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Spherical means in annular regions in the n-dimensional real hyperbolic spaces

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Abstract. Let $Z_{r,R}$ be the class of all continuous functions f on the annulus $\mathrm{Ann}(r, R)$ in the real hyperbolic space $\mathbb{B}^n$ with spherical means $M_s f(x) = 0$, whenever $s > 0$ and $x \in \mathbb{B}^n$ are such that the sphere $S_s(x) \subset \mathrm{Ann}(r, R)$ and $B_r(o) \subseteq B_s(x)$. In this article, we give a characterization for functions in $Z_{r,R}$. In the case $R = \infty$, this result gives a new proof of Helgason's support theorem for spherical means in the real hyperbolic spaces.

Keywords. Real hyperbolic spaces; spherical means; spherical harmonics.

1. Introduction

Let g be a continuous function on the open annulus $\{x \in \mathbb{R}^d : r < |x| < R\}$, where $0 \leq r < R \leq \infty$ and $d \geq 2$. We say that g satisfies the vanishing spherical means condition if

$$\int_{|x-y|=s} g(y)\mathrm{d}\sigma_s(y) = 0$$

for every sphere $\{y \in \mathbb{R}^d : |x - y| = s\}$ which is contained in the annulus and is such that the closed ball $\{y \in \mathbb{R}^d : |y| \leq r\}$ is contained in the closed ball $\{y \in \mathbb{R}^d : |x - y| \leq s\}$. Here $\mathrm{d}\sigma_s$ is the surface measure on the sphere $\{y \in \mathbb{R}^d : |x - y| = s\}$.

For a continuous function g on $\mathbb{R}^d$, let

$$g(x) = \sum_{k=0}^{\infty} \sum_{j=1}^{d_k} a_{kj}(\rho)\, Y_{kj}(\omega) \tag{1.1}$$

be the spherical harmonic expansion, where $x = \rho\omega,\ \rho = |x|,\ \omega \in S^{d-1}$ and $\{Y_{kj}(\omega) : j = 1, \ldots, d_k\}$ is an orthonormal basis for the space V_k of homogeneous harmonic polynomials in d variables of degree k restricted to the unit sphere S^{d-1}. Then the following interesting result, which can be thought of as a null space characterization of the Radon transform over spheres for continuous functions, has been proved in [2] by Epstein and Kleiner:

Theorem 1.1. *Let g be a continuous function on the annulus* $\{x \in \mathbb{R}^d : r < |x| < R\}$, $0 \leq r < R \leq \infty$. *Then g satisfies the vanishing spherical means condition if and only if*

$$a_{kj}(\rho) = \sum_{i=0}^{k-1} \alpha_{kj}^i \, \rho^{k-d-2i}, \quad \alpha_{kj}^i \in \mathbb{C},$$

for all $k \geq 1$, $1 \leq j \leq d_k$ *and* $a_0(\rho) = 0$, *whenever* $r < \rho < R$.

This result was first proved by Globevnik [4] for the case $n = 2$. For other related work, we refer to [1,9,13,14].

In a recent work of [8] the authors have proved a spectral Paley–Wiener theorem for the Heisenberg group by means of a support theorem for twisted spherical means on $\mathbb{C}^n$. The support theorem for twisted spherical means can be thought of as a special case of the problem described above for the twisted spherical means. The full analogue of Theorem 4.2 for these means has been investigated by authors in [10].

In this paper, we have investigated the following analogous problem for spherical means in real hyperbolic spaces. Let $\mathbb{B}^n = \{x \in \mathbb{R}^n : |x|^2 = \sum x_i^2 < 1\}$ be the open unit ball in $\mathbb{R}^n$, $n \geq 2$, endowed with the Poincare metric $\mathrm{d}s^2 = \lambda^2(\mathrm{d}x_1^2 + \cdots + \mathrm{d}x_n^2)$, where $\lambda = 2(1 - |x|^2)^{-1}$. Let $B_s(o) = \{x \in \mathbb{B}^n : d(x, o) \leq s\}$ be the closed geodesic ball of radius s with centre at the origin and $\mathrm{Ann}(r, R) = \{x \in \mathbb{B}^n : r < d(x, o) < R\}$, $0 \leq r < R \leq \infty$, be an open annulus in $\mathbb{B}^n$.

For $s > 0$, let μ_s denote the surface measure on the geodesic sphere $S_s(x) = \{y \in \mathbb{B}^n : d(x, y) = s\}$. Let f be a continuous function on $\mathbb{B}^n$. Define the spherical means of f by

$$M_s f(x) = \frac{1}{A(s)} \int_{S_s(x)} f(y) \mathrm{d}\mu(y), \quad x \in \mathbb{B}^n, \tag{1.2}$$

where $A(s) = (\Omega_n)^{-1}(\sinh s)^{-n+1}$.

Let $Z_{r,R}$ be the class of all continuous functions on $\mathrm{Ann}(r, R)$ with the spherical means $M_s f(x) = 0$, whenever $s > 0$, and $x \in \mathbb{B}^n$ are such that the sphere $S_s(x) \subset \mathrm{Ann}(r, R)$ and ball $B_r(o) \subseteq B_s(x)$.

Our main result is the following characterization theorem.

Theorem 1.2. *Let f be a continuous function on* $\mathrm{Ann}(r, R)$. *Then a necessary and sufficient condition for f to be in* $Z_{r,R}$ *is that its spherical harmonic coefficients* $a_{kj}(\rho)$ *satisfy*

$$a_{kj}(\rho) = \sum_{i=1}^{k} C_{kj}^i \frac{(1 - \rho^2)^{n+i-2}}{\rho^{n+k-2}}, \quad C_{kj}^i \in \mathbb{C},$$

for all $k \geq 1$, $1 \leq j \leq d_k$ *and* $a_0(\rho) \equiv 0$, *whenever* $\tanh \frac{r}{2} < \rho < \tanh \frac{R}{2}$.

