Proc. Indian Acad. Sci. (Math. Sci.) Vol. 121, No. 3, August 2011, pp. 249–257.

A classification of cubic symmetric graphs of order $16p^2$

MEHDI ALAEIYAN[1], B N ONAGH[2] and M K HOSSEINIPOOR[1]

[1]Department of Mathematics, Iran University of Science and Technology, Narmak, Tehran 16844, Iran
[2]Department of Mathematics, Golestan University, Gorgan, Iran
E-mail: alaeiyan@iust.ac.ir

MS received 1 June 2010; revised 12 March 2011

Abstract. A graph is called *symmetric* if its automorphism group acts transitively on its arc set. In this paper, we classify all connected cubic symmetric graphs of order $16p^2$ for each prime p.

Keywords. Regular coverings; symmetric graphs; invariant subspaces.

1. Introduction

Throughout this paper, graphs are assumed to be finite, simple, undirected and connected. For the group-theoretic concepts and notations not defined here we refer to [18].

For a graph X, we denote by $V(X)$, $E(X)$, $A(X)$ and Aut(X) the *vertex set*, the *edge set*, the *arc set* and *the full automorphism group* of X, respectively. For $u, v \in V(X)$, denote by $\{u, v\}$ the edge incident to u and v in X.

If a subgroup G of Aut(X) acts transitively on $V(X)$, $E(X)$ and $A(X)$, we say that X is G*-vertex-transitive*, G*-edge-transitive* and G*-arc-transitive*, respectively. In the special case, when $G = \mathrm{Aut}(X)$ we say that X is vertex-transitive, edge-transitive and arc-transitive (or *symmetric*), respectively.

Let G be a finite group and S a subset of G such that $1 \notin S$ and $S = S^{-1}$. The *Cayley graph* $X = \mathrm{Cay}(G, S)$ on G with respect to S is defined to have vertex set $V(X) = G$ and edge set $E(X) = \{\{g, sg\} | g \in G, s \in S\}$. Clearly, X is connected if and only if S generates G. The automorphism group Aut(X) of X contains the right regular representation G_{R} of G, the acting group of G by right multiplication, as a subgroup, and G_{R} is regular on $V(X)$, that is, G_{R} is transitive on $V(X)$ with trivial vertex stabilizers. A graph X is isomorphic to a Cayley graph on a group G if and only if its automorphism group Aut(X) has a subgroup isomorphic to G, acting regularly on the vertex set (see Lemma 4 of [17]). A Cayley graph Cay(G, S) is said to be *normal* if G_{R} is normal in Aut(Cay(G, S)).

An s-arc in a graph X is an ordered $(s+1)$-tuple $(v_0, v_1, \ldots, v_{s-1}, v_s)$ of vertices of X such that v_{i-1} is adjacent to v_i for $1 \leq i \leq s$ and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s$. A graph X is said to be s*-arc-transitive* if Aut(X) is transitive on the set of s-arcs in X. A graph X is said to be s*-regular* if Aut(X) acts regularly on the set of s-arcs in X. Tutte [19] showed that every finite connected cubic symmetric graph is s-regular for some s, $1 \leq s \leq 5$. A subgroup of Aut(X) is said to be s-regular if it acts regularly on the set of s-arcs in X. The classification of cubic symmetric graphs of different orders is given in many papers. So far, cubic symmetric graphs of orders $2p$, $2p^2$, $2p^3$, $2p^n$, $4p$, $4p^2$, $6p$, $6p^2$, $8p$, $8p^2$, $10p$, $10p^2$, $14p$ and $16p$ have been classified [4–8,15,16].

In this paper, we classify all cubic symmetric graphs of order $16p^2$ for each prime p.

2. Preliminaries

Let X be a graph and N be a subgroup of Aut(X). The *quotient graph* X/N or X_N of X induced by N is the graph defined as follows:

(i) the set of vertices Σ is the set of orbits under the action N on $V(X)$;

(ii) Let $A, B \in \Sigma$ be two vertices of X_N. $\{A, B\}$ is an edge of X_N if and only if there are $u \in A$ and $v \in B$ such that $\{u, v\} \in E(X)$.

