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Existence of solution of the pullback equation involving volume forms

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Abstract. Let $\Omega \subset \mathbb{R}^n$ be a smooth, bounded domain. We study the existence and regularity of diffeomorphisms of Ω satisfying the volume form equation

$$\phi^*(g) = f, \quad \text{in } \Omega,$$

where $f, g \in C^{m,\alpha}(\bar{\Omega}; \Lambda^n)$ are given positive volume forms.

Keywords. Jacobian equation; pullback equation; volume forms.

1. Introduction

In this article, we discuss the existence of diffeomorphisms $\phi : \bar{\Omega} \to \phi(\bar{\Omega}) \subseteq \bar{\Omega}$ satisfying the volume form equation

$$\phi^*(g) = f, \quad \text{in } \Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a smooth, bounded domain and f, g are positive volume forms defined on $\bar{\Omega}$. Identifying a volume form with its defining function, eq. (1.1) can be re-written as

$$(g \circ \phi) \det(D\phi) = f, \quad \text{in } \Omega, \tag{1.2}$$

which is why (1.1) is also referred to as the Jacobian equation. This equation has applications in problems of optimal mass transport and physics for example, in the problem of equilibrium of gases, see [4].

The study of eq. (1.1) dates back to Moser [10]. Following a conjecture of Palais, Moser studied the problem of equivalence of positive volume forms under automorphisms of a smooth, compact manifold without boundary. In doing so, Moser introduced a new method, now famously known as Moser's flow method, to prove that, on a smooth, compact manifold $\mathcal{M}$ without boundary, there exists $\phi \in \operatorname{Aut}(\mathcal{M})$ satisfying eq. (1.1), if and only if

$$\int_{\mathcal{M}} f = \int_{\mathcal{M}} g. \tag{1.3}$$

After Moser's article, the problem of existence of automorphisms satisfying (1.1) under condition (1.3) caught the attention of many, notably Riemann [11], Banyaga [3], Zehnder [15], Tartar [13], Dacorogna [4], to name a few. The problem of regularity for the volume form equation was first addressed by Dacorogna and Moser [6], see Theorem 5 for the statement. Later, the problem of existence and regularity of solutions for volume forms in various other function spaces were considered by Ye [14] and Rivière-Ye [12] and for general k-forms, see [1] and [2].

In the light of the aforementioned discussion, a natural question that arises at this point is what happens when condition (1.3) is not satisfied? Of course, in this case, the change of variable formula forbids us from expecting the existence of an automorphism of $\bar{\Omega}$ satisfying (1.1), but can one still find a diffeomorphism $\phi : \bar{\Omega} \to \phi(\bar{\Omega}) \subseteq \bar{\Omega}$ satisfying (1.1). Note that, it follows trivially from the change of variable formula that one needs

$$\int_{\Omega} f(x)\,\mathrm{d}x \leqslant \int_{\Omega} g(x)\,\mathrm{d}x \tag{1.4}$$

as a necessary condition in order to talk about the existence of a diffeomorphism, not necessarily an automorphism, that solves (1.1). To show that (1.4) is sufficient as well, is the aim of this article. Let us state the main theorem of the article at this point.

Theorem 1. *Let $m, n \in \mathbb{N}$, $0 < \alpha < 1$, and let $\Omega \subset \mathbb{R}^n$ be a smooth, bounded domain and $f, g \in C^{m,\alpha}(\bar{\Omega}; \Lambda^n)$ satisfy $f, g > 0$ in $\bar{\Omega}$. Then, there exists $\phi \in \mathrm{Diff}^{m+1,\alpha}(\bar{\Omega})$ satisfying*

$$\phi^*(g)(x) = f(x), \quad \textit{for all } x \in \Omega,$$

if and only if

$$\int_{\Omega} f(x)\,\mathrm{d}x \leqslant \int_{\Omega} g(x)\,\mathrm{d}x.$$

We conclude this section with few comments on the proof. At first glance, Theorem 1 may appear to be following from Theorem 5 simply by re-scaling the volume forms, but this not so, for when the forms are scaled back, the re-scaled diffeomorphism may leave the domain. To avoid this situation, it is necessary to control the flow appropriately and to solve the linearized equation in accordance with the control. This is done in Lemmas 2 and 3. Next, we solve the volume form equation without any gain in regularity (see Lemma 4). In the final step, we combine Lemma 4 with Theorem 5 to achieve the optimal regularity (see Theorems 6 and 7).

2. Notation

We adopt following notations and conventions in this article.

