Proc. Indian Acad. Sci. (Math. Sci.) Vol. 121, No. 3, August 2011, pp. 259–265.


# Deficiently extremal Gorenstein algebras

PAVINDER SINGH

Indian Institute of Science Education and Research, ITI Campus (Gas Rahat) Building, Govindpura, Bhopal 462 023, India
E-mail: pavinders@gmail.com

MS received 4 October 2010; revised 24 March 2011

**Abstract.** The aim of this article is to study the homological properties of deficiently extremal Gorenstein algebras. We prove that if $R/I$ is an odd deficiently extremal Gorenstein algebra with pure minimal free resolution, then the codimension of $R/I$ must be odd. As an application, the structure of pure minimal free resolution of a nearly extremal Gorenstein algebra is obtained.

**Keywords.** Extremal algebras; Stanley–Reisner rings; minimal resolutions; Betti numbers.

## 1. Introduction

Let $R = k[x_1, x_2, \dots, x_n]$ be the standard polynomial ring over a field $k$ and $I$ be a homogeneous ideal in $R$ of height $g$ and initial degree $p$. Then the Hilbert series $\mathbf{F}(R/I, t)$ of the algebra $R/I$ is of the form

$$\mathbf{F}(R/I, t) = \frac{\sum_{i=0}^{s} h_i t^i}{(1-t)^d},$$

where $d$ is the (Krull) dimension of $R/I$. The vector $(h_0, h_1, \dots, h_s)$ is known as the *h-vector* of $R/I$ and $s$ is its *length*. Suppose $R/I$ is a Gorenstein algebra, i.e. $\omega_S \simeq S[-l]$, where $S = R/I$ and $\omega_S$ is the canonical module of $S$. Then $s \geq 2(p-1)$ and $h_i = h_{s-i}$, i.e., $h$-vector is symmetric. The $k$-algebra $R/I$ is said to be *an extremal* (or *a p-extremal*) Gorenstein algebra if $s = 2(p-1)$. The extremal Gorenstein algebras have been first studied by Sally [7] for the case $p = 2$ and by Schenzel [8] for the general case. For the given codimension $g \geq 3$ and initial degree $p \geq 2$, a Gorenstein algebra $R/I$ with minimal multiplicity is extremal in the sense of Schenzel [8]. This has a nice structural implication: the minimal resolution of $R/I$ must be pure and almost linear, and so their Betti numbers are given by Herzog and Kühl [3] formulae. For a homogeneous Gorenstein ideal $I$ of initial degree $p \geq 2$ and height $g \geq 3$, Miller and Villarreal [5] has shown that

$$\nu(I_p) \leq \binom{p+g-1}{g-1} - \binom{p+g-3}{g-1},$$

and equality holds if and only if $R/I$ is an extremal Gorenstein algebra, where $\nu(I_p) =$ minimal number of generators of $I$ in degree $p$.

Let $\Delta$ be an *abstract simplicial complex* on the vertex set $V = [n]$, where $[n] = \{1, 2, \dots, n\}$, i.e., $\Delta$ is a collection of subsets of $V = [n]$ called *faces* such that each singleton $\{i\} \in \Delta$ and if $\sigma \in \Delta$ and $\tau \subseteq \sigma$, then $\tau \in \Delta$. To each simplicial complex $\Delta$ on $V = [n]$, one associates a (square-free) monomial ideal $I_\Delta$ in the standard polynomial ring $R = k[x_1, x_2, \dots, x_n]$ given by