A graph $\tilde{X}$ is called a *covering* of a graph X with covering projection $\wp : \tilde{X} \to X$ if there is a surjection $\wp : V(\tilde{X}) \to V(X)$ such that $\wp|_{N_{\tilde{X}}(\tilde{v})} : N_{\tilde{X}}(\tilde{v}) \to N_X(v)$ is a bijection for any vertex $v \in V(X)$ and $\tilde{v} \in \wp^{-1}(v)$. A covering $\tilde{X}$ of X with a covering projection $\wp$ is said to be *regular* (or *K-covering*) if there is a semiregular subgroup K of the automorphism group Aut$(\tilde{X})$ such that graph X is isomorphic to the quotient graph $\tilde{X}/K$, say by h, and the quotient map $\tilde{X} \to \tilde{X}/K$ is the composition $h \circ \wp$ of h and $\wp$; to emphasize this we sometimes write $\wp_N$ instead of just $\wp$. If $\tilde{X}$ is connected K becomes the covering transformation group. The fibre of an edge or a vertex is its preimage under $\wp$. An automorphism of $\tilde{X}$ is said to be *fibre-preserving* if it maps a fibre to a fibre, while every covering transformation maps a fibre onto itself. All of fibre-preserving automorphisms form a group called the *fibre-preserving group*.

Let $\tilde{X}$ be a K-covering of X with a covering projection $\wp$. If $\alpha \in$ Aut(X) and $\tilde{\alpha} \in$ Aut$(\tilde{X})$ satisfy $\wp \circ \tilde{\alpha} = \alpha \circ \wp$, we call $\tilde{\alpha}$ a *lift* of α, and α the *projection* of $\tilde{\alpha}$. In this case we also say that $\wp$ is *α-admissible*. A subgroup $G \leq$ Aut(X) lifts along $\wp$ if each $\alpha \in G$ lifts. The set of all lifts G forms a group $\tilde{G} \leq$ Aut$(\tilde{X})$, called the lift of G. A regular covering projection $\wp$ is called *arc-transitive* if some subgroup $G \leq$ Aut(X) lifts along $\wp$, where G is an arc-transitive subgroup (that is, G acts transitively on the arc set of X).

Two coverings $\tilde{X}_1$ and $\tilde{X}_2$ of X with projections $\wp_1$ and $\wp_2$ respectively, are said to be *isomorphic* if there exists an automorphism $\alpha \in$ Aut(X) and a graph isomorphism $\tilde{\alpha} : \tilde{X}_1 \to \tilde{X}_2$ such that $\tilde{\alpha}\wp_2 = \wp_1\alpha$. In particular, if α is the identity automorphism of X, then we say that $\wp_1$ and $\wp_2$ are *equivalent*.

Let X be a connected graph and K be a finite group, called the voltage group. Assign to each arc (u, v) of X a voltage $\xi(u, v) \in A(X)$ such that $\xi(v, u) = \xi(u, v)^{-1}$. This function ξ is called an (ordinary) voltage assignment of X. Let Cov$(X; \xi) = X \times_\xi K$ be the derived graph with vertex set $V(X) \times K$ and adjacency relation defined by $(u, g) \sim (v, g\xi(u, v))$ whenever $u \sim v$ in X. Then the first coordinate $\wp_\xi : X \times_\xi K \to X$ is regular. Giving a spanning tree T of the graph X, a voltage assignment ξ is said to be *T-reduced* if the voltages on the tree arcs are the identity. Gross and Tucker [11] showed that every regular covering $\tilde{X}$ of a graph X can be derived from a T-reduced voltage assignment ξ

with respect to an arbitrary fixed spanning tree T of X. It is clear that if ξ is reduced, the derived graph $X \times_\xi K$ is connected if and only if the voltages on the cotree arcs generate the voltage group K.

