomes a convex function in $\mathbb{R}^n$. For $x \in \mathbb{R}^n$ and $t \geq 0$, we define

$$\Gamma_t^x = \{\gamma : [0, t] \to \mathbb{R}^n : \gamma \text{ is absolutely continuous and } \gamma(t) = x\}.$$

Define

$$L(x_1, \ldots, x_{n-1}) = \frac{1}{2}\left(|x_1|^2 + \cdots + |x_{n-1}|^2\right) \text{ for } (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}.$$

Let $\bar{\gamma}(l) = (\gamma_1(l), \ldots, \gamma_{n-1}(l))$. For $(x, t) \in \mathbb{R}^n \times \mathbb{R}_+$ we define

$$\begin{aligned} u(x, t) &= \inf_{\Gamma_t^x} (m-1) \ln \Bigg[e^{\frac{1}{m-1}\{g(\gamma(0)) + \int_0^t L(\dot{\bar{\gamma}}(\theta))d\theta\}} \\ &\qquad + \frac{1}{m}\int_0^t |\dot{\gamma}_n|^{\frac{m}{m-1}} e^{\frac{1}{m-1}\int_\theta^t L(\dot{\bar{\gamma}}(l))dl} d\theta \Bigg]. \end{aligned} \tag{2.2}$$

Our main result is the following.

Theorem 2.1. *$u(x, t)$ defined in* (2.2) *is a Lipschitz continuous function and is a viscosity solution of* (2.1).

Proof. We prove the theorem by subsequent lemmas. □

Lemma 2.1. *Let $M_1 \leq g \leq M_2$ for all $x \in \mathbb{R}^n$. Then $M_1 \leq u(x, t) \leq M_2$ for all $(x, t) \in \mathbb{R}^n \times \mathbb{R}_+$. Also there exists a path $\phi \in \Gamma_t^x$ such that*

$$\begin{aligned} u(x, t) &= (m-1) \ln \Bigg[e^{\frac{1}{m-1}\{g(\phi(0)) + \int_0^t L(\dot{\bar{\phi}}(\theta))d\theta\}} \\ &\qquad + \frac{1}{m}\int_0^t |\dot{\phi}_n|^{\frac{m}{m-1}} e^{\frac{1}{m-1}\int_\theta^t L(\dot{\bar{\phi}}(l))dl} d\theta \Bigg]. \end{aligned}$$

Proof. Taking $\gamma \equiv x$ in (2.2) we have $u(x, t) \leq g(x) \leq M_2$. Again

$$u(x, t) \geq \inf_{\Gamma_t^x} g(\gamma(0)) \geq M_1.$$

This proves the first part. Let $\{\gamma_k\}$ be the minimizing sequence for (2.2). Since

$$\int_0^t L(\dot{\bar{\gamma}}_k(\theta))d\theta \geq \frac{1}{2}\sum_{i=1}^{n-1} t\left|\frac{x_i - (\gamma_k)_i(0)}{t}\right|^2$$

and

$$\int_0^t |(\dot{\gamma}_k)_n(\theta)|^{\frac{m}{m-1}} d\theta \geq t\left|\frac{x_n - (\gamma_k)_n(0)}{t}\right|^{\frac{m}{m-1}},$$

we note that $\{\gamma_k(0)\}$ remains in a compact set. Therefore we can get a subsequence, denoted by the same index $\{\gamma_k(0)\}$, and a path $\phi \in \Gamma_t^x$ such that

$$\begin{aligned} \gamma_k &\to \phi \ \text{uniformly in } [0,t], \\ \dot{\bar{\gamma}}_k &\to \dot{\bar{\phi}} \ \text{weakly in } L^2([0,t],\mathbb{R}^{n-1}), \\ (\dot{\gamma}_k)_n &\to \dot{\phi}_n \ \text{weakly in } L^{\frac{m}{m-1}}([0,t],\mathbb{R}). \end{aligned}$$

Again we note that

$$\frac{1}{m}\int_0^t |\dot{\gamma}_n|^{\frac{m}{m-1}} \mathrm{e}^{\frac{1}{m-1}\int_\theta^t L(\dot{\bar{\gamma}}(l))\mathrm{d}l}\mathrm{d}\theta$$

is convex in $(\cdot,\ldots,\cdot,\gamma_n)$ and $(\gamma_1,\ldots,\gamma_{n-1},\cdot)$. Since $\liminf\{a(n)+b(n)\} \geq \liminf a(n) + \liminf b(n)$, it is easy to see that ϕ is a minimizing path. □

Lemma 2.2 (*Dynamic Programming Principle*). *For* $0 \leq s < t$, *we have*

$$\begin{aligned} u(x,t) = \inf_{\Gamma_{t-s}^x} (m-1)\ln\Bigg[&\mathrm{e}^{\frac{1}{m-1}\{u(\gamma(0),s)+\int_0^{t-s} L(\dot{\bar{\gamma}}(\theta))\mathrm{d}\theta\}} \\ &+ \frac{1}{m}\int_0^{t-s} |\dot{\gamma}_n|^{\frac{m}{m-1}} \mathrm{e}^{\frac{1}{m-1}\int_\theta^{t-s} L(\dot{\bar{\gamma}}(l))\mathrm{d}l}\mathrm{d}\theta\Bigg]. \end{aligned} \tag{2.3}$$

Proof. Let $v(x,t)$ denote the right-hand side of (2.3). From Lemma 2.1, we can find $\phi \in \Gamma_t^x$ such that

$$\begin{aligned} u(x,t) = (m-1)\ln\Bigg[&\mathrm{e}^{\frac{1}{m-1}\{g(\phi(0))+\int_0^t L(\dot{\bar{\phi}}(\theta))\mathrm{d}\theta\}} \\ &+ \frac{1}{m}\int_0^t |\dot{\phi}_n|^{\frac{m}{m-1}} \mathrm{e}^{\frac{1}{m-1}\int_\theta^t L(\dot{\bar{\phi}}(l))\mathrm{d}l}\mathrm{d}\theta\Bigg]. \end{aligned}$$

Therefore

$$\begin{aligned} \mathrm{e}^{u(x,t)/(m-1)} &= \mathrm{e}^{\frac{1}{m-1}\{g(\phi(0))+\int_0^t L(\dot{\bar{\phi}}(\theta))\mathrm{d}\theta\}} \\ &\quad + \frac{1}{m}\int_0^t |\dot{\phi}_n|^{\frac{m}{m-1}} \mathrm{e}^{\frac{1}{m-1}\int_\theta^t L(\dot{\bar{\phi}}(l))\mathrm{d}l}\mathrm{d}\theta \\ &= \mathrm{e}^{\frac{1}{m-1}\{g(\phi(0))+\int_0^s L(\dot{\bar{\phi}}(\theta))\mathrm{d}\theta+\int_s^t L(\dot{\bar{\phi}}(\theta))\mathrm{d}\theta\}} \\ &\quad + \frac{1}{m}\int_0^s |\dot{\phi}_n|^{\frac{m}{m-1}} \mathrm{e}^{\frac{1}{m-1}\int_\theta^t L(\dot{\bar{\phi}}(l))\mathrm{d}l}\mathrm{d}\theta \\ &\quad + \frac{1}{m}\int_s^t |\dot{\phi}_n|^{\frac{m}{m-1}} \mathrm{e}^{\frac{1}{m-1}\int_\theta^t L(\dot{\bar{\phi}}(l))\mathrm{d}l}\mathrm{d}\theta \\ &\quad - \Bigg[\mathrm{e}^{\frac{1}{m-1}\{g(\phi(0))+\int_0^s L(\dot{\bar{\phi}}(\theta))\mathrm{d}\theta\}} \\ &\quad + \frac{1}{m}\int_0^s |\dot{\phi}_n|^{\frac{m}{m-1}} \mathrm{e}^{\frac{1}{m-1}\int_\theta^s L(\dot{\bar{\phi}}(l))\mathrm{d}l}\mathrm{d}\theta\Bigg] \times \mathrm{e}^{\frac{1}{m-1}\int_s^t L(\dot{\bar{\phi}}(\theta))\mathrm{d}\theta} \\ &\quad + \frac{1}{m}\int_s^t |\dot{\phi}_n|^{\frac{m}{m-1}} \mathrm{e}^{\frac{1}{m-1}\int_\theta^t L(\dot{\bar{\phi}}(l))\mathrm{d}l}\mathrm{d}\theta. \end{aligned}$$