1. $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.
2. Λ^n denotes the space of exterior n-forms in $\mathbb{R}^n$.
3. Let $m, n \in \mathbb{N}$, $0 \leqslant \alpha \leqslant 1$ and let $\Omega \subset \mathbb{R}^n$ be a bounded domain. We write

$$\begin{aligned} \mathrm{Diff}^{m,\alpha}(\bar{\Omega}) &:= \big\{\phi \in C^{m,\alpha}(\bar{\Omega}; \mathbb{R}^n) : \phi : \bar{\Omega} \to \phi(\bar{\Omega}) \text{ is a } \\ &\qquad C^{m,\alpha}\text{-diffeomorphism and } \phi(\bar{\Omega}) \subseteq \bar{\Omega}\big\}. \\ \mathrm{Aut}^{m,\alpha}(\bar{\Omega}) &:= \big\{\phi \in \mathrm{Diff}^{m,\alpha}(\bar{\Omega}) : \phi(\bar{\Omega}) = \bar{\Omega}\big\}. \end{aligned}$$

4. Often, we will identify a volume form with its defining function. For example, if $\omega \in C^{m,\alpha}(\Omega; \Lambda^n)$ is a volume form written as

$$\omega := g\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n, \quad \text{in } \Omega,$$

for some $g \in C^{m,\alpha}(\Omega)$, we say that g is the *defining function* of ω. Note that, $g = *\omega$ in Ω, where $*$ is the Hodge-star operator. We will often identify ω with g and use the same notation.

5. Let $m \in \mathbb{N} \cup \{\infty\}$, $n \in \mathbb{N}$ and let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then, $\rho \in C^m(\mathbb{R}^n)$ is said to be a *defining function* of Ω if

   (a) $\Omega = \{x \in \mathbb{R}^n : \rho(x) < 0\}$,
   (b) $\mathbb{R}^n \setminus \bar{\Omega} = \{x \in \mathbb{R}^n : \rho(x) > 0\}$ and
   (c) $D\rho(x) \neq 0$, for all $x \in \partial\Omega$.

   Note that, Ω is a C^m-domain if and only if there exists a C^m-defining function of Ω (see [9] for more information on defining functions).

3. Resolution of the volume form equation

We start with the following elementary lemma.

Lemma 2. *Let $m \in \mathbb{N}_0$, $n \in \mathbb{N}$, $0 < \alpha < 1$, and let $\Omega \subset \mathbb{R}^n$ be a smooth, bounded domain and $f \in C^{m,\alpha}(\bar{\Omega})$. Then, there exists $\omega \in C^{m+1,\alpha}(\bar{\Omega}; \mathbb{R}^n)$ satisfying*

$$\begin{cases} \operatorname{div} \omega(x) = f(x), & \textit{for all } x \in \Omega, \\ \langle \omega(x); \nu(x) \rangle \leqslant 0, & \textit{for all } x \in \partial\Omega, \end{cases} \tag{3.1}$$

if and only if

$$\int_\Omega f(x)\, \mathrm{d}x \leqslant 0.$$

Furthermore, if

$$\int_\Omega f(x)\, \mathrm{d}x = 0,$$

we can choose ω to satisfy

$$\omega = 0 \ \ on \ \ \partial\Omega.$$

Proof. Let us choose $h \in C^\infty(\mathbb{R}^n)$ such that

$$h(x) \leqslant 0, \ \text{for all } x \in \partial\Omega$$

and

$$\int_{\partial\Omega} h(x)\, \mathrm{d}S(x) = \int_\Omega f(x)\, \mathrm{d}x,$$

which is possible because $\int_\Omega f(x)\,\mathrm{d}x \leqslant 0$. We now find $u \in C^{m+2,\alpha}(\bar{\Omega}; \mathbb{R}^n)$ satisfying the Neumann problem

$$\begin{cases} \Delta u(x) = f(x), & \text{for all } x \in \Omega, \\ \frac{\partial u}{\partial \nu}(x) = h(x), & \text{for all } x \in \partial\Omega. \end{cases}$$

Defining $\omega \in C^{m+1,\alpha}(\bar{\Omega}; \mathbb{R}^n)$ by

$$\omega(x) := Du(x), \qquad \text{for all } x \in \bar{\Omega},$$

we have a solution of (3.1). For a proof of the second part of lemma, see [5]. □

The following result on ordinary differential equations shows a way to contain the flow inside a domain.