In [2] the authors have observed their result for Euclidean spherical means can be used to derive result for some cases, real hyperbolic spaces being one of them. In a recently published book (Chapter 10 of [14]) the authors have developed a general theory of transmutation operators from which our result Theorem 1.2 can be derived using a string of lemmas proved in this chapter. However, the expressions for the spherical harmonic coefficients $a_{kj}(\rho)$ as given in Theorem 1.2 have nowhere been listed before. Moreover, our approach to the proof of theorem is direct, transparent and brings out the underlying geometry of the real hyperbolic spaces clearly. The proof of the necessary part of this theorem is close to the work in p.108 of [13], on problems related to spherical means and the proof of the sufficient part is completely new.

The case of other real rank one symmetric spaces can be dealt with in a similar way.

2. Notation and preliminaries

We begin with the realization of real hyperbolic spaces (see [7], [11]). Let $O(1, n+1)$ be the group of all linear transformations which preserve the quadratic form

$$\langle y, y\rangle = y_0^2 - \sum_{i=1}^{n+1} y_i^2,\ y = (y_0, y_1, \ldots, y_{n+1})$$

on $\mathbb{R}^{n+2}$. This group is known as the Lorentz group and is equal to

$$\{g \in M_{n+2}(\mathbb{R}) : g^t J g = J,\ \ J = \operatorname{diag}(1, -1, \ldots, -1)\}.$$

In particular, $O(1, n+1)$ leaves invariant the cone

$$C = \left\{ y \in \mathbb{R}^{n+2} : \langle y, y\rangle = 0 \right\}.$$

With the inhomogeneous coordinates $\eta_i = y_i/y_0$, $i = 1, \ldots, n+1$, the relation $\langle y, y\rangle = 0$ would imply that η is in $S^n = \{\eta \in \mathbb{R}^{n+1} :\ |\eta| = 1\}$. Thus a point on C gets identified with a point on the sphere S^n. Conversely for $\eta \in S^n$, $\eta^* = (1, \eta_1, \ldots, \eta_{n+1})$ gives a point on the cone C. As $g \in O(1, n+1)$ acts on η^* and $g\eta^* \in C$, g acts on S^n via the above identification. More explicitly, $g\eta^*$ can be identified with the point $\left(\frac{(g\eta^*)_1}{(g\eta^*)_0}, \ldots, \frac{(g\eta^*)_{n+1}}{(g\eta^*)_0}\right)$ in S^n. ($(g\eta^*)_0$ is nonzero, as η^* is nonzero and $g\eta^* \in C$.)

Let $O_\pm(1, n+1) \cong O(1, n+1)/\{\pm I\}$ be the subgroup of $O(1, n+1)$ which leaves invariant the positive cone

$$C^+ = \left\{ y = (y_0, y_1, \ldots, y_{n+1}) \in \mathbb{R}^{n+2} : \langle y, y\rangle = y_0^2 - \sum_{i=1}^{n+1} y_i^2 > 0,\ \ y_0 > 0 \right\}.$$

Equivalently,

$$O_\pm(1, n+1) = \{g \in M_{n+2}(\mathbb{R}) : g^t J g = J,\ \ J = \operatorname{diag}(1, -1, \ldots, -1),\ \ g_{00} > 0\},$$

where g_{00} is the top left entry in the matrix of g. In particular, $O_\pm(1, n+1)$ leaves the cone $C^0 = \left\{ y \in \mathbb{R}^{n+2} : \langle y, y\rangle = 0,\ \ y_0 > 0 \right\}$ invariant. Moreover, as the action of g and $-g$ in $O(1, n+1)$ on the sphere S^n coincide, $O_\pm(1, n+1)$ also acts on S^n. In fact, this is the group of Mobius transforms on S^n. The real hyperbolic space $\mathbb{B}^n$ is then isomorphic to the quotient space $SO_\pm(1, n)/SO(n)$. This isomorphism is established as follows.

We identify $S^n \setminus \{e_{n+1}\}$ with $\mathbb{R}^n$ under the stereographic projection from the point $e_{n+1} = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$ onto the plane $\eta_{n+1} = 0$. Then the $O_\pm(1, n+1)$ action on S^n induces an action on $\mathbb{R}^n \cup \{\infty\}$ and vice versa. It turns out that the subgroup of $O_{\pm}(1, n+1)$ which stabilizes $\mathbb{B}^n$ is isomorphic to $O_{\pm}(1, n)$. This can be seen as follows.

Let $x = (x_1, \dots, x_n) \in \mathbb{B}^n$. Then the inverse stereographic projection of $\eta \in S^n$ of x is given by

$$\eta_i = \frac{2x_i}{1+|x|^2},\ i = 1, \dots, n \quad \text{and} \quad \eta_{n+1} = \frac{|x|^2 - 1}{|x|^2 + 1}. \tag{2.3}$$

Therefore, $x \in \mathbb{B}^n$ if and only if $\eta_{n+1} < 0$. Thus a subgroup of $O_{\pm}(1, n+1)$ stabilizes the open unit ball $\mathbb{B}^n$ if and only if it stabilizes the lower hemisphere $\{\eta \in S^n : \eta_{n+1} < 0\}$. This subgroup in turn is isomorphic to $O_{\pm}(1, n)$ (see [7]). The elements of this subgroup realized as elements of $O_{\pm}(1, n+1)$ look like

$$\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix},$$

with $g \in O_{\pm}(1, n)$. Moreover, this action of $O_{\pm}(1, n)$ on $\mathbb{B}^n$ is transitive and the orthogonal group $O(n)$ thought of as

$$\begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix}$$

inside $O_{\pm}(1, n)$ is the isotropy subgroup of the point origin in the ball $\mathbb{B}^n$. Thus $\mathbb{B}^n$ is isomorphic to the quotient space $O_{\pm}(1, n)/O(n)$. Likewise, $\mathbb{B}^n \cong SO_{\pm}(1, n)/SO(n)$. Let $G = SO_{\pm}(1, n)$ and $K = SO(n)$. Henceforth, we will work with the representation G/K of $\mathbb{B}^n$. Using the G-invariant metric $\mathrm{d}y_0^2 - \mathrm{d}y_1^2 - \cdots - \mathrm{d}y_n^2$ on the positive cone $y_0^2 - y_1^2 - \cdots - y_n^2 = 1,\ y_0 > 0$, $\mathbb{B}^n$ can be endowed with a G-invariant Riemannian metric given by $\mathrm{d}s^2 = \lambda^2 |\mathrm{d}x|^2$. The distance $d(x, y)$ between points $x, y \in \mathbb{B}^n$, in this metric, is then given by

$$\tanh \frac{1}{2} d(x, y) = \frac{|x - y|}{\sqrt{1 - 2x \cdot y + |x|^2 |y|^2}}.$$

This makes $(\mathbb{B}^n, d)$ into a Riemannian symmetric space. Group theoretically, $\mathbb{B}^n = G/K$ is a real rank one symmetric space.