$$I_\Delta = \langle x^\sigma : \sigma \notin \Delta \rangle,$$

where $x^\sigma = \prod_{i \in \sigma} x_i$. Note that $I_\Delta$ is minimally generated by $x^\sigma$, where $\sigma$ runs over minimal non-faces of $\Delta$. The monomial ideal $I_\Delta$ is called the *Stanley–Reisner ideal* associated to $\Delta$, and the quotient ring $R/I_\Delta = k[\Delta]$ is called the *Stanley–Reisner ring* of $\Delta$. The combinatorial properties of the simplicial complex $\Delta$ are intimately related to the algebraic properties of its Stanley–Reisner ring $k[\Delta]$. The Stanley–Reisner rings having 2-linear resolutions have been completely characterized by Fröberg [2]. This result has been extended to Stanley–Reisner rings $k[\Delta]$ having 2-pure but not 2-linear by Bruns and Hibi [1]. In fact, the Stanley–Reisner rings with 2-pure but not 2-linear resolutions are Gorenstein.

Consider the Gorenstein algebra $R/I$, where $I$ is a homogeneous ideal in the polynomial ring $R = k[x_1, x_2, \dots, x_n]$ of height $g$ and initial degree $p$. We say that the $k$-algebra $R/I$ is *c-deficiently extremal Gorenstein algebra* if $s = 2(p-1) + c$, where $s$ = length of the $h$-vector of $R/I$. If $c$ is odd, we say that $k$-algebra is an *odd deficiently extremal Gorenstein* algebra. One can see that, a $c$-deficiently extremal Gorenstein algebra is $c$-steps away from being an extremal Gorenstein algebra. Thus 0-deficient Gorenstein algebra is nothing but an extremal Gorenstein algebra whereas a 1-deficient Gorenstein algebra is a nearly extremal Gorenstein algebra introduced in Kumar *et al* [4]. We shall investigate the homological properties of $c$-deficiently extremal Gorenstein algebras vis-á-vis extremal or nearly extremal Gorenstein algebras.

## 2. Deficiently extremal Gorenstein algebras

For $c \geq 1$, we have the following characterization theorem for $c$-deficiently extremal Gorenstein algebras

**Theorem 2.1.** *Let $R = k[x_1, x_2, \dots, x_n]$ be a standard polynomial ring over a field $k$ and $I$ be a homogeneous Gorenstein ideal in $R$ of height $g$ and initial degree $p$. Then the following conditions are equivalent*:

(1) *$R/I$ is a $c$-deficiently extremal Gorenstein algebra.*
(2) *The minimal graded free resolution of $R/I$ is of the form*

$$0 \to F_g \to F_{g-1} \to \cdots \to F_2 \to F_1 \to R \to R/I \to 0,$$

*where $F_i = \bigoplus_{j=0}^{c} R[-(p+i+j-1)]^{b_{ij}}$ for $1 \leq i \leq g-1$ and $F_g = R[-(2p-2+g+c)]$, and $\beta_{i,p+i+j-1} = b_{ij}$ are graded Betti numbers.*
(3) *The Hilbert series of $R/I$ is of the form*

$$\mathbf{F}(R/I, t) = \frac{\sum_{i=0}^{p-2} \binom{i+g-1}{g-1}(t^i + t^{2p-2+c-i}) + \binom{g+p-2}{g-1} t^{p-1} + \sum_{j=0}^{c-1} h_{p+j} t^{p+j}}{(1-t)^d},$$

*where $d = \dim(R/I)$ and $0 < h_{p+j} < \binom{g+p+j-1}{g-1}$; for $j = 0, 1, \dots, c-1$.*

*Proof.* It can be assumed that $k$ is an infinite field. Since $R/I$ is a Gorenstein algebra, there is a regular system of parameters $\mathbf{y} = \{y_1, \ldots, y_d\}$ of $R/I$ such that each $y_i$ is of degree 1 in $R$ and $\bar{R} = R/\mathbf{y}R$ is a polynomial ring in $g$ variables such that $\bar{R}/\bar{I} = R/(\mathbf{y}, I)$ is Artinian. Thus going modulo $\mathbf{y}$, we may assume that $R$ is a standard polynomial ring in $g$-variables and $R/I$ is Artinian. Since $R/I$ is Gorenstein, the minimal free resolution of $R/I$ is of the form