Lemma 3. *Let $m, n \in \mathbb{N}$, $0 < \alpha < 1$, and let $\Omega \subset \mathbb{R}^n$ be a smooth, bounded domain and $F \in C([0,1]; C^{m,\alpha}(\bar{\Omega}; \mathbb{R}^n))$ satisfy*

$$\langle F(t,x); \nu(x)\rangle \leqslant 0, \qquad \textit{for all } (t,x) \in [0,1] \times \partial\Omega,$$

where ν is the unit outward normal to $\partial\Omega$. Then, there exists a unique $\Phi \in C^1\big([0,1]; C^{m,\alpha}\big(\bar{\Omega}; \bar{\Omega}\big)\big)$ satisfying, for all $x \in \bar{\Omega}$,

$$\begin{cases} \frac{\partial \Phi}{\partial t}(t,x) = F(t, \Phi(t,x)), & \textit{for all } t \in (0,1], \\ \Phi(0,x) = x. \end{cases}$$

Furthermore, if

$$\Phi(s, x_0) \in \partial\Omega, \qquad \textit{for some } (s, x_0) \in (0,1) \times \bar{\Omega},$$

it follows that

$$\langle F(s, \Phi(s,x_0)); \nu\left(\Phi(s,x_0)\right)\rangle = 0.$$

Proof. Let $E : C^{m,\alpha}(\bar{\Omega}; \mathbb{R}^n) \to C_0^{m,\alpha}(\mathbb{R}^n; \mathbb{R}^n)$ be the extension operator and let us define $G \in C\ ([0,1]; C_0^{m,\alpha}(\mathbb{R}^n; \mathbb{R}^n))$ by

$$G := E \circ F, \qquad \text{in } [0,1].$$

Evidently,

$$\langle G(t,x); \nu(x)\rangle \leqslant 0, \qquad \text{for all } (t,x) \in [0,1] \times \partial\Omega.$$

It follows from standard results of ordinary differential equations that there exists a unique $\Phi \in C^1\ ([0,1]; C^{m,\alpha}\ (\mathbb{R}^n; \mathbb{R}^n))$ satisfying, for all $x \in \mathbb{R}^n$,

$$\begin{cases} \frac{\partial \Phi}{\partial t}(t,x) = G(t, \Phi(t,x)), & \text{for all } t \in (0,1], \\ \Phi(0,x) = x. \end{cases}$$

We now claim that

$$\Phi(t, x) \in \bar{\Omega}, \quad \text{for all } (t, x) \in [0, 1] \times \bar{\Omega}. \tag{3.2}$$

Let $x_0 \in \bar{\Omega}$ be fixed. We now divide the proof of (3.2) into two steps.

Step 1. For all $(t, x) \in [0, 1] \times \partial\Omega$,

$$\langle G(t, x); \nu(x)\rangle < 0.$$

We first consider the case when $x_0 \in \Omega$. Let us define

$$\mathcal{S} := \{p \in [0, 1] : \Phi(t, x_0) \in \bar{\Omega}, \quad \text{for all } t \in [0, p]\}.$$

Clearly $0 \in \mathcal{S}$ and let us set $m := \sup \mathcal{S}$. Then, $\Phi(t, x_0) \in \bar{\Omega}$, for all $t \in [0, m]$. It is enough to show that $m = 1$. Let us suppose, on the contrary, that $m < 1$. Note that, we have $\Phi(m, x_0) \in \partial\Omega$. For, if $\Phi(m, x_0) \in \Omega$, it follows from the continuity of Φ that, for some $\delta > 0$, $\Phi(t, x_0) \in \Omega$ for all $t \in [m, m+\delta]$ which, in turn, implies that $\Phi(t, x_0) \in \bar{\Omega}$, for all $t \in [0, m+\delta]$. In other words, we have $m+\delta \in \mathcal{S}$, which contradicts the fact that m is an upper bound of $\mathcal{S}$. Hence, $\Phi(m, x_0) \in \partial\Omega$. This implies that $m \neq 0$ i.e. $0 < m < 1$. We now define $g \in C^1((0, 1); \mathbb{R})$ by

$$g(t) := (\rho \circ \Phi)(t, x_0), \quad \text{for all } t \in (0, 1),$$

where $\rho \in C^\infty(\mathbb{R}^n; \mathbb{R})$ is the defining function of Ω, see §2. Now,

$$g'(m) = \left\langle D\rho(\Phi(m, x_0)); \frac{\partial\Phi}{\partial t}(m, x_0)\right\rangle = \langle \nu\,(\Phi(m, x_0))\,; G(m, \Phi(m, x_0))\rangle < 0.$$

Therefore, there exists $\epsilon > 0$ such that

$$g(t) < g(m) = 0, \quad \text{for all } t \in (m, m+\epsilon] \subset (0, 1),$$

which implies that

$$\Phi(t, x_0) \in \Omega, \quad \text{for all } t \in (m, m+\epsilon].$$

In other words, we have

$$\Phi(t, x_0) \in \bar{\Omega}, \text{ for all } t \in [0, m+\epsilon],$$

which is a contradiction to the fact that m is an upper bound of $\mathcal{S}$. Hence, we have deduced that $m = 1$ and this proves the first step when $x_0 \in \Omega$.