Further, let $G = K\overline{A_+}K$ be the Cartan decomposition of G, where

$$A = \left\{ \begin{pmatrix} \cosh \frac{t}{2} & 0 & \sinh \frac{t}{2} \\ 0 & I_{n-1} & 0 \\ \sinh \frac{t}{2} & 0 & \cosh \frac{t}{2} \end{pmatrix} : t \in \mathbb{R} \right\},$$

is a maximal abelian subgroup of G and A_+ is a chosen positive Weyl chamber $\{a_t : t > 0\}$. Let M be the centralizer $\{k \in K : ka = ak, \forall a \in A\}$ of A in K. Therefore, M is given by

$$M = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 1 \end{pmatrix} : m \in SO(n-1) \right\}.$$

Thus the boundary S^{n-1} of $\mathbb{B}^n$ gets identified with K/M under the map $\sigma M \to \sigma \cdot e_n$, $\sigma \in K$ where $e_n = (0, 0, \dots, 1) \in \mathbb{R}^n$ and the elements of G/K can be thought of as pairs $(a_t, \omega),\ t \geq 0,\ \omega \in S_{n-1}$. The point (a_t, ω) then is identified with the point

$(\cosh \frac{t}{2},\ \sinh \frac{t}{2}.\omega)$ on the positive cone in $\mathbb{R}^{n+1}$ and this point in turn, is identified with the point $\tanh \frac{t}{2}\omega$ in $\mathbb{B}^n$.

Next, we recall certain standard facts about spherical harmonics, for more details see p. 12 of [12].

Let $\hat{K}_M$ denote the set of all the equivalence classes of irreducible unitary representations of K which have a nonzero M-fixed vector. It is well known that each representation in $\hat{K}_M$ has in fact a unique nonzero M-fixed vector, up to a scalar multiple.

For $\delta \in \hat{K}_M$, which is realized on V_δ, let $\{e_1, \ldots, e_{d(\delta)}\}$ be an orthonormal basis of V_δ, with e_1 as the M-fixed vector. Let $t_\delta^{ji}(\sigma) = \langle e_i, \delta(\sigma)e_j\rangle$, $\sigma \in K$ and $\langle , \rangle$ stand for the inner product on V_δ. By Peter–Weyl theorem, it follows that $\{\sqrt{d(\delta)}t_\delta^{1j} : 1 \leq j \leq d(\delta), \delta \in \hat{K}_M\}$ is an orthonormal basis of $L^2(K/M)$.

We would further need a concrete realization of the representations in $\hat{K}_M$, which can be done in the following way.

Let $\mathbb{Z}_+$ denote the set of all non-negative integers. For $k \in \mathbb{Z}_+$, let P_k denote the space of all homogeneous polynomials P in n variables of degree k. Let $H_k = \{P \in P_k : \Delta P = 0\}$, where Δ is the standard Laplacian on $\mathbb{R}^n$. The elements of H_k are called solid spherical harmonics of degree k. It is easy to see that the natural action of K leaves the space H_k invariant. In fact the corresponding unitary representation π_k is in $\hat{K}_M$. Moreover, $\hat{K}_M$ can be identified, up to unitary equivalence, with the collection $\{\pi_k : k \in \mathbb{Z}_+.\}$

Define the spherical harmonics on the sphere S^{n-1} by $Y_{kj}(\omega) = \sqrt{d_k}t_{\pi_k}^{1j}(\sigma)$, where $\omega = \sigma \cdot e_n \in S^{n-1}$, $\sigma \in K$ and d_k is the dimension of H_k. Then $\{Y_{kj} : 1 \leq j \leq d_k, k \in \mathbb{Z}_+\}$ forms an orthonormal basis for $L^2(S^{n-1})$. Therefore, for a continuous function f on $\mathbb{B}^n$, writing $y = \rho\,\omega$, where $0 < \rho < 1$ and $\omega \in S^{n-1}$, we can expand the function f in terms of spherical harmonics as in (1.1). For each non negative integer k, the k-th spherical harmonic projection, $\Pi_k(f)$ of the function f is defined by

$$\Pi_k(f)(y) = \sum_{j=1}^{d_k} a_{kj}(\rho)\, Y_{kj}(\omega), \tag{2.4}$$

where a_{kj}'s are the spherical harmonic coefficients of the function f.

3. Auxiliary results

We begin with the observation that the K-invariance of the annulus and the measure μ_s implies that for any f in $Z_{r,R}$ and $k \in \mathbb{Z}_+$, $\Pi_k(f)$, as defined in eq. (2.4), also belongs to $Z_{r,R}$. In fact the following stronger result is true.