Define $\psi(\theta) = \phi(\theta + s)$ for $\theta \in [0, t - s]$. Then

$$\begin{aligned} e^{u(x,t)/(m-1)} &\geq e^{\frac{1}{m-1}\{u(\psi(0),s)+\int_0^{t-s} L(\dot{\bar{\psi}}(\theta))d\theta\}} \\ &\quad + \frac{1}{m}\int_0^{t-s} |\dot{\psi}_n(\theta)|^{\frac{m}{m-1}} e^{\frac{1}{m-1}\int_\theta^{t-s} L(\dot{\bar{\psi}}(l))dl} d\theta \\ &\geq e^{v(x,t)/(m-1)}. \end{aligned}$$

Therefore $u(x, t) \geq v(x, t)$. Consider $\epsilon > 0$. Then there exists $\phi \in \Gamma_{t-s}^x$ such that

$$\begin{aligned} e^{\frac{v(x,t)+\epsilon}{m-1}} &\geq e^{\frac{1}{m-1}\{u(\phi(0),s)+\int_0^{t-s} L(\dot{\bar{\phi}}(\theta))d\theta\}} \\ &\quad + \frac{1}{m}\int_0^{t-s} |\dot{\phi}_n(\theta)|^{\frac{m}{m-1}} e^{\frac{1}{m-1}\int_\theta^{t-s} L(\dot{\bar{\phi}}(l))dl} d\theta. \end{aligned} \tag{2.4}$$

Again using Lemma 2.1 we have an absolutely continuous path $\psi \in \Gamma_s^{\phi(0)}$ such that

$$\begin{aligned} e^{\frac{u(\phi(0),s)}{m-1}} &= e^{\frac{1}{m-1}\{g(\psi(0))+\int_0^{s} L(\dot{\bar{\psi}}(\theta))d\theta\}} \\ &\quad + \frac{1}{m}\int_0^{s} |\dot{\psi}_n(\theta)|^{\frac{m}{m-1}} e^{\frac{1}{m-1}\int_\theta^{s} L(\dot{\bar{\psi}}(l))dl} d\theta. \end{aligned} \tag{2.5}$$

Now define a path $\chi \in \Gamma_t^x$ by

$$\chi(\theta) = \begin{cases} \psi(\theta), & \text{for } \theta \in [0, s], \\ \phi(\theta - s), & \text{for } \theta \in (s, t]. \end{cases}$$

Then using (2.4), (2.5) and the same steps as above we have

$$e^{\frac{v(x,t)+\epsilon}{m-1}} \geq e^{\frac{u(x,t)}{m-1}},$$

$\epsilon > 0$ being arbitrary we have $v(x, t) \geq u(x, t)$. This completes the proof. □

Lemma 2.3. *The function u defined in* (2.2) *is Lipschitz in* $\mathbb{R}^n \times \mathbb{R}_+$ *and* $\lim_{t\to 0} u(x, t) = g(x)$.

Proof. Let $x_1, x_2 \in \mathbb{R}^n$, and from Lemma 2.1 there exists $\phi_i \in \Gamma_t^{x_i}$ $(i = 1, 2)$ such that

$$\begin{aligned} e^{\frac{u(x_i,t)}{m-1}} &= e^{\frac{1}{m-1}\{g(\phi_i(0))+\int_0^{t} L(\dot{\bar{\phi}}_i(\theta))d\theta\}} \\ &\quad + \frac{1}{m}\int_0^{t} |\dot{\phi}_{i\,n}(\theta)|^{\frac{m}{m-1}} e^{\frac{1}{m-1}\int_\theta^{t} L(\dot{\bar{\phi}}_i(l))dl} d\theta. \end{aligned}$$

Since $u(x, t) \leq M_2$ we note that $e^{\frac{1}{m-1}\int_0^t L(\dot{\bar{\phi}}_i(\theta))d\theta} \leq e^{\frac{M_2+|M_1|}{m-1}}$. Let K_1 be the Lipschitz constant of $e^{\frac{x}{m-1}}$ in $[M_1, M_2]$. Define $\psi_i \in \Gamma_t^{x_i}$ by

$$\psi_1(\theta) = x_1 - \int_\theta^t \dot{\phi}_2 d\theta \text{ for } \theta \in [0, t]$$

and

$$\psi_2(\theta) = x_2 - \int_\theta^t \dot{\phi}_1 d\theta \text{ for } \theta \in [0, t].$$

Therefore $\psi_1(0) - \phi_2(0) = x_1 - x_2$ and $\psi_2(0) - \phi_1(0) = x_2 - x_1$. Hence

$$\begin{aligned}
e^{u(x_1,t)/(m-1)} - e^{u(x_2,t)/(m-1)} &\leq e^{\frac{1}{m-1}\{g(\psi_2(0))+\int_0^t L(\dot{\bar{\psi}}_2(\theta))d\theta\}} \\
&\quad - e^{\frac{1}{m-1}\{g(\phi_1(0))+\int_0^t L(\dot{\bar{\phi}}_1(\theta))d\theta\}} \\
&= e^{\frac{1}{m-1}\int_0^t L(\dot{\bar{\phi}}_1(\theta))d\theta}\left[e^{\frac{g(\psi_2(0))}{m-1}} - e^{\frac{g(\phi_1(0))}{m-1}}\right] \\
&\leq e^{\frac{M_2+|M_1|}{m-1}} K_1 K_2 |x_1 - x_2|,
\end{aligned}$$

where K_2 is the Lipschitz constant of $g(\cdot)$. Similarly we can show that

$$e^{u(x_2,t)/(m-1)} - e^{u(x_1,t)/(m-1)} \leq e^{M_2/(m-1)} K_1 K_2 |x_1 - x_2|.$$

Therefore there exists a constant K such that

$$|e^{u(x_1,t)/(m-1)} - e^{u(x_2,t)/(m-1)}| \leq K|x_1 - x_2|. \tag{2.6}$$

From (2.3) we have $u(x,t) \leq u(x,s)$ for $0 \leq s < t$. From Lemma 2.1 there exists $\phi \in \Gamma^x_{t-s}$ such that

$$\begin{aligned}
e^{\frac{u(x,t)}{m-1}} &= e^{\frac{1}{m-1}\{u(\phi(0),s)+\int_0^{t-s} L(\dot{\bar{\phi}}(\theta))d\theta\}} \\
&\quad + \frac{1}{m}\int_0^{t-s} |\dot{\phi}_n(\theta)|^{\frac{m}{m-1}} e^{\frac{1}{m-1}\int_\theta^{t-s} L(\dot{\bar{\phi}}(l))dl} d\theta.
\end{aligned}$$