Let us now assume that $x_0 \in \partial\Omega$. Let $(y_r)_{r\in\mathbb{N}}$ be a sequence in Ω that converges to x_0. Then, it follows from the aforementioned discussion that

$$\Phi(t, y_r) \in \bar{\Omega}, \quad \text{for all } t \in [0, 1],\ r \in \mathbb{N}.$$

Letting $r \to \infty$, we deduce that

$$\Phi(t, x_0) \in \bar{\Omega}, \quad \text{for all } t \in [0, 1],$$

which proves the first step. In the second step, we consider the general case.

Step 2. For all $(t, x) \in [0, 1] \times \partial\Omega$,

$$\langle G(t, x); \nu(x)\rangle \leqslant 0.$$

Define $H : [0, 1] \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ by

$$H(t, x, \epsilon) := G(t, x) - \epsilon\, D\rho(x), \qquad \text{for all } (t, x, \epsilon) \in [0, 1] \times \mathbb{R}^n \times \mathbb{R}.$$

Using the theorem on continuous dependence of solutions with respect to parameters (see [8]), we find $r > 0$ such that, for all $\epsilon \in (-r, r)$, there exists $X_\epsilon \in C^1([0, 1]; \mathbb{R}^n)$ satisfying

$$\begin{cases} X_\epsilon'(t) = H(t, X_\epsilon(t), \epsilon), & \text{for all } t \in (0, 1],\\ X_\epsilon(0) = x_0, \end{cases}$$

and

$$\lim_{\epsilon\to 0} X_\epsilon(t) = \Phi(t, x_0), \qquad \text{for all } t \in [0, 1].$$

Since for all $(t, x, \epsilon) \in [0, 1] \times \partial\Omega \times (0, r)$,

$$\begin{aligned} \langle H(t, x, \epsilon); \nu(x)\rangle &= \langle G(t, x); \nu(x)\rangle - \epsilon\,\langle D\rho(x); \nu(x)\rangle\\ &= \langle G(t, x); \nu(x)\rangle - \epsilon < 0, \end{aligned}$$

it follows from Step 1 that

$$X_\epsilon(t) \in \bar{\Omega}, \qquad \text{for all } t \in [0, 1] \text{ and } \epsilon \in (0, r).$$

Therefore, letting $\epsilon \to 0$, we have that

$$\Phi(t, x_0) \in \bar{\Omega}, \qquad \text{for all } t \in [0, 1],$$

which proves Claim (3.2). Hence, we have, for all $x \in \bar{\Omega}$,

$$\frac{\partial\Phi}{\partial t}(t, x) = G(t, \Phi(t, x)) = F(t, \Phi(t, x)), \qquad \text{for all } t \in (0, 1].$$

To prove the last statement, let us again consider the function g. It follows from (3.2) that

$$g(s) = 0 \quad \text{and} \quad g(t) \leqslant 0, \qquad \text{for all } t \in [0, 1].$$

In other words, s is a global maxima of g and hence

$$g'(s) = \langle F(s, \Phi(s, x_0)); \nu\,(\Phi(s, x_0))\rangle = 0.$$

This completes the proof. □

We first prove a special and weaker version of Theorem 1.

Lemma 4. *Let $m, n \in \mathbb{N}$, $0 < \alpha < 1$, and let $\Omega \subset \mathbb{R}^n$ be a smooth, bounded domain and $f \in C^{m,\alpha}(\bar{\Omega})$ satisfy $f > 0$ in $\bar{\Omega}$. Then, there exists $\phi \in \mathrm{Diff}^{m,\alpha}(\bar{\Omega})$ satisfying*

$$\det(D\phi(x)) = f(x), \qquad \textit{for all } x \in \Omega,$$

if and only if

$$\int_\Omega f(x)\,\mathrm{d}x \leqslant \mathcal{L}^n(\Omega).$$

Proof. Using Lemma 2, we find a $\omega \in C^{m+1,\alpha}(\bar{\Omega};\mathbb{R}^n)$ satisfying

$$\begin{cases} \operatorname{div}\omega(x) = f(x) - 1, & \text{for all } x \in \Omega,\\ \langle \omega(x); \nu(x)\rangle \leqslant 0, & \text{for all } x \in \partial\Omega. \end{cases} \tag{3.3}$$