$$0 \to R[-d_g] \to \bigoplus_{j=1}^{b_{g-1}} R[-d_{(g-1)j}] \to \cdots \to \bigoplus_{j=1}^{b_1} R[-d_{1j}]$$
$$\to R \to R/I \to 0,$$

with $d_{i1} \leq \ldots \leq d_{ib_i}$ and $\bigoplus_{j=1}^{b_{g-i}} R[-(d_g - d_{(g-i)j})] \simeq \bigoplus_{j=1}^{b_i} R[-d_{ij}]$ for $1 \leq i \leq g-1$ (see Theorem 4.3.11 of [9]). Also the Socle of $R/I$, $\mathrm{Soc}(R/I) \simeq k[-(d_g - g)]$ (see Lemma 4.3.3 of [9]). Let $R/I$ be a $c$-deficiently extremal Gorenstein algebra. Since $R/I$ is also Artinian, its Socle lies in the last non-zero graded component. Thus $d_g - g = s = 2p - 2 + c$ or $d_g = 2p + g - 2 + c$. From the minimality of the resolution and the duality isomorphisms $\bigoplus_{j=1}^{b_{g-i}} R[-(d_g - d_{(g-i)j})] \simeq \bigoplus_{j=1}^{b_i} R[-d_{ij}]$ for $1 \leq i \leq g-1$, we see that $d_{11} = p$, $d_{i1} \geq p + i - 1$ and $d_g - d_{(g-i)b_{(g-i)}} = d_{i1}$ for $1 \leq i \leq g-1$. Thus, we have

$$\begin{aligned} d_{(g-i)b_{(g-i)}} &= d_g - d_{i1} \leq 2p + g - 2 + c - (p + i - 1) \\ &= p + g - 1 + c - i \end{aligned}$$

for $i = 1, 2, \ldots, g-1$. Replacing $i$ by $g - i$ in the above inequality, we see that

$$d_{ib_i} \leq p + i + c - 1$$

for $i = 1, 2, \ldots, g-1$. Now, we have

$$p + i - 1 \leq d_{i1} \leq d_{i2} \leq \cdots \leq d_{ib_i} \leq p + i + c - 1$$

for $i = 1, 2, \ldots, g-1$. Therefore, $d_{ij}$ can take the values $p + i - 1, p + i, \ldots, p + i + c - 1$, and hence the minimal resolution of $R/I$ has the required form. This proves that (1) implies (2).

Now assume that the minimal resolution of $R/I$ is as in (2). Then the Socle of $R/I$ lives in degree $d_g - g = 2p - 2 + c$ as $d_g = 2p + g - 2 + c$. Since $R/I$ is an Artinian Gorenstein algebra, its Socle lies in the last non-zero graded component. Thus we have $s = 2p - 2 + c$, which shows that (2) implies (1).

Now we shall show that (1) implies (3). As $R$ is a polynomial ring in $g$-variables and $I$ is a Gorenstein ideal of initial degree $p$ with $R/I$ Artinian, we have $h_i = H(R/I, i) = \binom{g+i-1}{g-1} = h_{2p-2+c-i}$ for $i = 0, 1, \ldots, p-1$, and $0 < h_{p+j} < \binom{g+p+j-1}{g-1}$ for $j = 0, 1, \ldots, c-1$. Thus the Hilbert series of $R/I$ has the required form given in (3). Finally, (3) implies (1) is clear. □

We see that if $c = 0$, then $R/I$ becomes an extremal Gorenstein algebra. In this case $R/I$ has pure and almost $p$-linear resolution and its Hilbert series is given by

$$\mathbf{F}(R/I, t) = \frac{\sum_{i=0}^{p-2} \binom{i+g-1}{g-1}(t^i + t^{2p-2-i}) + \binom{g+p-2}{g-1} t^{p-1}}{(1-t)^d}, \quad d = \dim(R/I).$$