Assume that a connected graph X and a subgroup $G \leq \mathrm{Aut}(X)$ are given. Choose a spanning tree T of X and a set of arcs $\{x_1, \ldots, x_r\} \subseteq A(X)$ containing exactly one arc from each edge in $E(X \backslash T)$. Let $\mathcal{B}_T$ be the corresponding basis of the first homology group $H_1(X, \mathbb{Z}_p)$ determined by $\{x_1, \ldots, x_r\}$. Further, denote by $G^{\#h} = \{\alpha^{\#h} \mid \alpha \in G\} \leq GL(H_1(X, \mathbb{Z}_p))$ the induced action of G on $H_1(X, \mathbb{Z}_p)$, and let $M_G \leq \mathbb{Z}_p^{r\times r}$ be the matrix representation of $G^{\#h}$ with respect to the basis $\mathcal{B}_T$. By M_G^t we denote the dual group consisting all transposes of matrices in M_G.

The following proposition is a special case of Proposition 6.3, Corollary 6.5 of [12]:

PROPOSITION 2.1

Let T be a spanning tree of a connected graph X and let the set $\{x_1, \ldots, x_r\} \subseteq A(X)$ contain exactly one arc from each cotree edge. Let $\xi : A(X) \to \mathbb{Z}_p^2$ be a voltage assignment on X which is trivial on T, and let $Z(\xi) = [\xi(x_1), \ldots, \xi(x_r)]^t \in \mathbb{Z}_p^{r\times 2}$. Then the following hold.

(a) *A group $G \leq \mathrm{Aut}(X)$ lifts along $\wp_\xi : \mathrm{Cov}(X; \xi) \to X$ if and only if the induced subspace $Z(\xi)$ is an M_G^t-invariant 2-dimensional subspace.*
(b) *If $\xi' : A(X) \to \mathbb{Z}_p^2$ is another voltage assignment satisfying* (a), *then* $\mathrm{Cov}(X; \xi')$ *is equivalent to* $\mathrm{Cov}(X; \xi)$ *if and only if $\langle Z(\xi)\rangle = \langle Z(\xi')\rangle$, as subspaces. Moreover,* $\mathrm{Cov}(X; \xi')$ *is isomorphic to* $\mathrm{Cov}(X; \xi)$ *if and only if there exists an automorphism* $\alpha \in \mathrm{Aut}(X)$ *such that M_α^t maps $Z(\xi')$ to $Z(\xi)$.*

The following proposition is a straightforward result of Theorem 1.1 of [9].

PROPOSITION 2.2

There is no connected $\mathbb{Z}_{2p^2}$-covering of the three dimensional hypercube Q_3, whose fibre-preserving group is arc-transitive.

PROPOSITION 2.3 (Theorem 9 of [12])

Let X be a connected symmetric graph of prime valency and G an s-regular subgroup of $\mathrm{Aut}(X)$ *for some $s \geq 1$. If a normal subgroup N of G has more than two orbits, then it is semiregular and G/N is an s-regular subgroup of* $\mathrm{Aut}(X_N)$, *where X_N is the quotient graph of X induced by N. Furthermore, X is an N-regular covering of X_N.*

Let $X = \mathrm{Cay}(G, S)$ be a Cayley graph on a group G with respect to a subset S of G. Set $A = \mathrm{Aut}(X)$ and $\mathrm{Aut}(G, S) = \{\alpha \in \mathrm{Aut}(G) \mid S^\alpha = S\}$. Recall that X is called normal when $G_R \lhd A$.

PROPOSITION 2.4 (Proposition 1.5 of [21])

The Cayley graph X is normal if and only if $A_1 = \mathrm{Aut}(G, S)$, where A_1 is the stabilizer of the vertex $1 \in V(X) = G$ in A.

3. Main results

We identify the vertex set of the generalized Petersen graph GP(8, 3) with $V = \{1, 2, \ldots, 16\}$ and the edge set with the union of *outer edges* $E_1 = \{\{i, 1 + i \pmod 8\} \mid i \in \{1, \ldots, 8\}\}$, the *inner edges* $E_2 = \{\{8 + i, 9 + ((i + 2) \bmod 8)\} \mid i \in \{1, \ldots, 8\}\}$, and the *spokes* $E_3 = \{\{i, i + 8\} \mid i \in \{1, \ldots, 8\}\}$, see figure 1.