Lemma 3.1. *Let $f \in Z_{r,R}$. Then each of the spherical harmonic projection $\Pi_k(f) \in Z_{r,R}$, in fact $a_{kj}(\rho)Y_{km}(\omega) \in Z_{r,R}$ $\forall\, j, m,\ 1 \leq j, m \leq d_k$ and $\forall\, k \geq 0$.*

Proof. Since the measure μ_s and space Ann(r, R) both are rotation invariant, it is easy to verify that, if $f \in Z_{r,R}$, then the function $f(\tau \cdot y) \in Z_{r,R}$ for each $\tau \in K$. Since the space H_k is K-invariant, for $\tau \in K$ and a spherical harmonic Y_{kj}, we have

$$Y_{kj}(\tau^{-1}\omega) = \sum_{m=1}^{d_k} \overline{t_{\pi_k}^{mj}(\tau)}Y_{km}(\omega).$$

Hence from eq. (1.1), the function $f(\tau^{-1}.)$ can be decomposed as

$$f(\tau^{-1}\rho\omega) = \sum_{k\geq 0}\sum_{j,m=1}^{d_k} a_{kj}(\rho)\overline{t_{\pi_k}^{mj}(\tau)}Y_{km}(\omega).$$

The set $\{\sqrt{d_k}t_{\pi_k}^{mj} : 1 \leq j, m \leq d_k, k \geq 0\}$ form an orthonomal basis for $L^2(K)$, therefore,

$$a_{kj}(\rho)Y_{km}(\omega) = d_k \int_K f(\tau^{-1}\rho\omega)t_{\pi_k}^{mj}(\tau)\mathrm{d}\tau \in Z_{r,R}.$$

Subsequently, each projection $\Pi_k(f)$ belongs to $Z_{r,R}$. □

Next, we need the following explicit expression for the action of G on $\mathbb{B}^n$, which has been derived in [6].

Lemma 3.2. *Let $g \in G$ and $x \in \mathbb{B}^n$. Then $g \cdot (x_1, \ldots, x_n) = (y_1, \ldots, y_n)$, where*

$$y_j = \frac{\frac{(1+|x|^2)}{2}g_{j0} + \sum_{l=1}^n g_{jl}x_l}{\frac{1-|x|^2}{2} + \frac{(1+|x|^2)}{2}g_{00} + \sum_{l=1}^n g_{0l}x_l}, \quad j = 1, \ldots, n. \tag{3.5}$$

Proof. By eq. (2.3), a point $x \in \mathbb{B}^n$ is mapped to the point $\eta \in S^n$ via the inverse stereographic projection. By definition, for $g \in G$,

$$g \cdot \eta = \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ \eta \end{pmatrix} = \alpha,$$

where $\alpha = (\alpha_0, \ldots, \alpha_n, \eta_{n+1})$ and $\alpha_j = g_{j0} + \sum_{l=1}^n g_{jl}\eta_l,\ l = 0, 1, \ldots, n$. Since the cone C^0 is G-invariant, it follows that $\alpha_0 > 0$. In the inhomogeneous coordinates introduced earlier, the point α gets identified with the point $\left(\frac{\alpha_1}{\alpha_0}, \ldots, \frac{\alpha_n}{\alpha_0}, \frac{\eta_{n+1}}{\alpha_0}\right)$ on the sphere S^n. The image of this point, under the stereographic projection is the point $y = (y_1, \ldots, y_n) \in \mathbb{B}^n$, where

$$y_j = \frac{\alpha_j/\alpha_0}{1 - \eta_{n+1}/\alpha_0}, \quad j = 1, \ldots, n.$$

That is,

$$y_j = \frac{g_{j0} + \sum_{l=1}^n g_{jl}\eta_l}{g_{00} + \sum_{l=1}^n g_{0l}\eta_l - \eta_{n+1}}, \quad j = 1, \ldots, n.$$

Since we know that

$$\eta_l = \frac{2x_l}{1+|x|^2},\ l = 1, \ldots, n, \quad \eta_{n+1} = \frac{|x|^2 - 1}{|x|^2 + 1},$$

a simple computation gives

$$y_j = \frac{\frac{(1+|x|^2)}{2}g_{j0} + \sum_{l=1}^n g_{jl}x_l}{\frac{1-|x|^2}{2} + \frac{(1+|x|^2)}{2}g_{00} + \sum_{l=1}^n g_{0l}x_l}, \quad j = 1, \ldots, n.$$

□

As in the proof of the Euclidean case [2], to characterize functions in $Z_{r,R}$ it would be enough to characterize the spherical harmonic coefficients of smooth functions in $Z_{r,R}$. This can be done using the following approximation argument.

Let φ_ϵ be nonnegative, K-biinvariant, smooth, compactly supported approximate identity on G/K. Let $f \in Z_{r,R}$. Then f can be thought of as a right K-invariant function on G. Define

$$S_\epsilon(f)(g) = \int_G f(gh^{-1})\varphi_\epsilon(h)\mathrm{d}h, \quad g \in G.$$

Then $S_\epsilon(f)$ is smooth and it is easy to see that $S_\epsilon(f) \in Z_{r+\epsilon,R-\epsilon}$ for each $\epsilon > 0$. Since f is continuous, $S_\epsilon(f)$ converges to f uniformly on compact sets. Therefore, for each k,

$$\lim_{\epsilon\to 0} \Pi_k(S_\epsilon(f)) = \Pi_k(f).$$

Hence, we can assume, without loss of generality, that the functions in $Z_{r,R}$ are also smooth in the annulus Ann(r, R).

We next introduce right K-invariant differential operators on G which leave invariant the space $Z_{r,R}$. These differential operators arise naturally from the Lie algebra $\mathfrak{g}$ of G, in the following way. They also appear prominently in the work of Volchkov on ball means in real hyperbolic spaces (p. 108 of [13]).

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G. Here $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{p}$ its orthogonal complement in $\mathfrak{g}$ with respect to the Killing form $B(-,-)$. Let $X_i = E_{0i} + E_{i0},\ i = 1, \ldots, n$ and $X_{ij} = E_{ij} - E_{ji},\ 1 \le i < j \le n$, where $E_{ij} \in gl(n+1, \mathbb{R})$ is the matrix with entry 1 at the ij-th place and zero elsewhere. Then $\{X_i : i = 1, \ldots, n\}$ and $\{X_{ij} : 1 \le i < j \le n\}$ form bases of $\mathfrak{p}$ and $\mathfrak{k}$ respectively.