Therefore from Lemma 2.2 we have

$$\begin{aligned}
&e^{u(x,t)/(m-1)} - e^{u(x,s)/(m-1)} \\
&= e^{\frac{1}{m-1}\{u(\phi(0),s)+\int_0^{t-s} L(\dot{\bar{\phi}}(\theta))d\theta\}} - e^{\frac{u(x,s)}{m-1}} \\
&\quad + \frac{1}{m}\int_0^{t-s} |\dot{\phi}_n(\theta)|^{\frac{m}{m-1}} e^{\frac{1}{m-1}\int_\theta^{t-s} L(\dot{\bar{\phi}}(l))dl} d\theta \\
&\geq e^{\frac{1}{m-1}\int_0^{t-s} L(\dot{\bar{\phi}}(\theta))d\theta}\left[e^{\frac{u(\phi(0),s)}{m-1}} - e^{\frac{u(x,s)}{m-1}}\right] \\
&\quad + e^{\frac{u(x,s)}{m-1}}\left[e^{\frac{1}{m-1}\int_0^{t-s} L(\dot{\bar{\phi}}(\theta))d\theta} - 1\right] \\
&\quad + \frac{1}{m}\int_0^{t-s} |\dot{\phi}_n(\theta)|^{\frac{m}{m-1}} d\theta \\
&\geq -K_3|x - \phi(0)| + \frac{K_4}{m-1}\int_0^{t-s} L(\dot{\bar{\phi}}(\theta))d\theta \\
&\quad + \frac{1}{m}\int_0^{t-s} |\dot{\phi}_n(\theta)|^{\frac{m}{m-1}} d\theta \text{ (for some constant } K_3, K_4) \\
&\geq -K_3|x - \phi(0)| + \frac{K_4(t-s)}{m-1} L\left(\frac{\bar{\phi}(t-s) - \bar{\phi}(0)}{t-s}\right) \\
&\quad + \frac{t-s}{m}\left|\frac{\phi_n(t-s) - \phi_n(0)}{t-s}\right|^{\frac{m}{m-1}} \\
&\geq -(t-s)\sup_{\eta \leq K_3}\sup_{p}\left\{\langle p, \eta\rangle - \frac{K_4}{m-1}L(\bar{p}) - \frac{1}{m}|p_n|^{\frac{m}{m-1}}\right\} \\
&= -K_5|t-s| \text{ (for some constant } K_5).
\end{aligned}$$

This proves that for any $s, t \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$,

$$|e^{u(x,t)/(m-1)} - e^{u(x,s)/(m-1)}| \leq K_5|t-s|. \tag{2.7}$$

Now let $(x_1, s), (x_2, t) \in \mathbb{R}^n \times \mathbb{R}_+$. Then

$$\begin{aligned} |e^{u(x_1,s)/(m-1)} - e^{u(x_2,t)/(m-1)}| &\leq |e^{u(x_1,s)/(m-1)} - e^{u(x_2,s)/(m-1)}| \\ &\quad + |e^{u(x_2,s)/(m-1)} - e^{u(x_2,t)/(m-1)}| \\ &\leq \max\{K, K_5\}(|x_1 - x_2| + |t-s|). \end{aligned}$$

Since the range of $u(\cdot,\cdot)$ is in $[M_1, M_2]$ and $e^{\frac{u}{m-1}}$ is not 0, we can have a constant M such that

$$|u(x_1,s) - u(x_2,t)| \leq M\left(|x_1 - x_2| + |t-s|\right). \tag{2.8}$$

□

Lemma 2.4. *u is a viscosity solution of HJE* (2.1).

Proof. Let $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}_+$ and ν be a C^1-function such that $u(x_0,t_0) = \nu(x_0,t_0)$ and $u - \nu$ has a local maximum at (x_0,t_0). Let B be a ball centered at (x_0,t_0) such that $u(x,t) \leq \nu(x,t)$ for $(x,t) \in B$. Let $0 \leq t < t_0$, $\lambda_i > 0$, $h = t_0 - t$ and $x = x_0 - h \cdot \lambda$ with $\lambda = (\lambda_1, \ldots, \lambda_n)$. Define

$$\bar{\Gamma}_t^x \stackrel{\text{def}}{=} \{\gamma \in \Gamma_t^x \ : \ \gamma \text{ is a straight line}\}.$$

Therefore for small h we have

$$\begin{aligned} e^{\frac{\nu(x_0,t_0)}{m-1}} &= e^{\frac{u(x_0,t_0)}{m-1}} \\ &\leq \inf_{\bar{\Gamma}_h^{x_0}} \left[e^{\frac{1}{m-1}\{u(\gamma(0),t_0-h)+\int_0^h L(\dot{\bar{\gamma}}(\theta))d\theta\}} \right. \\ &\qquad \left. + \frac{1}{m}\int_0^h |\dot{\gamma}_n|^{\frac{m}{m-1}} e^{\frac{1}{m-1}\int_\theta^h L(\dot{\bar{\gamma}}(l))dl} d\theta \right] \\ &= \inf_{y\in\mathbb{R}^n} \left[e^{\frac{1}{m-1}\{u(y,t_0-h)+hL(\overline{\frac{x_0-y}{h}})\}} \right. \\ &\qquad \left. + \frac{m-1}{m}\left|\frac{(x_0)_n - y_n}{h}\right|^{\frac{m}{m-1}} \left(\frac{e^{\frac{1}{m-1}hL(\overline{\frac{x_0-y}{h}})} - 1}{L(\overline{\frac{x_0-y}{h}})} \right) \right] \\ &\leq \left[e^{\frac{1}{m-1}\{u(x,t_0-h)+hL(\bar{\lambda})\}} + \frac{m-1}{m}|\lambda_n|^{\frac{m}{m-1}} \left(\frac{e^{\frac{1}{m-1}hL(\bar{\lambda})} - 1}{L(\bar{\lambda})} \right) \right] \\ &\leq \left[e^{\frac{1}{m-1}\{\nu(x,t_0-h)+hL(\bar{\lambda})\}} + \frac{m-1}{m}|\lambda_n|^{\frac{m}{m-1}} \left(\frac{e^{\frac{1}{m-1}hL(\bar{\lambda})} - 1}{L(\bar{\lambda})} \right) \right]. \end{aligned}$$

Therefore if v is C^2 using Taylor's expansion we have from above

$$
\begin{aligned}
e^{\frac{v(x_0,t_0)}{m-1}} \;\le\; & e^{\frac{v(x_0,t_0)}{m-1}} \\
& + \frac{1}{m-1}\left[\sum_{i=1}^{n} -\lambda_i h v_{x_i}(x_0,t_0) - h v_t(x_0,t_0) + hL(\bar{\lambda})\right] e^{\frac{v(x_0,t_0)}{m-1}} \\
& + \mathcal{O}(h^2) + \frac{m-1}{m}|\lambda_n|^{\frac{m}{m-1}}\left(\frac{e^{\frac{1}{m-1}hL(\bar{\lambda})}-1}{L(\bar{\lambda})}\right).
\end{aligned}
$$

Hence dividing both sides by h and letting $h \downarrow 0$ we have

$$
\begin{aligned}
0 \;\le\; & \frac{1}{m-1}\left[\sum_{i=1}^{n} -\lambda_i v_{x_i}(x_0,t_0) - v_t(x_0,t_0) + L(\bar{\lambda})\right] e^{\frac{v(x_0,t_0)}{m-1}} \\
& + \frac{1}{m-1}\frac{m-1}{m}|\lambda_n|^{\frac{m}{m-1}}.
\end{aligned}
$$

Similarly we could choose $\lambda_i \le 0$ to get the above estimate. Hence λ being arbitrary we have

$$
v_t(x_0,t_0) + \sup_{\lambda}\left[\langle \lambda, Dv(x_0,t_0)\rangle - L(\bar{\lambda}) - e^{-\frac{v(x_0,t_0)}{m-1}}\frac{m-1}{m}|\lambda_n|^{\frac{m}{m-1}}\right] \le 0,
$$

i.e

$$
v_t(x_0,t_0) + \frac{1}{2}\sum_{i=1}^{n-1}|v_{x_i}|^2 + \frac{e^{v(x_0,t_0)}}{m}|v_{x_n}|^m \le 0.
$$

Since the result can be deduced for v in C^1, this proves that u is a subsolution. □