If $c = 1$, then $R/I$ is a nearly extremal Gorenstein algebra and its Hilbert series is given by

$$\mathbf{F}(R/I, t) = \frac{\sum_{i=0}^{p-1} \binom{i+g-1}{g-1}(t^i + t^{2p-1-i})}{(1-t)^d}, \quad d = \dim(R/I).$$

Since an extremal Gorenstein algebra has pure minimal free resolution, its Betti numbers are specified by Herzog and Kühl formulae. By manipulating the techniques of Herzog and Kühl, Kumar *et al* [4] obtained a numerical identity satisfied by the graded Betti numbers of a nearly extremal Gorenstein algebra, and in particualr $\nu(I_p) = \binom{p+g-1}{g-1} - \binom{p+g-2}{g-1}$. Again, we would like to remark that the techniques of Herzog and Kühl cannot be extended beyond the nearly extremal (or 1-deficient) case.

Before proceeding further, we shall consider the following example.

*Example* 2.2. Let $R = k[x, y]$, the standard polynomial ring over a field $k$ and $I = \langle x^2 - xy + y^2, x^3 \rangle$, the homogeneous ideal in $R$ of height $g = 2$ and initial degree $p = 2$. Then

$$R/I = k \oplus (kx \oplus ky) \oplus (kx^2 \oplus ky^2) \oplus kx^2y$$

is an Artinian algebra. Thus $R/I$ is Cohen–Macaulay with minimal graded free resolution

$$0 \to R[-5] \to R[-2] \oplus R[-3] \to R \to R/I \to 0.$$

From the minimal resolution of $R/I$, we have $\mathrm{Soc}(R/I) \simeq k[-3]$. Thus, $R/I$ is a Cohen–Macaulay algebra of Type 1, and hence $R/I$ is Gorenstein. In view of Theorem 2.1, $R/I$ is a nearly (or 1-deficient) extremal Gorenstein algebra.

We now shall describe a result of Bruns and Hibi [1] which characterizes the Stanley–Reisner rings having 2-pure but not 2-linear resolutions.

**Theorem 2.3.** *Let $\Delta$ be a simplicial complex on a vertex set $V = [n]$. Then the Stanley–Reisner ring $k[\Delta]$ has a 2-pure but not 2-linear resolution exactly in the following two cases*:

(1) *$\Delta$ is a multicone over a 1-dimensional cycle*, *or*
(2) *$\Delta$ is a multi-join of copies of two points.*

*In the first case*, *the minimal resolution of $k[\Delta]$ is of the form*

$$\begin{aligned} 0 &\to R[-(g+2)] \to R[-g]^{b_{g-1}} \to \cdots \to R[-(i+1)]^{b_i} \\ &\to \cdots \to R[-2]^{b_1} \to R \to k[\Delta] \to 0 \end{aligned}$$

*whereas, in the latter case, the minimal resolution of* $k[\Delta]$ *is the Koszul resolution of* $I_\Delta$ *of the form*

$$0 \to R[-2g] \to R[-2(g-1)]^{b_{g-1}} \to \cdots \to R[-2i]^{b_i} \\ \to \cdots \to R[-2]^{b_1} \to R \to k[\Delta] \to 0.$$

*Proof.* For a complete proof, see Theorem 2.1 of [1]. □

One can see that the Stanley–Reisner rings described in Theorem 2.3 are Gorenstein. Note that in the first case $k[\Delta]$ is a Gorenstein ring with deficiency $c = 0$, i.e., $k[\Delta]$ is an extremal Gorenstein ring with $h$-vector $(1, g, 1)$, whereas in the second case, $k[\Delta]$ is a Gorenstein ring with deficiency $c = g - 2$, $s = g$ and the $h$-vector $(h_0, h_1, \ldots, h_g)$ is given by $h_i = \binom{g}{i}$, where $i = 0, 1, \ldots, g$ and $\sum_{i=0}^{g}(-1)^i h_i = \sum_{i=0}^{g}(-1)^i\binom{g}{i} = (1+(-1))^g = 0$. In particular, $(-1)^{\frac{g}{2}}\mathbf{F}(R/I, -1) = 0$, $g$ even, which gives the Charney–Davis conjecture [6] in case $k[\Delta]$ has pure minimal free resolution.