Let $f \in C^\infty(\mathbb{B}^n)$. Then f can be thought of as the right K-invariant function on G. For a given $X \in \mathfrak{g}$, let $\tilde{X}$ be the differential operator given by

$$(\tilde{X}f)(gK) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(\exp tXgK). \tag{3.6}$$

For $X = X_p \in \mathfrak{p}$, let

$$\tau_{t,p} = \exp tX_p = \begin{pmatrix} \cosh t & 0 & \sinh t & 0 \\ 0 & I_{p-1} & 0 & 0 \\ \sinh t & 0 & \cosh t & 0 \\ 0 & 0 & 0 & I_{n-p} \end{pmatrix},$$

for $t \in \mathbb{R}$. Let $x \in \mathbb{B}^n$. Then by Lemma 3.2, $\tau_{t,p} \cdot x = y \in \mathbb{B}^n$, where $y_j = x_j u(t,x)$, if $j \neq p$ and $y_p = (x_p \cosh t + (1+|x|^2)\frac{\sinh t}{2})u(t,x)$, $u(t,x) = (\cosh^2 \frac{t}{2} + x_p \sinh t + |x|^2 \sinh^2 \frac{t}{2})^{-1}$. Rewrite $\tau_{t,p} \cdot x$ as $\tau(t,x)$. Then τ is a differentiable function on $\mathbb{R} \times \mathbb{R}^n$ into $\mathbb{R}^n$ and from (3.6), we have

$$\frac{\partial}{\partial t}(f o \tau(t,x)) = f'(\tau(t,x))\frac{\partial \tau}{\partial t}(t,x) = \sum_{j=1}^n \frac{\partial f}{\partial y_j}\frac{\partial y_j}{\partial t}.$$

Evaluating the above equation at $t = 0$, we get

$$\left.\frac{\partial}{\partial t}(f o \tau(t,x))\right|_{t=0} = \sum_{j=1}^n \left.\frac{\partial f}{\partial y_j}\right|_{t=0} \left.\frac{\partial y_j}{\partial t}\right|_{t=0} = \sum_{j=1}^n \frac{\partial f}{\partial x_j} \left.\frac{\partial y_j}{\partial t}\right|_{t=0}. \tag{3.7}$$

A straightforward calculation then gives

$$\left.\frac{\partial y_j}{\partial t}\right|_{t=0} = \begin{cases} -x_p x_j, & \text{if } j \neq p; \\ \frac{1}{2}(1+|x|^2) - x_p^2, & \text{if } j = p. \end{cases}$$

Substituting these values in (3.7), we get

$$\tilde{X}_p = \frac{1}{2}(1+|x|^2)\frac{\partial}{\partial x_p} - \sum_{j=1}^{n} x_p x_j \frac{\partial}{\partial x_j}, \quad p = 1, \ldots, n.$$

The following lemma is a crucial step towards the proof of our main result.

Lemma 3.3. *Suppose f is a smooth function belonging to $Z_{r,R}$. Then the function $\tilde{X}_p f \in Z_{r,R}$, $\forall\ p,\ 1 \leq p \leq n$.*

Proof. For a fixed $t \in \mathbb{R}$, define

$$\epsilon_1 = \sup_{y \in B_r(o)} d(\tau_{t,p}.y, y) \quad \text{and} \quad \epsilon_2 = \sup_{y \in B_R(0)} d(\tau_{t,p}.y, y).$$

Then, it is easy to see that the translated function $\tau_{t,p} f$ defined by $\tau_{t,p} f(y) = f(\tau_{t,p}.y),\ y \in \mathbb{B}^n$ belongs to $Z_{r+\epsilon_1, R-\epsilon_2}$. Therefore,

$$\int_{S_s(x)} f(\tau_{t,p}.\xi)\mathrm{d}\mu_s(\xi) = \int_{S_s(\tau_{t,p}.x)} f(\xi)\mathrm{d}\mu_s(\xi) = 0,$$

whenever $S_s(x) \subset \text{Ann}(r+\epsilon_1, R-\epsilon_2)$ and $B_{r+\epsilon_1}(0) \subset B_s(x)$. As $t \to 0$, this implies

$$\int_{S_s(x)} \left.\frac{\partial f}{\partial t}\right|_{t=0} (\tau_{t,p}.\xi)\mathrm{d}\mu_s(\xi) = 0,$$

whenever $S_s(x) \subset \text{Ann}(r, R)$ and $B_r(0) \subseteq B_s(x)$. Hence $\tilde{X}_p f \in Z_{r,R}$. □

A repeated application of Lemma 3.3, leads naturally to a family of differential operators which we now introduce. These operators also appear in the work of Volchkov (p.108 of [13]) in the problems on averages over geodesic balls in real hyperbolic spaces. Let $C^1(0, 1)$ denote the space of all differentiable functions on (0, 1). For $m \in \mathbb{Z}$, the set of integers, define a differential operator A_m on $C^1(0, 1)$ by

$$(A_m f)(t) := \frac{t^m}{(1-t^2)^{m-1}} \frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\frac{1}{t} - t\right)^m f(t)\right]. \tag{3.8}$$

The Laplace–Beltrami operator $\mathcal{L}_x$ on $\mathbb{B}^n$ (p. 31 of [5]) is given by

$$\mathcal{L}_x = \frac{(1-|x|^2)^n}{4} \sum_i \frac{\partial}{\partial x_i}\left(\sum_i (1-|x|^2)^{2-n} \frac{\partial}{\partial x_i}\right).$$

The radial part $\mathcal{L}_s$ of $\mathcal{L}_x$ is given by

$$\mathcal{L}_s = \frac{\partial^2}{\partial s^2} + (n-1)\coth s\ \frac{\partial}{\partial s}$$

and satisfies the Darboux equation $M_s\mathcal{L}_x = \mathcal{L}_s M_s$ (p. 159 of [5]). For any positive integer k, define an operator $\mathcal{L}_k$ by

$$\mathcal{L}_k = \mathcal{L}_x - 4(k-1)(n+k-2)\mathrm{Id}. \tag{3.9}$$

Let $f(x) = a(\rho)Y_k(\omega)$, where Y_k is a spherical harmonic of degree k. Then, a simple calculation shows that

$$\mathcal{L}_k f(x) = (A_{k-1}A_{2-k-n}a)(\rho)Y_k(\omega), \quad x = \rho\omega.$$

Lemma 3.4. *Let* $x = \rho\omega$, $0 < \rho < 1$ *and* $\omega \in S^{n-1}$ *and* $k \geq 0$. *Suppose the function* $f(x) = a(\rho)Y_k(\omega) \in Z_{r,R}$. *Then*

(i) $(A_{2-k-n}a)(\rho)Y_{(k-1)j}(\omega) \in Z_{r,R}$, $k \geq 1$ *and* $1 \leq j \leq d_{k-1}(n)$,
(ii) $(A_k a)(\rho)Y_{(k+1)i}(\omega) \in Z_{r,R}$, $k \geq 0$ *and* $1 \leq i \leq d_{k+1}(n)$,
(iii) $(A_{1-k-n}A_k a)(\rho)Y_k(\omega) \in Z_{r,R}$, $k \geq 0$ *and*
(iv) $\mathcal{L}_k f(x) = (A_{k-1}A_{2-k-n}a)(\rho)Y_k(\omega) \in Z_{r,R}$, $k \geq 1$.