The automorphism group of the generalized Petersen graph GP(8, 3) has order 96 and $\mathrm{Aut(GP}(8, 3)) = \langle \rho, \omega, \sigma \rangle$, where

$$\begin{aligned} \rho &= (1, 2, 3, 4, 5, 6, 7, 8)(9, 10, 11, 12, 13, 14, 15, 16), \\ \omega &= (2, 8, 9)(3, 16, 14)(4, 13, 6)(7, 12, 10), \\ \sigma &= (1, 14, 7, 12, 5, 10, 3, 16)(2, 11, 8, 9, 15, 4, 13). \end{aligned}$$

Note that Aut(GP(8, 3)) has two proper arc-transitive subgroups, namely $K_1 = \langle H, \tau \rangle$ and $K_2 = \langle H, \eta \rangle$, where $H = \langle \rho^2, \omega \rangle$, $\tau = (1, 9)(2, 14)(3, 11)(4, 16)(5, 13)(6, 10)(7, 15)(8, 12)$ and $\eta = (1, 9)(2, 12)(3, 15)(4, 10)(5, 13)(6, 16)(7, 11)(8, 14)$.

Let T be a spanning tree of GP(8, 3) containing all spokes and inner edges except for the edge $\{11, 16\}$ (see figure 2). Let x denote the arc (16, 11) and let x_i denote the arc $(i, 1 + i \pmod 8)$, $i \in \{1, \ldots, 8\}$. Let ξ be a K-voltage assignment defined by $\xi = 0$ on T and $\xi(x), \xi(x_1), \ldots, \xi(x_8)$ on the cotree arcs $x, x_1, \ldots, x_8$, respectively, where 0 is the identity element of K and $\xi(x), \xi(x_i) \in K$ $(1 \le i \le 8)$.

By [1, 2] we have the following.

Lemma 3.1. *Let X be a cubic symmetric graph of order* $16p^2$, *where p is a prime. If* $p \le 11$, *then X is isomorphic to one of the graphs in table* 1.

We have the following lemma when $p \ge 13$.

Lemma 3.2. *Suppose that X is a cubic symmetric graph of order* $16p^2$, *where* $p \ge 13$ *is a prime. Set* $A := \mathrm{Aut}(X)$. *Moreover suppose that* $Q := O_p(A)$ *is the maximal normal p-subgroup of A. Then* $|Q| = p^2$ *and moreover, X is a Q-regular covering of the generalized Petersen graph* GP(8, 3).

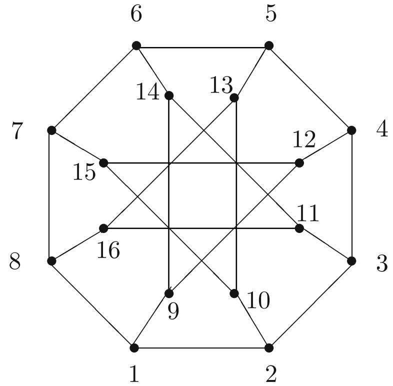

Figure 1. GP(8,3).

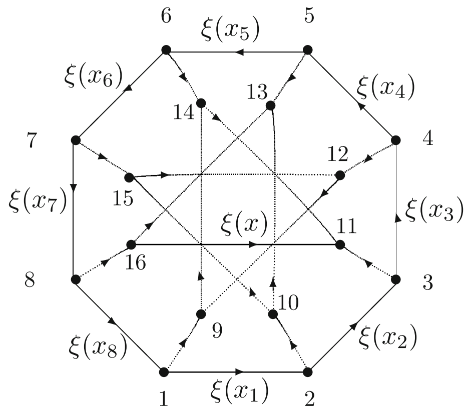

Figure 2. A spanning tree and voltage assignment on GP(8,3).