Now we shall prove that if an odd deficiently extremal Gorenstein algebra has pure minimal free resolution, then its codimension must be odd. In particular, we obtain the structure of pure minimal free resolution of a nearly extremal Gorenstein algebra.

**Theorem 2.4.** *Let* $I$ *be a homogeneous Gorenstein ideal of height* $g$ *and initial degree* $p$ *in* $R = k[x_1, \ldots, x_n]$ *such that* $R/I$ *is a* $c$*-deficiently extremal Gorenstein algebra with* $c = 2r + 1$, *i.e.,* $c$ *is odd. If* $R/I$ *has pure minimal free resolution, then* $g = 2m + 1$, *for some* $m$, *i.e.,* $g$ *is odd and* $(m + 1)$*-th syzygy module in the minimal free resolution has no linear relations.*

*Proof.* As $R/I$ is a $c$-deficiently extremal Gorenstein algebra, by Theorem 2.1 the pure minimal free resolution of $R/I$ is of the form

$$0 \to R[-d_g] \to R[-d_{g-1}]^{\beta_{g-1}} \\ \to \cdots \to R[-d_2]^{\beta_2} \to R[-d_1]^{\beta_1} \to R \to R/I \to 0,$$

where $p = d_1 < d_2 < \cdots < d_g = 2p+g-2+c$ and $p+i-1 \leq d_i \leq p+i-1+c$. Now from the duality isomorphisms $R[-(d_g - d_{g-i})]^{\beta_{g-i}} \simeq R[-d_i]^{\beta_i}$, for $i = 1, 2, \ldots, g-1$ (see Theorem 4.3.11 of [9]), we see that $d_g - d_{g-1} = d_1$, which implies that $d_{g-1} = d_g - d_1 = p + g - 2 + c$. Now we have

$$p = d_1 < d_2 < \cdots < d_{g-1} = p + g - 2 + c.$$

Since $c = 2r+1 > 0$, we see from the above chain of inequalities that there exist integers $l_1, l_2, \ldots, l_{q+1} \geq 1$, $(q > 1)$, such that

$$\sum_{j=1}^{q+1} l_j = g - 1, \quad d_i = p + i - 1 \text{ for } 1 \leq i \leq l_1,$$

$$d_{(\sum_{j=1}^{\lambda} l_j)+i} = d_{(\sum_{j=1}^{\lambda} l_j)+1} + i - 1 \text{ for } 1 \leq i \leq l_{\lambda+1},$$

$$d_{(\sum_{j=1}^{\lambda} l_j)} + 1 < d_{(\sum_{j=1}^{\lambda} l_j)+1} \text{ for } \lambda = 1, 2, \ldots, q, \quad \text{and}$$
$$d_{(\sum_{j=1}^{q+1} l_j)} = d_{g-1} = p + g - 2 + c.$$

For $j = 1, 2, \ldots, q$, define $\gamma_j = d_{(\sum_{i=1}^{j} l_i)+1} - d_{(\sum_{i=1}^{j} l_i)}$. Then $\gamma_j \geq 2$ and $d_{(\sum_{i=1}^{j} l_i)+1} = \gamma_j + d_{(\sum_{i=1}^{j} l_i)}$, and so from this and the above equalities, we see that

$$d_{\sum_{j=1}^{\lambda} l_j} = p + \sum_{j=1}^{\lambda}(l_j - 1) + \sum_{j=1}^{\lambda-1} \gamma_j \quad \text{and}$$
$$d_{\sum_{j=1}^{\lambda} l_j + i} = p + \sum_{j=1}^{\lambda}(l_j - 1) + \sum_{j=1}^{\lambda} \gamma_j + (i - 1)$$