Proof. Let $k \geq 1$. Let $P(x) = \rho^k Y_k(\omega)$ and $\tilde{a}(\rho) = \rho^{-k}a(\rho)$. Then $f = \tilde{a}P$, where $P \in H_k$. By Lemma 3.3, the function $2\tilde{X}_p f \in Z_{r,R}$ $\forall\ 1 \leq p \leq n$. A straightforward calculation then gives

$$2\tilde{X}_p f = \left(\frac{(1-\rho^2)}{\rho}\frac{\partial\tilde{a}}{\partial\rho} - 2k\tilde{a}\right)x_p P + (1+\rho^2)\tilde{a}\frac{\partial P}{\partial x_p}. \tag{3.10}$$

Further,

$$x_p P = P_{k+1} + \frac{|x|^2}{n+2(k-1)}\frac{\partial P}{\partial x_p},$$

where $P_{k+1} \in H_{k+1}$ (for a proof, see [2]). Let $l = 2-k-n$, then (3.10) gives

$$2\tilde{X}_p f = \left(\frac{(1-\rho^2)}{\rho}\frac{\partial\tilde{a}}{\partial\rho} - 2k\tilde{a}\right)\left(P_{k+1} + \frac{\rho^2}{k-l}\frac{\partial P}{\partial x_p}\right) + (1+\rho^2)\tilde{a}\frac{\partial P}{\partial x_p}.$$

After a rearrangement of terms, we get

$$\begin{aligned} 2(k-l)\tilde{X}_p f &= (k-l)\left(\frac{(1-\rho^2)}{\rho}\frac{\partial\tilde{a}}{\partial\rho} - 2k\tilde{a}\right)P_{k+1} \\ &\quad + \left(\rho(1-\rho^2)\frac{\partial\tilde{a}}{\partial\rho} - 2k\rho^2\tilde{a} + (k-l)(1+\rho^2)\tilde{a}\right)\frac{\partial P}{\partial x_p}. \end{aligned}$$

Since $\tilde{a}(\rho) = \rho^{-k}\, a(\rho)$, $\dfrac{\partial\tilde{a}}{\partial\rho} = -k\rho^{-k-1}a + \rho^{-k}\dfrac{\partial a}{\partial\rho}$. Using this in the above equation, we have

$$\begin{aligned} 2(k-l)\tilde{X}_p f &= (k-l)\left((1-\rho^2)\frac{\partial a}{\partial\rho} - k\frac{(1+\rho^2)}{\rho}a\right)\rho^{-k-1}P_{k+1} \\ &\quad + \left((1-\rho^2)\frac{\partial a}{\partial\rho} - l\frac{(1+\rho^2)}{\rho}a\right)\rho^{-k}\frac{\partial P}{\partial x_p}. \end{aligned} \tag{3.11}$$

Also the operator A_m, given by (3.8), can be rewritten as

$$A_m = (1-t^2)\frac{\mathrm{d}}{\mathrm{d}t} - m\frac{(1+t^2)}{t}.$$

Thus (3.11) can be rephrased as

$$2(k-l)\tilde{X}_p f = (A_k a)(\rho)\rho^{-k-1}P_{k+1} + (A_{2-k-n}a)(\rho)\rho^{-k+1}\frac{\partial P}{\partial x_p} \in Z_{r,R},$$

whenever $1 \le p \le n$. Consequently, by Lemma 3.1, we get $(A_k a)(\rho)\rho^{-k-1}P_{k+1} \in Z_{r,R}$ and $(A_{2-k-n}a)(\rho)\rho^{-k+1}\frac{\partial P}{\partial x_p} \in Z_{r,R}$ and in particular, $(A_{2-k-n}a)(\rho)\ Y_{(k-1)j}(\omega)$ and $(A_k a)(\rho)Y_{(k+1)i}(\omega)$ belong to $Z_{r,R}$.

The assertions (iii) and (iv) can be obtained by composing (i) and (ii). □

4. Proof of the main result

In this section, we prove our main result Theorem 1.2. We first take up the necessary part of the theorem.

PROPOSITION 4.1

Let f be a radial function in $Z_{r,R}$. Then $f \equiv 0$ on Ann(r, R).

Proof. By hypothesis

$$\int_{S_s(x)} f(\rho)\mathrm{d}\mu_s(y) = 0,$$

whenever $x \in \mathbb{B}^n$ is such that the sphere $S_s(x) \subseteq \mathrm{Ann}(r, R)$ and ball $B_r(o) \subseteq B_s(x)$. Evaluating at $x = 0$, this implies

$$\int_{S_s(0)} f(|y|)\mathrm{d}\mu_s(y) = 0, \quad \text{whenever } R > s > r.$$

Thus $f(\tanh\frac{s}{2}) = 0,\ R > s > r$. □

PROPOSITION 4.2

Let $f(\rho\omega) = a(\rho)Y_k(\omega) \in Z_{r,R},\ k \ge 1$. Then $a(\rho)$ is given by

$$a(\rho) = \sum_{i=1}^{k} C_i\frac{(1-\rho^2)^{n+i-2}}{\rho^{n+k-2}},\ C_i \in \mathbb{C},\ \textit{whenever}\ \tanh\frac{r}{2} < \rho < \tanh\frac{R}{2}. \tag{4.12}$$

Proof. We use induction on k. For $k = 1$, let $f(\rho\omega) = a(\rho)Y_1(\omega) \in Z_{r,R}$. Using Lemma 3.4(ii), it follows that $(A_{1-n}a)(\rho)Y_0(\omega)$ belongs to $Z_{r,R}$. Therefore, by Proposition 4.1,

$(A_{1-n}a)(\rho) = 0$ on Ann(r, R). On solving this differential equation, we get $a(\rho) = C\left(\frac{1}{\rho} - \rho\right)^{n-1}$.