We now define $F \in C^{m,\alpha}\left([0,1]\times\bar{\Omega};\mathbb{R}^n\right)$ by

$$F(t,x) := \frac{\omega(x)}{t + (1-t) f(x)}, \qquad \text{for all } (t,x) \in [0,1]\times\bar{\Omega}.$$

Note that, it follows from eq. (3.3) that

$$\langle F(t,x); \nu(x)\rangle \leqslant 0, \qquad \text{for all } (t,x) \in [0,1]\times\partial\Omega.$$

Therefore, using Lemma 3, we find $\Phi \in C^1\left([0,1]; C^{m,\alpha}\left(\bar{\Omega};\bar{\Omega}\right)\right)$ such that, for all $x \in \bar{\Omega}$,

$$\begin{cases} \frac{\partial\Phi}{\partial t}(t,x) &= F(t,\Phi(t,x)), \qquad \text{for all } t \in (0,1],\\ \Phi(0,x) &= x. \end{cases} \tag{3.4}$$

Clearly, for each $t \in [0,1]$, $\Phi(t,\cdot) \in \mathrm{Diff}^{m,\alpha}\left(\bar{\Omega}\right)$. Furthermore, using Liouville formula (see [7]), we have, for all $(t,x) \in (0,1)\times\Omega$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\det\left(D_x\Phi(t,x)\right) = \operatorname{div}_x F(t,x)\det\left(D_x\Phi(t,x)\right). \tag{3.5}$$

Using eqs (3.3), (3.4) and (3.5), we now deduce that, for all $(t,x) \in (0,1)\times\Omega$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\det\left(D_x\Phi(t,x)\right)\left(t + (1-t) f(\Phi(t,x))\right)\right] = 0.$$

Therefore, defining $\phi : \bar{\Omega} \to \bar{\Omega}$ by

$$\phi(x) := \Phi(1,x), \qquad \text{for all } x \in \bar{\Omega},$$

we see that $\phi \in \mathrm{Diff}^{m,\alpha}(\bar{\Omega})$ is a solution of the problem. This proves the lemma. □

Let us now recall the result of Dacorogna and Moser [6] at this point.

Theorem 5 [6]. *Let $m, n \in \mathbb{N}$, $0 < \alpha < 1$, and let $\Omega \subset \mathbb{R}^n$ be a smooth, bounded domain and $f, g \in C^{m,\alpha}(\bar{\Omega};\Lambda^n)$ satisfy $f, g > 0$ in $\bar{\Omega}$. Then, there exists $\phi \in \mathrm{Aut}^{m+1,\alpha}(\bar{\Omega})$ satisfying*

$$\begin{cases} \phi^*(g)(x) = f(x), & \textit{for all } x \in \Omega,\\ \phi(x) = x, & \textit{for all } x \in \partial\Omega, \end{cases}$$

if and only if

$$\int_\Omega f(x)\,\mathrm{d}x = \int_\Omega g(x)\,\mathrm{d}x.$$

We now improve the regularity of the solution obtained in Lemma 4 by combining the lemma with Theorem 5 as follows.

Theorem 6. *Let $m, n \in \mathbb{N}$, $0 < \alpha < 1$, and let $\Omega \subset \mathbb{R}^n$ be a smooth, bounded domain and $f \in C^{m,\alpha}(\bar{\Omega})$ satisfy $f > 0$ in $\bar{\Omega}$. Then, there exists $\phi \in \mathrm{Diff}^{m+1,\alpha}(\bar{\Omega})$ satisfying*

$$\det(D\phi(x)) = f(x), \quad \textit{for all } x \in \Omega,$$

if and only if

$$\int_{\Omega} f(x)\,\mathrm{d}x \leqslant \mathcal{L}^n(\Omega).$$

Proof. Using Theorem 5, we find $v \in \mathrm{Aut}^{m+1,\alpha}(\bar{\Omega})$ such that

$$\begin{cases} \det(Dv(x)) = \left(\frac{\mathcal{L}^n(\Omega)}{\int_{\Omega} f(z)\,\mathrm{d}z}\right) f(x), & \text{for all } x \in \Omega, \\ v(x) = x, & \text{for all } x \in \partial\Omega. \end{cases} \tag{3.6}$$