Now we prove that u is a supersolution. Let $(x_0,t_0) \in \mathbb{R}^n \times \mathbb{R}_+$ be a point and v be a C^1 function on $\mathbb{R}^n \times \mathbb{R}_+$ such that $v(x_0,t_o) = u(x_0,t_0)$ and $u - v$ has a local minimum at (x_0,t_0). We have to show that

$$
v_t(x_0,t_0) + \frac{1}{2}\sum_{i=1}^{n-1}|v_{x_i}|^2 + \frac{e^{v(x_0,t_0)}}{m}|v_{x_n}|^m \ge 0. \tag{2.9}
$$

If this is not true then there exists $\theta > 0$ such that

$$
v_t(x,t) + \frac{1}{2}\sum_{i=1}^{n-1}|v_{x_i}|^2 + \frac{e^{v(x,t)}}{m}|v_{x_n}|^m \le -\theta \quad \forall\,(x,t) \in B_1 \tag{2.10}
$$

and

$$
v(x,t) \le u(x,t) \quad \forall\,(x,t) \in B_1, \tag{2.11}
$$

where B_1 is a small closed ball around (x_0,t_0). From (2.3) we note that for small $h > 0$ there exists a path $\gamma \in \Gamma_h^{x_0}$ such that

$$
\gamma(l) \in B_1 \;\; \forall\, l \in [0,h]
$$

and

$$e^{\frac{u(x_0,t_0)}{m-1}} = \Bigg[e^{\frac{1}{m-1}\{u(\gamma(0),t_0-h)+\int_0^h L(\dot{\bar{\gamma}}(\theta))d\theta\}} + \frac{1}{m}\int_0^h |\dot{\gamma}_n|^{\frac{m}{m-1}} e^{\frac{1}{m-1}\int_\theta^h L(\dot{\bar{\gamma}}(l))dl} d\theta \Bigg]. \tag{2.12}$$

Define $\phi(s) = \gamma(s - t_0 + h)$ for $s \in [t_0 - h, t_0]$. Then

$$\begin{aligned} &e^{\frac{v(\phi(t_0),t_0)}{m-1}} - e^{\frac{1}{m-1}\{v(\phi(t_0-h),t_0-h)+\int_{t_0-h}^{t_0} L(\dot{\bar{\phi}}(\theta))d\theta\}} \\ &= \int_{t_0-h}^{t_0} \frac{d}{ds} e^{\frac{1}{m-1}\{v(\phi(s),s)+\int_s^{t_0} L(\dot{\bar{\phi}}(\theta))d\theta\}} \\ &= \int_{t_0-h}^{t_0} \frac{1}{m-1} e^{\frac{1}{m-1}\{v(\phi(s),s)+\int_s^{t_0} L(\dot{\bar{\phi}}(\theta))d\theta\}} [v_t + \langle Dv, D\phi\rangle - L(\dot{\bar{\phi}})]ds. \end{aligned}$$

From (2.10) we have

$$\begin{aligned} &v_t(\phi(s), s) + \langle Dv(\phi(s), s), D\phi(s)\rangle - L(\dot{\bar{\phi}}(s)) \\ &-\frac{m-1}{m} e^{-\frac{v(\phi(s),s)}{m-1}} |\dot{\phi}_n(s)|^{\frac{m}{m-1}} \le -\theta, \end{aligned}$$

i.e.

$$\begin{aligned} &\frac{1}{m-1} e^{\frac{v(\phi(s),s)}{m-1}} [v_t(\phi(s), s) + \langle Dv(\phi(s), s), D\phi(s)\rangle - L(\dot{\bar{\phi}}(s))] \\ &\le -\frac{\theta}{m-1} e^{\frac{v(\phi(s),s)}{m-1}} + \frac{1}{m} |\dot{\phi}_n(s)|^{\frac{m}{m-1}}, \end{aligned}$$

i.e.

$$\begin{aligned} &e^{\frac{v(\phi(t_0),t_0)}{m-1}} - e^{\frac{1}{m-1}\{v(\phi(t_0-h),t_0-h)+\int_{t_0-h}^{t_0} L(\dot{\bar{\phi}}(\theta))d\theta\}} \\ &\le -\int_{t_0-h}^{t_0} \frac{\theta}{m-1} e^{\frac{1}{m-1}\{v(\phi(s),s)+\int_s^{t_0} L(\dot{\bar{\phi}}(\theta))d\theta\}} \\ &\quad + \frac{1}{m}\int_{t_0-h}^{t_0} |\dot{\phi}_n(s)|^{\frac{m}{m-1}} e^{\frac{1}{m-1}\int_s^{t_0} L(\dot{\bar{\phi}}(l))dl} ds. \end{aligned}$$

Therefore using (2.11) and (2.12) we have

$$0 \le -\int_{t_0-h}^{t_0} \frac{\theta}{m-1} e^{\frac{1}{m-1}\{v(\phi(s),s)+\int_s^{t_0} L(\dot{\bar{\phi}}(\theta))d\theta\}}.$$

This is a contradiction. Therefore (2.9) is true. Hence u is a supersolution.

Remark 2.1. We note that the Hamiltonian in (2.1) is increasing in u and so the viscosity solution of (2.1) is unique. Hence u is the unique solution of (2.1) and given by (2.2).

Remark 2.2. Let $1 < i < n$ and $H_1 : \mathbb{R}^i \to \mathbb{R}_+$, $H_2 : \mathbb{R}^{n-i} \to \mathbb{R}_+$ be two convex homogeneous functions of order $n, m > 1$ respectively that vanish only at 0 and have homogeneous Legendre transformations. It is easy to note that we can give a Hopf-Lax type representation for the viscosity solution of HJE of type

$$\begin{aligned} u_t + H_1(u_{x_1}, \ldots, u_{x_i}) + \mathrm{e}^u H_2(u_{x_{i+1}}, \ldots, u_{x_n}) &= 0, \\ u(x,0) &= g(x). \end{aligned}$$

Let L_1, L_2 be the Legendre transformations of H_1, H_2 respectively. Then

$$\begin{aligned} u(x,t) &= \inf_{\Gamma_t^x}(m-1)\ln\Bigg[\mathrm{e}^{\frac{1}{m-1}\{g(\gamma(0))+\int_0^t L_1(\dot{\chi}_1(\theta))\mathrm{d}\theta\}} \\ &\quad + \frac{1}{m-1}\int_0^t L_2(\dot{\chi}_2(\theta))\mathrm{e}^{\frac{1}{m-1}\int_\theta^t L_1(\dot{\chi}_1(l))\mathrm{d}l}\mathrm{d}\theta\Bigg], \end{aligned}$$

where $\chi_1(\cdot)$ denotes the path corresponding to the first i coordinates of $\gamma(\cdot)$ and $\chi_2(\cdot)$ denotes the path corresponding to the rest of the coordinates of $\gamma(\cdot)$.