Next, we assume the result is true for k. Suppose $f(\rho\omega) = a(\rho)Y_{k+1}(\omega) \in Z_{r,R}$. An application of Lemma 3.4(ii) gives $A_{1-k-n}a(\rho)Y_k(\omega) \in Z_{r,R}$. Using the result for k and the definition of A_{1-k-n}, it follows that

$$\frac{\rho^{1-k-n}}{(1-\rho^2)^{-k-n}}\frac{\partial}{\partial\rho}\left(\left(\frac{1}{\rho}-\rho\right)^{1-k-n}a(\rho)\right) = \sum_{i=1}^{k} C_i\frac{(1-\rho^2)^{n+i-2}}{\rho^{n+k-2}}.$$

Simplifying this equation and integrating both the sides with respect to ρ, we obtain

$$\left(\frac{1}{\rho}-\rho\right)^{1-k-n}a(\rho) = \sum_{i=1}^{k} D_i\frac{1}{(1-\rho^2)^{k-i+2}} + D_{k+1}, \quad D_i \in \mathbb{C}.$$

Hence

$$a(\rho) = \sum_{i=1}^{k+1} D_i\frac{(1-\rho^2)^{n+i-2}}{\rho^{n+k-1}},$$

whenever $\tanh\frac{r}{2} < \rho < \tanh\frac{R}{2}$. □

Now, we shall prove the sufficient part of Theorem 1.2. For this, without loss of generality, we may assume that $R = \infty$. The idea of the proof is to use the asymptotic behavior of the hypergeometric function and compare it with that of the coefficients given in (4.12). In the proof, we need the following result from Erdelyi *et al* (pp. 75–76 of [3]).

Lemma 4.1. *The general solution of the hypergeometric differential equation*

$$z(1-z)U'' + \{\gamma - (\alpha+\beta+1)z\}U' - \alpha\beta U = 0, \tag{4.13}$$

where α, β, γ are independent of z, in the neighborhood of ∞ is given in the following way. If $\alpha - \beta$ is not an integer then

$$U(z) = \lambda_1 z^{-\alpha} + \lambda_2 z^{-\beta} + O\left(z^{-\alpha-1}\right) + O\left(z^{-\beta-1}\right),$$

where $\lambda_1(x)$ and $\lambda_2(x)$ are non zero. Otherwise, $z^{-\alpha}$ or $z^{-\beta}$ has to be multiplied by a factor of $\log z$.

Theorem 4.2. *Let $y = \rho\omega$, $\omega \in S^{n-1}$ and* $\tanh\frac{r}{2} < \rho < \infty$. *Let $h(y) = a(\rho)Y_k(\omega)$ with*

$$a(\rho) = \sum_{i=1}^{k} C_i\frac{(1-\rho^2)^{n+i-2}}{\rho^{n+k-2}}, \quad C_i \in \mathbb{C}.$$

Then the function $h \in Z_{r,\infty}$.

Proof. We use the induction hypothesis on k. For $k = 1$, consider the function $h(y) = \left(\frac{1}{\rho} - \rho\right)^{n-1} Y_1(\omega)$. In this case, a straightforward calculation gives $(A_0 A_{1-n}) \left(\frac{1}{\rho} - \rho\right)^{n-1} \equiv 0$. Thus, we can write

$$\mathcal{L}_1 h(y) = (A_0 A_{1-n}) \left(\frac{1}{\rho} - \rho\right)^{n-1} Y_1(\omega) = (A_0 A_{1-n}) h(y) = 0.$$

Hence, from eq. (3.9), it follows that $\mathcal{L}_y h(y) = 0$. Again by Darboux's equation $\mathcal{L}_s M_s h = M_s \mathcal{L}_y h$, the above leads to $\mathcal{L}_s(M_s h) = 0$. Define $F_1(s, x) = M_s h(x)$. For fixed x, F_1 as a function of s satisfies the differential equation

$$\frac{\partial^2 F_1}{\partial s^2} + (n-1) \coth s \, \frac{\partial F_1}{\partial s} = 0. \tag{4.14}$$

Setting $z = -\sinh^2 \frac{s}{2}$, we get

$$\frac{\partial F_1}{\partial s} = \frac{\partial F_1}{\partial z} \frac{\partial z}{\partial s} = -\frac{1}{2} \sinh s \frac{\partial F_1}{\partial z},$$

$$\frac{\partial^2 F_1}{\partial s^2} = \frac{1}{4} (\sinh s)^2 \frac{\partial^2 F_1}{\partial z^2} - \frac{1}{2} \cosh s \frac{\partial F_1}{\partial z}.$$

After substituting these values in (4.14), we obtain

$$-z(1-z) \frac{\partial^2 F_1}{\partial z^2} - \left(\frac{n}{2} - nz\right) \frac{\partial F_1}{\partial z} = 0. \tag{4.15}$$

Comparing this equation with (4.13), we get $\gamma = \frac{n}{2}$, $\alpha + \beta + 1 = n$, $\alpha\beta = 0$. For $\alpha = 0$, $\beta = n - 1$. The solution of (4.15) as $|z| \to \infty$ is given by

$$F_1(z, x) = \lambda_1(x) + \lambda_2(x) z^{-(n-1)} \log z + O(z^{-(n-1)-1} \log z), \tag{4.16}$$

where $\lambda_1(x)$ and $\lambda_2(x)$ are non zero. On the other hand, for $x = g \cdot o$, $g \in G$, we have

$$\begin{aligned} M_s h(x) &= \frac{1}{A(s)} \int_{S_s(x)} h(y) \mathrm{d}\mu_s(y), \\ &= \frac{1}{A(s)} \int_{S_s(o)} h(g \cdot y) \mathrm{d}\mu_s(y). \end{aligned}$$