Since

$$0 < \frac{\int_{\Omega} f(z)\,\mathrm{d}z}{\mathcal{L}^n(\Omega)} \leqslant 1,$$

it follows from Lemma 4 that there exists $u \in \mathrm{Diff}^{m+1,\alpha}(\bar{\Omega})$ satisfying

$$\det(Du(x)) = \frac{\int_{\Omega} f(z)\,\mathrm{d}z}{\mathcal{L}^n(\Omega)}, \quad \text{for all } x \in \Omega.$$

We now define $\phi : \bar{\Omega} \to \mathbb{R}^n$ by

$$\phi(x) := (u \circ v)(x), \quad \text{for all } x \in \bar{\Omega}.$$

Clearly, $\phi \in \mathrm{Diff}^{m+1,\alpha}(\bar{\Omega})$ and for all $x \in \Omega$, we have

$$\begin{aligned} \det(D\phi(x)) &= \det(Du(v(x)))\det(Dv(x)) \\ &= \frac{\int_{\Omega} f(z)\,\mathrm{d}z}{\mathcal{L}^n(\Omega)} \frac{\mathcal{L}^n(\Omega)}{\int_{\Omega} f(z)\,\mathrm{d}z} f(x) = f(x). \end{aligned}$$

This proves the theorem. □

We are now ready to prove the main theorem, Theorem 1. For convenience, let us restate the theorem.

Theorem 7. *Let $m, n \in \mathbb{N}$, $0 < \alpha < 1$, and let $\Omega \subset \mathbb{R}^n$ be a smooth, bounded domain and $f, g \in C^{m,\alpha}(\bar{\Omega}; \Lambda^n)$ satisfy $f, g > 0$ in $\bar{\Omega}$. Then, there exists $\phi \in \mathrm{Diff}^{m+1,\alpha}(\bar{\Omega})$ satisfying*

$$\phi^*(g)(x) = f(x), \quad \textit{for all } x \in \Omega,$$

if and only if

$$\int_{\Omega} f(x)\,\mathrm{d}x \leqslant \int_{\Omega} g(x)\,\mathrm{d}x.$$

Proof. Using Theorem 5, we find $v \in \mathrm{Aut}^{m+1,\alpha}(\bar{\Omega})$ satisfying

$$\begin{cases} \det(Dv(x)) = \left(\frac{\mathcal{L}^n(\Omega)}{\int_\Omega g(z)\,\mathrm{d}z}\right) g(x), & \text{for all } x \in \Omega, \\ v(x) = x, & \text{for all } x \in \partial\Omega. \end{cases} \tag{3.7}$$

As

$$\int_\Omega \frac{\mathcal{L}^n(\Omega)}{\int_\Omega g(z)\,\mathrm{d}z} f(x)\,\mathrm{d}x = \left(\frac{\int_\Omega f(x)\,\mathrm{d}x}{\int_\Omega g(x)\,\mathrm{d}x}\right) \mathcal{L}^n(\Omega) \leqslant \mathcal{L}^n(\Omega),$$

we use Theorem 6 to find $u \in \mathrm{Diff}^{m+1,\alpha}(\bar{\Omega})$ satisfying

$$\det(Du(x)) = \left(\frac{\mathcal{L}^n(\Omega)}{\int_\Omega g(z)\,\mathrm{d}z}\right) f(x), \qquad \text{for all } x \in \Omega.$$

We now define $\phi : \bar{\Omega} \to \mathbb{R}^n$ by

$$\phi(x) := (v^{-1} \circ u)(x), \qquad \text{for all } x \in \bar{\Omega}.$$

Evidently, $\phi \in \mathrm{Diff}^{m+1,\alpha}(\bar{\Omega})$ and

$$\det(Du(x)) = \det(Dv(\phi(x)))\det(D\phi(x)), \qquad \text{for all } x \in \Omega,$$

which implies that, for all $x \in \Omega$,

$$\left(\frac{\mathcal{L}^n(\Omega)}{\int_\Omega g(z)\,\mathrm{d}z}\right) f(x) = \left(\frac{\mathcal{L}^n(\Omega)}{\int_\Omega g(z)\,\mathrm{d}z}\right) g(\phi(x))\det(D\phi(x)).$$

In other words, we have deduced that

$$\phi^*(g)(x) = f(x), \qquad \text{for all } x \in \Omega.$$

This proves the theorem. □

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