3. Characterization for minimizing paths

For $\gamma \in \Gamma_t^x$ define

$$\begin{aligned} J(\gamma) &\stackrel{\text{def}}{=} \Bigg[\mathrm{e}^{\frac{1}{m-1}\{g(\gamma(0))+\int_0^t L(\dot{\gamma}(\theta))\mathrm{d}\theta\}} \\ &\quad + \frac{1}{m}\int_0^t |\dot{\gamma}_n|^{\frac{m}{m-1}}\,\mathrm{e}^{\frac{1}{m-1}\int_\theta^t L(\dot{\gamma}(l))\mathrm{d}l}\mathrm{d}\theta\Bigg]. \end{aligned} \tag{3.1}$$

The minimizer of J defines u and from Lemma 2.1 we note that there exists a minimizer $Z(\cdot)$ of J in Γ_t^x. In this section we deduce some characterizations for the minimizing paths that would reduce the set of controls in (2.2). Let $\alpha(\cdot)$ be a smooth curve from $[0,t]$ to $\mathbb{R}$ with the property that $\alpha(0) = 0 = \alpha(t)$. Let $\tau > 0$ and we define $Y_\tau(l) = (Z_1(l), \ldots, Z_{n-1}(l), Z_n(l) + \tau\alpha(l))$. Then

$$\begin{aligned} J(Y_\tau) &= \Bigg[\mathrm{e}^{\frac{1}{m-1}\{g(Y_\tau(0))+\int_0^t L(\dot{Y}_\tau(\theta))\mathrm{d}\theta\}} \\ &\quad + \frac{1}{m}\int_0^t |\dot{Z}_n + \tau\dot{\alpha}|^{\frac{m}{m-1}}\,\mathrm{e}^{\frac{1}{m-1}\int_\theta^t L(\dot{Y}_\tau(l))\mathrm{d}l}\mathrm{d}\theta\Bigg] \end{aligned}$$

First we note that if $Z_i(0) = x_i$ for some i, $1 \le i \le n$, then $Z_i(l) = x_i$ for all $l \in [0,t]$ since it is a minimizer of J. So it is better to assume that $Z_i(0) \neq x_i$ just by removing all those i for which this does not hold. Z being a minimizer, we have

$$\left.\frac{\mathrm{d}J(Y_\tau)}{\mathrm{d}\tau}\right|_{\tau=0} = 0.$$

Therefore from the above we have

$$\int_0^t |\dot{Z}_n|^{\frac{1}{m-1}} \operatorname{sign}(\dot{Z}_n(\theta))\dot{\alpha}(\theta) e^{\frac{1}{m-1}\int_\theta^t L(\dot{\dot{Z}}(l))\mathrm{d}l} \mathrm{d}\theta = 0,$$

α being arbitrary and we have

$$|\dot{Z}_n|^{\frac{1}{m-1}} \operatorname{sign}(\dot{Z}_n(\theta)) e^{\frac{1}{m-1}\int_\theta^t L(\dot{\dot{Z}}(l))\mathrm{d}l} = \text{constant} = A \tag{3.2}$$

almost everywhere in $[0, t]$. We can assume $\dot{Z}_n$ to be continuous in $[0, t]$ and hence (3.2) holds everywhere in $[0, t]$. Again sign of $\dot{Z}_n$ depends on sign of A. If sign of A is negative $\dot{Z}_n < 0$ is everywhere which readily implies that Z_n is decreasing in $[0, t]$. Since $Z_n(0) \neq x_n$, we can assume that $\dot{Z}_n(l) \neq 0$ for all $[0, t]$. Also we note that $\dot{Z}_n$ is monotone in $[0, t]$. Now taking logarithm on both sides of (3.2) and differentiating we get

$$\frac{\mathrm{d}^2 Z_n(\theta)}{\mathrm{d}\theta^2} = \dot{Z}_n(\theta) L(\dot{\dot{Z}}(\theta)), \quad \text{a.e. in } [0, t]. \tag{3.3}$$

Fix i, $1 \leq i < n$, such that $Z_i(0) \neq x_i$. Then as before perturbing in the i-th direction we have

$$(mG + Z_n(l) - Z_n(0))\dot{Z}_i(l) = mB_i, \quad \text{a.e. in } [0, t], \tag{3.4}$$

where $G = \frac{e^{\frac{g(Z(0))}{m-1}}}{\dot{Z}_n(0)}$ and B_i is some constant. This implies that $\dot{Z}_i$ is continuous in $[0, t]$. Now we note that if $B_i = 0$, Z_n being a monotone function we have $\dot{Z}_i = 0$ a.e. in $[0, t]$ what contradicts the assumption that $Z_i(0) \neq x_i$. This implies that $B_i \neq 0$ and hence $(mG + Z_n(l) - Z_n(0)) \neq 0$ and $\dot{Z}_i(l) \neq 0$ for all $l \in [0, t]$. Since $(mG + Z_n(l) - Z_n(0))$ is a monotone function, $\dot{Z}_i$ is also monotone and hence differentiable. After differentiating (3.4), we get

$$\dot{Z}_n(l) = -\frac{mB_i}{(\dot{Z}_i)^2}\frac{\mathrm{d}^2 Z_i(l)}{\mathrm{d}l^2}, \text{ a.e. in } [0, t]. \tag{3.5}$$

Therefore from (3.3) and (3.5) we get

$$\frac{\mathrm{d}^2 Z_n(l)}{\mathrm{d}l^2} = -\frac{m}{2}\sum_{i=1}^{n-1} B_i \frac{\mathrm{d}^2 Z_i(l)}{\mathrm{d}l^2} \quad \text{a.e. in } [0, t].$$

After integration we have

$$\dot{Z}_n(l) - \dot{Z}_n(0) = -\frac{m}{2}\sum_{i=1}^{n-1} B_i(\dot{Z}_i(l) - \dot{Z}_i(0)),$$

i.e.

$$\sum_{i=1}^{n-1} B_i \dot{Z}_i(l) = \sum_{i=1}^{n-1} B_i \dot{Z}_i(0) - \frac{2}{m}[\dot{Z}_n(l) - \dot{Z}_n(0)]. \tag{3.6}$$

Now using (3.6) we have from (3.4),

$$\left[\sum_{i=1}^{n-1} B_i \dot{Z}_i(0) - \frac{2}{m}(\dot{Z}_n(l) - \dot{Z}_n(0))\right](mG + Z_n(l) - Z_n(0)) = m\sum_{i=1}^{n-1} B_i^2,$$

i.e.

$$\left[\frac{m}{2}\sum_{i=1}^{n-1} B_i \dot{Z}_i(0) + \dot{Z}_n(0) - \dot{Z}_n(l)\right](mG + Z_n(l) - Z_n(0)) = \frac{m^2}{2}\sum_{i=1}^{n-1} B_i^2.$$

Define

$$C \stackrel{\text{def}}{=} \frac{m}{2}\sum_{i=1}^{n-1} B_i \dot{Z}_i(0) + \dot{Z}_n(0), \tag{3.7}$$

$$D \stackrel{\text{def}}{=} \frac{m^2}{2}\sum_{i=1}^{n-1} B_i^2. \tag{3.8}$$

Therefore integrating the above equality we have

$$\begin{aligned} mG + Z_n(l) - Z_n(0) - \frac{D}{C} &= \left(mG + x_n - Z_n(0) - \frac{D}{C}\right) \\ &\quad \times \mathrm{e}^{-\frac{C}{D}\{C(t-l) - x_n + Z_n(l)\}}. \end{aligned} \tag{3.9}$$

We can rewrite (3.9) as follows:

$$\begin{aligned} -mG + Z_n(0) + \frac{D}{C} &= Z_n(l) - \left(mG + x_n - Z_n(0) - \frac{D}{C}\right) \\ &\quad \times \mathrm{e}^{-\frac{C}{D}\{C(t-l) - x_n + Z_n(l)\}} \\ &= F(Z_n(l)) \text{ (say)}. \end{aligned}$$

We show that F is invertible. First we note from (3.4) that

$$B_i = G\dot{Z}_i(0) \quad \forall i.$$

Therefore

$$\begin{aligned} mG\frac{C}{D} - 1 &= \frac{\frac{m}{2}\sum_{i=1}^{n-1} G\dot{Z}_i^2(0) + \dot{Z}_n(0)}{\frac{m}{2}\sum_{i=1}^{n-1} G\dot{Z}_i^2(0)} - 1 \\ &> 0. \end{aligned}$$

Therefore for the fix l we have

$$F'(z) = 1 + \left[mG\frac{C}{D} - 1 + \frac{C}{D}(x_n - Z_n(0))\right]\mathrm{e}^{-\frac{C}{D}\{C(t-l) - x_n + z\}} > 0.$$

This implies that F is invertible in z and the inverse depends on l, $Z(0)$, $DZ(0)$. Let us denote this inverse by $F^{l,Z(0),DZ(0)}$ and hence from (3.9) we have

$$Z_n(l) = F^{l,Z(0),DZ(0)}\left(-mG + Z_n(0) + \frac{D}{C}\right). \tag{3.10}$$

So we define a class of function as follows:

$$\mathcal{V}_t^x \stackrel{\text{def}}{=} \{F^{p,q} : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R} \ : \text{continuous paths}\}, \tag{3.11}$$

where the following conditions are satisfied.