From the above equation, it follows that

$$M_s h(x) = O\left(a\left(\tanh \frac{s}{2}\right)\right), \text{ as } s \to \infty. \tag{4.17}$$

From (4.17) one can also conclude that any function of type $h(y) = a(\rho) Y_k(\omega)$, must satisfy the relation $M_s h(x) = O\left(a\left(\tanh \frac{s}{2}\right)\right)$. In fact, for $k = 1$,

$$\left|a\left(\tanh \frac{s}{2}\right)\right| = \left|\cosh \frac{s}{2} \sinh \frac{s}{2}\right|^{-(n-1)} = |z(1-z)|^{-\frac{(n-1)}{2}}. \tag{4.18}$$

From (4.17) and (4.18), we have $F_1(z,x) = O(z^{-(n-1)})$, as $|z| \to \infty$. In view of (4.16), we infer that $F_1(z,x) = 0$, whenever $|z| > \sinh^2 r$. Thus $M_s h(x) = 0$, whenever $x \in \mathbb{B}^n$ is such that the ball $B_r(o) \subseteq B_s(x)$ and $r < s < \infty$, which proves the result for $k = 1$.

To complete the induction argument, we assume the result is true for $k-1$ and then prove for k. For this, consider the function

$$h(y) = a(\rho)Y_k(\omega) = \frac{(1-\rho^2)^{n+i-2}}{\rho^{n+k-2}} Y_k(\omega),$$

for each i, $1 \le i \le k$. Using Lemma 3.4(i) and the case $(k-1)$, it follows that

$$(A_{2-k-n}a)(\rho)Y_{k-1}(\omega) = \frac{(1-\rho^2)^{n+i-2}}{\rho^{n+k-3}} Y_{k-1}(\omega) \in Z_{r,\infty}.$$

Applying Lemma 3.4(ii), it follows that $\mathcal{L}_k h(y) = (A_{k-1}A_{2-k-n})a(\rho)Y_k(\omega)$ belongs to $Z_{r,\infty}$. Since we know that

$$\mathcal{L}_k h(y) = \mathcal{L}_y h(y) - 4(k-1)(n+k-2)h(y),$$

therefore, evaluating mean and using Darboux's equation, we obtain

$$\mathcal{L}_s(M_s h(x)) - 4(k-1)(n+k-2)M_s h(x) = 0,$$

whenever $x \in \mathbb{B}^n$ is such that the ball $B_r(o) \subseteq B_s(x)$ and $r < s < \infty$. Let $F_k(s,x) = M_s h(x)$. For fixed x, F_k as a function of s satisfies the differential equation

$$\frac{\partial^2 F_k}{\partial s^2} + (n-1)\coth s\, \frac{\partial F_k}{\partial s} - 4(k-1)(n+k-2)F_k = 0.$$

Using the change of variable $z = -\sinh^2 \frac{s}{2}$, the above equation becomes

$$-z(1-z)\frac{\partial^2 F_k}{\partial z^2} - \left(\frac{n}{2} - nz\right)\frac{\partial F_k}{\partial z} - 4(k-1)(n+k-2)F_k = 0. \tag{4.19}$$

Comparing this equation with (4.13), we have $\gamma = \frac{n}{2}, \alpha + \beta + 1 = n, \alpha\beta = -4(k-1)(n+k-2)$. On solving, we find that $\alpha = \frac{n-1+\nu}{2}$, $\beta = \frac{n-1-\nu}{2}$, where $\nu = \sqrt{(n-1)^2 + 16(k-1)(n+k-2)}$. If $\nu \notin \mathbb{Z}$, then the solution of (4.19) as $|z| \to \infty$ is given by

$$F_k(z,x) = \lambda_1(x)z^{-\alpha} + \lambda_2(x)z^{-\beta} + O(z^{-\alpha-1}) + O(z^{-\beta-1}), \tag{4.20}$$

where $\lambda_1(x)$ and $\lambda_2(x)$ are non zero, otherwise $z^{-\alpha}$ or $z^{-\beta}$ has to be multiplied by a factor of $\log z$. From the given expression of the function h, we find that

$$M_s h(x) = O\left(a\left(\tanh\frac{s}{2}\right)\right), \text{ as } s \to \infty.$$

Using $z = -\sinh^2 \frac{s}{2}$, it follows that

$$\left|a\left(\tanh\frac{s}{2}\right)\right| = \left|\frac{\left(\operatorname{sech}^2\frac{s}{2}\right)^{n+i-2}}{\left(\tanh\frac{s}{2}\right)^{n+k-2}}\right| = \frac{|1-z|^{\frac{n+k-2}{2}-(n+i-2)}}{|z|^{\frac{n+k-2}{2}}}.$$

That is,

$$F_k(z, x) = O(z^{-(n+i-2)}), \quad i = 1, \ldots, k \text{ as } |z| \to \infty, \tag{4.21}$$

which contradicts with the expression of $F_k(z, x)$ given by (4.20). Therefore, $F_k(z, x) = 0$, whenever $|z| > \sinh^2 r$. Hence, we conclude that $M_s h(x) = 0$, whenever $x \in \mathbb{B}^n$ is such that the ball $B_r(o) \subseteq B_s(x)$ and $r < s < \infty$, which proves the result for any positive integer k. This completes the proof. □

As a corollary of Theorem 1.2, we have the following Helgason support theorem (see p. 156 of [5]).

Theorem 4.3. *Let f be a function on $\mathbb{B}^n$. Suppose for each $m \in \mathbb{Z}_+$, the function $e^{md(x,\,0)} f(x)$ is bounded. Then f is supported in a closed geodesic ball $B_r(o)$ if and only if $f \in Z_{r,\infty}$.*

Proof. The decay condition on function f implies that for all k and j, $a_{kj}(|x|) = 0$, whenever $|x| > \tanh \frac{r}{2}$. This proves f is supported in the ball $B_r(o)$. □

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