for $\lambda = 1, 2, \ldots, q$ and $i = 1, 2, \ldots, l_{\lambda+1}$. Now we have

$$p + g - 2 + c = d_{g-1} = d_{\sum_{j=1}^{q+1} l_j} = p + \sum_{j=1}^{q+1}(l_j - 1) + \sum_{j=1}^{q} \gamma_j$$
$$= p + \sum_{j=1}^{q+1} l_j - (q + 1) + \sum_{j=1}^{q} \gamma_j.$$

This implies that $\sum_{j=1}^{q}(\gamma_j - 1) = c$.

From the minimality of the resolution and the duality isomorphisms $R[-(d_g - d_i)]^{\beta_{g-i}} \simeq R[-d_i]^{\beta_i}$ for $i = 1, 2, \ldots, g - 1$ (see Theorem 4.3.11 of [9]), we see that $\gamma_j = \gamma_{q-j}$ for $j = 1, 2, \ldots, q - 1$, and $l_j = l_{q+1-j}$ for $j = 1, 2, \ldots, q$. Since $\gamma_j - 1 = \gamma_{q-j} - 1$ for $j = 1, 2, \ldots, q - 1$ and $\sum_{j=1}^{q}(\gamma_j - 1) = c = 2r + 1$ is odd, we have $q = 2\mu + 1$, i.e., $q$ is odd and $\gamma_{\mu+1} - 1$ is odd. Again from the duality isomorphisms, we see that $d_g - d_{\sum_{j=1}^{\mu+1} l_j} = d_{\sum_{j=1}^{\mu+1} l_j + 1}$. Thus we have

$$2p + g - 2 + c - \left(p + \sum_{j=1}^{\mu+1}(l_j - 1) + \sum_{j=1}^{\mu} \gamma_j\right) = p + \sum_{j=1}^{\mu+1}(l_j - 1) + \sum_{j=1}^{\mu+1} \gamma_j,$$

which gives

$$g = 2\left(\sum_{j=1}^{\mu+1}(l_j - 1)\right) + 2\left(\sum_{j=1}^{\mu} \gamma_j\right) + \gamma_{\mu+1} + c + 2.$$

Since $c$ is odd and $\gamma_{\mu+1}$ is even, we see that $g = 2m + 1$, for some $m$, i.e., $g$ is odd. Also, since $\sum_{j=1}^{q+1} l_j = g - 1$, $q = 2\mu + 1$ and $l_j = l_{q+1-j}$, for $j = 1, 2, \ldots, q$, we see that $\sum_{j=1}^{\mu+1} l_j = \frac{g-1}{2} = m$. Thus $(m + 1)$-th syzygy module has no linear relations. □

COROLLARY 2.5

*With notations as in Theorem* 2.4, *let* $R/I$ *be a nearly extremal Gorenstein algebra having pure minimal free resolution. Then* $g = 2m+1$, *for some* $m$, *i.e.*, $g$ *is odd and the pure minimal free resolution of* $R/I$ *is of the form*

$$\begin{aligned} 0 &\to R[-d_g] \to R[-d_{g-1}]^{\beta_{g-1}} \\ &\to \cdots \to R[-d_{m+2}]^{\beta_{m+2}} \to R[-d_{m+1}]^{\beta_{m+1}} \\ &\to R[-d_m]^{\beta_m} \to R[-d_{m-1}]^{\beta_{m-1}} \\ &\to \cdots \to R[-d_1]^{\beta_1} \to R \to R/I \to 0, \end{aligned}$$

*where* $d_i = p+i-1$ *and* $d_{m+i} = p+m+i$; *for* $i = 1, 2, \ldots, m$ *and* $d_g = 2p+g-1$.

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