(a) $l \in \mathbb{R}_+$, $p, q \in \mathbb{R}^n$ and $(q_1, \ldots, q_{n-1}) \neq 0$.
(b) $q_n \neq 0$ and if C, D are defined by (3.7), (3.8) respectively, by replacing $Z(0)$, $DZ(0)$ with p, q respectively. Let $G(p, q) = \frac{e^{\frac{g(p)}{m-1}}}{q_n}$. Then for $l \in [0, t]$ the following are satisfied:

$$\begin{aligned} & mG(p,q) + F^{p,q}\left(l, -mG(p,q) + p_n + \frac{D}{C}\right) - p_n - \frac{D}{C} \\ & = \left(mG(p,q) + x_n - p_n - \frac{D}{C}\right) \\ & \quad \times e^{-\frac{C}{D}\{C(t-l) - x_n + F^{p,q}(l, -mG(p,q) + p_n + \frac{D}{C})\}}. \end{aligned}$$

(c) $F^{p,q}\big(0, -mG(p,q) + p_n + \frac{D}{C}\big) = p_n$ and $F^{p,q}\big(t, -mG(p,q) + p_n + \frac{D}{C}\big) = x_n$.
(d) For all $l \in [0, t]$, $F^{p,q}\big(l, -mG(p,q) + p_n + \frac{D}{C}\big) \neq p_n - mG(p,q)$.

If $(q_1, \ldots, q_{n-1}) = 0$ then $\mathcal{V}_t^x$ denotes all straight lines γ with $\gamma(t) = x_n$. For $(q_1, \ldots, q_{n-1}) = 0$ we define

$$\bar{\mathcal{V}}_t^x = \{(x_1, \ldots, x_{n-1}, \gamma) \ : \ \gamma \in \mathcal{V}_t^x\}.$$

For $q_n = 0$ we define

$$\bar{\mathcal{V}}_t^x = \{(Z_1, \ldots, Z_{n-1}, x_n)\},$$

where Z_i denotes straight lines with property $Z_i(t) = x_i$. For q other than the above two cases we define

$$\bar{\mathcal{V}}_t^x = \{(Z_1, \ldots, Z_n) \ : \ Z_n \in \mathcal{V}_t^x\},$$

where Z_i is defined as follows:

$$Z_i(l) = x_i - \int_l^t \frac{mG(p,q)q_i}{(mG(p,q) + Z_n(s) - p_n)} \mathrm{d}s,$$

if $q_i \neq 0$ otherwise $Z_i(l) = x_i$. The minimizer of J is contained in $\bar{\mathcal{V}}_t^x$. We also note that the only straight lines that $\bar{\mathcal{V}}_t^x$ contains is given by the first two cases. At the end of the paper we will provide an example to show that there are points (x, t) where J does not attend minimum on the straight lines.

4. Value function for HJE decreasing in *u*

In this section we consider a Hamiltonian of the form

$$H(u, p) = \frac{1}{2}\sum_{i=1}^{n-1} |p_i|^2 + \frac{e^{-u}}{m}|p_n|^m, \quad \text{for } (u, p) \in \mathbb{R} \times \mathbb{R}^n, \tag{4.1}$$

and the HJE

$$\begin{aligned} u_t + H(u, Du) &= 0 \text{ for } (x,t) \in \mathbb{R}^n \times \mathbb{R}_+, \\ u(x,0) &= g(x). \end{aligned} \tag{4.2}$$

Let Γ_t^x be as before. For $(x,t) \in \mathbb{R}^n \times \mathbb{R}_+$, we define

$$\begin{aligned} u(x,t) \stackrel{\text{def}}{=} \inf_{\Gamma_t^x} -(m-1)\ln\Bigg[& \mathrm{e}^{\frac{-1}{m-1}\{g(\gamma(0))+\int_0^t L(\dot{\gamma}(\theta)\mathrm{d}\theta)\}} \\ & - \frac{1}{m}\int_0^t |\dot{\gamma}_n(\theta)|^{\frac{m}{m-1}} \mathrm{e}^{-\frac{1}{m-1}\int_\theta^t L(\dot{\gamma}(l))\mathrm{d}l}\mathrm{d}\theta \Bigg]. \end{aligned} \tag{4.3}$$

(We have taken the convention $-\ln(a) = \infty$ for $a \leq 0$.)

Theorem 4.1. *The function u defined by* (4.3) *is Lipschitz continuous and a viscosity solution of* (4.2).

Proof. Proof of the theorem is same as the proof of Theorem 2.1. For convenience we give some sketch. Lemmas 2.1 and 2.2 could be shown along the same lines. For Lemma 2. we show only the Lipschitz continuity in t. First we note that $u(x,t) \leq u(x,s)$ for $0 \leq s \leq t$. Again if ϕ is a minimizing path for $u(x,t)$, then

$$\mathrm{e}^{\frac{-1}{m-1}\{u(\phi(0),s)+\int_0^{t-s} L(\dot{\phi}(\theta)\mathrm{d}\theta)\}} \geq \mathrm{e}^{\frac{-u(x,t)}{m-1}} \geq \mathrm{e}^{\frac{-M_2}{m-1}}.$$

Therefore

$$\mathrm{e}^{\frac{-1}{m-1}\int_0^{t-s} L(\dot{\phi}(\theta)\mathrm{d}\theta)} \geq \mathrm{e}^{\frac{M_1-M_2}{m-1}}.$$

In a similar manner we can show that

$$\left| \mathrm{e}^{-\frac{u(x_1,t)}{m-1}} - \mathrm{e}^{-\frac{u(x_2,t)}{m-1}} \right| \leq K|x_1 - x_2| \ \ \forall\, x_1, x_2 \in \mathbb{R}^n,$$

For some constant K, consider

$$\begin{aligned} \mathrm{e}^{-\frac{u(x,s)}{m-1}} - \mathrm{e}^{-\frac{u(x,t)}{m-1}} &= \left[\mathrm{e}^{-\frac{u(x,s)}{m-1}} - \mathrm{e}^{-\frac{u(\phi(0),s)}{m-1}}\right] \\ &\quad + \mathrm{e}^{-\frac{u(\phi(0),s)}{m-1}}\left[1 - \mathrm{e}^{-\frac{1}{m-1}\int_0^{t-s} L(\dot{\phi})\mathrm{d}\theta}\right] \\ &\quad + \frac{1}{m}\int_0^{t-s} |\dot{\phi}_n|^{\frac{m}{m-1}} \mathrm{e}^{-\frac{1}{m-1}\int_\theta^{t-s} L(\dot{\phi})\mathrm{d}l}\mathrm{d}\theta \\ &\geq -K|x - \phi(0)| \\ &\quad + \mathrm{e}^{\frac{-1}{m-1}\{u(\phi(0),s)+\int_0^{t-s} L(\dot{\phi}(\theta)\mathrm{d}\theta)\}}\left[\mathrm{e}^{\frac{1}{m-1}\int_0^{t-s} L(\dot{\phi})\mathrm{d}\theta} - 1\right] \\ &\quad + \frac{1}{m}\int_0^{t-s} |\dot{\phi}_n|^{\frac{m}{m-1}} \mathrm{e}^{-\frac{1}{m-1}\int_\theta^{t-s} L(\dot{\phi})\mathrm{d}l}\mathrm{d}\theta \\ &\geq -K|x - \phi(0)| + \mathrm{e}^{\frac{-M_2}{m-1}}\int_0^{t-s} L(\dot{\phi}(\theta)\mathrm{d}\theta \\ &\quad + \mathrm{e}^{\frac{M_1-M_2}{m-1}}\frac{1}{m}\int_0^{t-s} |\dot{\phi}_n|^{\frac{m}{m-1}}. \end{aligned}$$

Rest of the proof is similar to Lemma 2.3. Proof of subsolution and supersolution would be similar to Lemma 2.4. □

We can also characterize the minimizer for (4.3) in the same way as §3. But our goal is to show that u defined by (4.3) is the unique viscosity solution for (4.2).

4.1 *Uniqueness*

First we consider the HJE

$$u_t + \frac{(m-1)m^{\frac{1}{m-1}}}{2(m-1-m^{\frac{1}{m-1}}u)}\sum_{i=1}^{n-1}|u_{x_i}|^2 + |u_{x_n}|^m = 0 \ \text{ for } (x,t)\in\mathbb{R}^n\times\mathbb{R}_+,$$
$$u(x,0) = h(x). \tag{4.4}$$

The above HJE is valid whenever $m^{\frac{1}{m-1}}u \neq m-1$. Define

$$\mathcal{F}(t) = \int_0^t \frac{e^{-\frac{s}{m-1}}}{m^{\frac{1}{m-1}}}ds \quad \forall\, t\in\mathbb{R}. \tag{4.5}$$

Therefore $\mathcal{F}(\cdot)$ is an increasing C^1 function. First we show that if u is a viscosity solution of (4.2) then $\mathcal{F}(u)$ is a viscosity solution of (4.4) with initial condition $\mathcal{F}(g)$. Let $(x_0,t_0)\in\mathbb{R}^n\times\mathbb{R}_+$ and ν be a C^1 function such that $\mathcal{F}(u)-\nu$ has a local maximum at (x_0,t_0) with $\mathcal{F}(u(x_0,t_0)) = \nu(x_0,t_0)$. Then $u-\mathcal{F}^{-1}(\nu)$ has a local maximum at (x_0,t_0). Since u is subsolution of (4.2), we have

$$(\mathcal{F}^{-1}(\nu(x_0,t_0)))_t + H(\mathcal{F}^{-1}(\nu(x_0,t_0)), D\mathcal{F}^{-1}(\nu(x_0,t_0)) \geq 0.$$

A direct calculation shows that

$$\nu_t + \frac{(m-1)m^{\frac{1}{m-1}}}{2(m-1-m^{\frac{1}{m-1}}\nu)}\sum_{i=1}^{n-1}|\nu_{x_i}|^2 + |\nu_{x_n}|^m \geq 0,$$

at the point (x_0,t_0). We note that $\mathcal{F}'(\mathcal{F}^{-1}(s)) = \frac{m-1-m^{\frac{1}{m-1}}s}{(m-1)m^{\frac{1}{m-1}}} > 0$ if s is in the image of $\mathcal{F}$. This proves that $\mathcal{F}(u)$ is a subsolution for (4.4). Similarly we can show that $\mathcal{F}(u)$ is a supersolution for (4.4).

We now prove that viscosity solution of (4.4) is unique whenever the value function is Lipschitz, bounded with supremum strictly less than $\frac{m-1}{m^{\frac{1}{m-1}}}$. Let u and v be two viscosity solutions (4.4) with the same initial data. Then there exists $\sigma > 0$ such that

$$\sup_{\mathbb{R}^n\times\mathbb{R}_+}(u-v) \geq \sigma > 0. \tag{4.6}$$

Now we consider a function $\Phi : (\mathbb{R}^n)^2\times(\bar{\mathbb{R}}_+)^2 \to \mathbb{R}$ defined as follows

$$\Phi(x,y,t,s) = u(x,t) - v(y,s) - \frac{\|x-y\|^2+|t-s|^2}{2\alpha^2}$$
$$- \lambda(t+s) - \epsilon(\|x\|^2+\|y\|^2), \tag{4.7}$$

where $\alpha, \lambda, \epsilon > 0$. Note that, u and v being bounded the supremum of Φ is attended inside a compact set, say at $(\bar{x}, \bar{y}, \bar{t}, \bar{s})$. Again for $\lambda \in (0, \lambda_0]$, $\epsilon \in (0, \epsilon_0]$, for λ_0, ϵ_0 small, we can have

$$\Phi(\bar{x}, \bar{y}, \bar{t}, \bar{s}) \geq \sup_{\mathbb{R}^n \times \mathbb{R}_+} \Phi(x, x, t, t) \geq \frac{\sigma}{2}, \tag{4.8}$$

and this implies that $u(\bar{x}, \bar{t}) > v(\bar{y}, \bar{s})$. Since $\Phi(\bar{x}, \bar{y}, \bar{t}, \bar{s}) \geq \Phi(0, 0, 0, 0)$, we have

$$\lambda(\bar{t} + \bar{s}) + \frac{\|\bar{x} - \bar{y}\|^2 + |\bar{t} - \bar{s}|^2}{2\alpha^2} + \epsilon(\|\bar{x}\|^2 + \|\bar{y}\|^2) \leq C, \tag{4.9}$$

for some constant C depending on the bounds of u and v. This implies that

$$\begin{aligned} \epsilon(\|\bar{x}\| + \|\bar{y}\|) &= \epsilon^{\frac{1}{4}} \epsilon^{\frac{3}{4}} (\|\bar{x}\| + \|\bar{y}\|) \\ &\leq \frac{\epsilon^{\frac{1}{2}}}{2} + \frac{\epsilon^{\frac{3}{2}} (\|\bar{x}\| + \|\bar{y}\|)^2}{2} \\ &\leq C_1 \epsilon^{\frac{1}{2}}. \end{aligned} \tag{4.10}$$

Let $h(x)$ be the initial condition for u, v. Therefore from (4.8) we have

$$\begin{aligned} \frac{\sigma}{2} &\leq u(\bar{x}, \bar{t}) - v(\bar{y}, \bar{s}) \\ &= u(\bar{x}, \bar{t}) - h(\bar{x}) + h(\bar{x}) - v(\bar{x}, \bar{t}) + v(\bar{x}, \bar{t}) - v(\bar{y}, \bar{s}) \\ &\leq K_1 \bar{t} + K_2 \bar{t} + K_2(\|\bar{x} - \bar{y}\| + |t - s|), \end{aligned}$$

where K_1, K_2 are the Lipschitz constants of u, v respectively. Using (4.9) we note that for small $\alpha > 0$, $\bar{t} > 0$. Similarly we have $\bar{s} > 0$. Also note that the choice of α is independent of ϵ, λ.

Now we observe that $u - \nu$ has a maximum at $(\bar{x}, \bar{t})$ for

$$\nu(x, t) \stackrel{\text{def}}{=} v(\bar{y}, \bar{s}) + \frac{\|x - \bar{y}\|^2 + |t - \bar{s}|^2}{2\alpha^2} + \lambda(t + \bar{s}) + \epsilon(\|x\|^2 + \|\bar{y}\|^2) + K,$$

where K is a constant that ensures $u(\bar{x}, \bar{t}) = \nu(\bar{x}, \bar{t})$. Since u is a viscosity solution we have from (4.4),

$$\lambda + \frac{\bar{t} - \bar{s}}{\alpha^2} + H_1(u(\bar{x}, \bar{t}), \frac{\bar{x} - \bar{y}}{\alpha^2} + 2\epsilon\bar{x}) \leq 0, \tag{4.11}$$

where

$$\begin{aligned} H_1(u, p) &= \frac{(m-1) m^{\frac{1}{m-1}}}{2(m - 1 - m^{\frac{1}{m-1}} u)} \\ &\quad \times \sum_{i=1}^{n-1} |p_i|^2 + |p_n|^m, \quad \text{for } (u, p) \in \left[\frac{m-1}{m^{\frac{1}{m-1}}}, \infty\right) \times \mathbb{R}^n. \end{aligned}$$

We also observe that $v - \nu$ has a minimum at $(\bar{y}, \bar{s})$ for

$$\nu(y, s) \stackrel{\text{def}}{=} u(\bar{x}, \bar{t}) - \frac{\|\bar{x} - y\|^2 + |\bar{t} - s|^2}{2\alpha^2} - \lambda(\bar{t} + s) - \epsilon(\|\bar{x}\|^2 + \|y\|^2) + K,$$

where K is a constant that ensures $v(\bar{y}, \bar{s}) = \nu(\bar{y}, \bar{s})$. Since v is a viscosity solution of (4.4), we have

$$-\lambda + \frac{\bar{t} - \bar{s}}{\alpha^2} + H_1\left(v(\bar{y}, \bar{s}), \frac{\bar{x} - \bar{y}}{\alpha^2} - 2\epsilon \bar{y}\right) \geq 0. \tag{4.12}$$

Now subtracting (4.12) from (4.11) we have

$$2\lambda + H_1\left(u(\bar{x}, \bar{t}), \frac{\bar{x} - \bar{y}}{\alpha^2} + 2\epsilon \bar{x}\right) - H_1\left(v(\bar{y}, \bar{s}), \frac{\bar{x} - \bar{y}}{\alpha^2} - 2\epsilon \bar{y}\right) \leq 0. \tag{4.13}$$

As $\epsilon \downarrow 0$, we note that $\epsilon\bar{x}, \epsilon\bar{y} \to 0$ from (4.10). Also from (4.9), we note that $\bar{x} - \bar{y} \to z \in \mathbb{R}^n$, may be for a subsequence, as $\epsilon \downarrow 0$. u, v being bounded, as $\epsilon \downarrow 0$ let $u(\bar{x}, \bar{t}) \to \bar{u}$ and $v(\bar{y}, \bar{s}) \to \bar{v}$ where $\bar{u} \geq \bar{v}$. Therefore letting $\epsilon \downarrow 0$ in (4.13) we have

$$2\lambda + H_1\left(\bar{u}, \frac{z}{\alpha^2}\right) - H_1\left(\bar{v}, \frac{z}{\alpha^2}\right) \leq 0.$$

But this is a contradiction as the left-hand side is > 0 due to monotonicity of H_1 in u. Therefore our assumption (4.6) is wrong and hence $u \leq v$. Similarly we can prove that $v \leq u$. This also proves that (4.2) has a unique viscosity solution.

Therefore we have the following theorem:

Theorem 4.2. *The Hamilton-Jacobi equation* (4.2) *has a unique bounded*, *Lipschitz continuous viscosity solution and this is given by* (4.3).

Remark 4.1. Let H_1, H_2 be the same as in Remark 2.2. Then

$$u_t + H_1(u_{x_1}, \ldots, u_{x_i}) + \mathrm{e}^{-u} H_2(u_{x_{i+1}}, \ldots, u_{x_n}) = 0$$

has a unique viscosity solution. The Hopf-Lax type representation is given by

$$u(x,t) = \inf_{\Gamma_t^x} -(m-1)\ln\Big[\mathrm{e}^{\frac{-1}{m-1}\{g(\gamma(0)) + \int_0^t L_1(\dot{\chi}_1(\theta))\mathrm{d}\theta\}} - \frac{1}{m-1}\int_0^t L_2(\dot{\chi}_2(\theta))\mathrm{e}^{\frac{-1}{m-1}\int_\theta^t L_1(\dot{\chi}_1(l))\mathrm{d}l}\mathrm{d}\theta\Big].$$

Remark 4.2. Let H_1, H_2 be as above. Then it is easy to check that for $\alpha \neq 0$ the unique viscosity solution of

$$u_t + H_1(u_{x_1}, \ldots, u_{x_i}) + \mathrm{e}^{\alpha u} H_2(u_{x_{i+1}}, \ldots, u_{x_n}) = 0$$

has representation

$$u(x,t) = \inf_{\Gamma_t^x} \frac{(m-1)}{\alpha}\ln\Big[\mathrm{e}^{\frac{\alpha}{m-1}\{g(\gamma(0)) + \int_0^t L_1(\dot{\chi}_1(\theta))\mathrm{d}\theta\}} + \frac{\alpha}{m-1}\int_0^t L_2(\dot{\chi}_2(\theta))\mathrm{e}^{\frac{\alpha}{m-1}\int_\theta^t L_1(\dot{\chi}_1(l))\mathrm{d}l}\mathrm{d}\theta\Big].$$

5. Example

Consider the HJE

$$\left.\begin{array}{lcl} u_t + \frac{|u_{x_1}|^2}{2} + \frac{e^u}{2}|u_{x_2}|^2 &=& 0 \text{ in } \mathbb{R}^2 \times \mathbb{R}_+ \\ u(x,0) &=& g_1(x_1) + g_2(x_2) \end{array}\right\}, \tag{5.1}$$

where $g_1 : \mathbb{R} \to \mathbb{R}$ is defined by

$$g_1(x) = \begin{cases} 0, & \text{if } |x| \geq 1, \\ 1+x, & \text{if } -1 \leq x \leq 0, \\ 1-x, & \text{if } 0 \leq x \leq 1 \end{cases}$$

and $g_2 : \mathbb{R} \to \mathbb{R}$ is defined by

$$g_2(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ \ln(1+x), & \text{if } 0 \leq x \leq 1, \\ \ln 2, & \text{if } 1 \leq x. \end{cases}$$

Define

$$u_1(x,t) = \inf_{y_1 \in \mathbb{R}} \left\{ g_1(y_1) + \frac{t}{2}\left|\frac{x_1 - y_1}{t}\right|^2 \right\} + g_2(x_2) \tag{5.2}$$

and

$$u_2(x,t) = \inf_{y_2 \in \mathbb{R}} \ln \left\{ \mathrm{e}^{g_1(x_1)+g_2(y_2)} + \frac{t}{2}\left|\frac{x_2 - y_2}{t}\right|^2 \right\}. \tag{5.3}$$

If the minimizer of J (defined for HJE (5.1)) is attended on straight lines for all points $(x,t) \in \mathbb{R}^2 \times \mathbb{R}_+$, then

$$u(x,t) = \min\{u_1(x,t),\ u_2(x,t)\}$$

would be the viscosity solution of (5.1). We note that $u_1(x,t) = g_2(x_2)$ if $|x| \geq 1$. Also for $0 \leq x \leq 1$, by direct calculation we have for $t \leq 2$,

$$u_1(x,t) = \begin{cases} 1 - x_1 - \dfrac{t}{2} + g_2(x_2), & \text{if } 0 \leq x_1 \leq t-1, \\ \dfrac{(1-x_1)^2}{2t} + g_2(x_2), & \text{if } t-1 \leq x_1 \leq 1. \end{cases}$$

Similarly for $x_2 \leq 0$, $u_2(x,t) = g_1(x_1)$. Consider $t > 0$ small such that $te^2 < 1$. By direct calculation we see that for $0 \leq x_2 \leq 1$,

$$u_2(x,t) = \begin{cases} \ln\left(\mathrm{e}^{g_1(x_1)} + \dfrac{x_2^2}{2t}\right), & \text{if } 0 \leq x_2 \leq \mathrm{e}^{g_1(x_1)}t, \\ g_1(x_1) + \ln\left(1 + x_2 - \dfrac{t}{2}\mathrm{e}^{g_1(x_1)}\right), & \text{if } \mathrm{e}^{g_1(x_1)}t \leq x_2 \leq 1. \end{cases}$$

Now in $\{(x_1,x_2) \mid 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1\}$, $g_1', g_2' \neq 0$ and so it is easy to show that none of u_1, u_2 is a viscosity solution. So u cannot be a viscosity solution if it is defined as a minimum of u_1, u_2. This proves that the minimizer of J may not be attained on straight lines.

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