Proof. Let X be a cubic symmetric graph of order $16p^2$, where $p \geq 13$ is a prime. Since X is symmetric, by [19], X is at most 5-regular. Thus $|A| = 2^s \cdot 3 \cdot p^2$, for some $4 \leq s \leq 8$. Also A is solvable. Namely if it was not then, by the classification of $\{2, 3, p\}$-simple groups, its composition factors would have to be A_5 or $PSL(2, 7)$, a contradiction to $p \geq 13$ (see pp. 12–14 of [10]). Let $Q = O_p(A)$ be the maximal normal p-subgroup of A. Now, we intend to prove that $|Q| = p^2$.

We first suppose that $|Q| = 1$. Let N be a minimal normal subgroup of A. Since A is solvable, N is an elementary abelian 2-group, 3-group or p-group. Since $|Q| = 1$, N can not be an elementary abelian p-group. Also, N can not be an elementary abelian 3-group because otherwise $N \leq A_v$, where $v \in V(X)$ and N is not semiregular, which contradicts Proposition 2.3. Thus, N is an elementary abelian 2-group. It is easy to check that N has more than two orbits on $V(X)$ and then, by Proposition 2.3, it is semiregular. Therefore, $|N| = 2, 4, 8$ or 16. If $|N| = 16$, then the number of vertices of the quotient graph X_N is an odd number p^2, a contradiction. So, $|N| = 2, 4$ or 8. In each case, with repeated use of Proposition 2.3, we get a contradiction. Since the details of the proofs are similar, we investigate only the case when $|N| = 2$.

Table 1. Cubic symmetric graphs of order $16p^2$ with $p \leq 11$.

Graph	Order	s-Regular	Girth	Diameter	Bipartite?
F_{64}	16.2^2	1	8	6	Yes
F_{144A}	16.3^2	1	8	7	Yes
F_{144B}	16.3^2	2	10	8	Yes
F_{400A}	16.5^2	1	8	10	Yes
F_{400B}	16.5^2	2	10	13	Yes
F_{784A}	16.7^2	1	10	17	Yes
F_{784B}	16.7^2	2	10	19	Yes
F_{1936A}	16.11^2	1	8	15	Yes
F_{1936B}	16.11^2	2	10	29	Yes

Now suppose that $|N| = 2$. By Proposition 2.3, A/N is an s-regular subgroup of $\mathrm{Aut}(X_N)$. Let M/N be a minimal normal subgroup of A/N. Clearly, A/N is solvable and so M/N is elementary abelian. Again, by Proposition 2.3, M/N is an elementary abelian p-group or 2-group. Suppose that M/N is an elementary abelian p-group. Let P be a Sylow p-subgroup of M. Since $|M| = 2p$ or $2p^2$ and $p \geq 13$, P is normal and characteristic in M. Therefore, A has a normal subgroup of order p or p^2, a contradiction to $|Q| = 1$. Thus M/N must be an elementary abelian 2-group. By Proposition 2.3, M/N is semiregular and then $|M/N| = 2$ or 4. In this step, we investigate only $|M/N| = 2$. The proof for $|M/N| = 4$ is similar. Now suppose that $|M/N| = 2$. By Proposition 2.3, A/M is an s-regular subgroup of $\mathrm{Aut}(X_M)$. Let L/M be a minimal normal subgroup of A/M. As earlier, L/M can not be an elementary abelian 3-group or p-group. Therefore, by Proposition 2.3 $|L/M| = 2$. Let K/L be a minimal normal subgroup of A/L. So K/L is solvable and $|K/L| = 2$. It follows that the quotient graph X_L has odd number of vertices and valency 3, which is impossible. Therefore, $|Q| \neq 1$.

Finally, if $|Q| = p$, then Q has more than two orbits and then, by Proposition 2.3, A/Q is an s-regular subgroup of $\mathrm{Aut}(X_Q)$. Let N/Q be a minimal normal subgroup of A/Q. One can see that N/Q must be an elementary abelian 2-group. In this case by the same argument as in the preceding paragraph a similar contradiction is obtained. Therefore, $|Q| = p^2$. By Proposition 2.3, X_Q is a cubic symmetric graph of order 16. Then X_Q must be isomorphic to the generalized Petersen graph GP(8, 3). Indeed, X is a Q-regular covering of the generalized Petersen graph GP(8, 3), where $|Q| = p^2$. □

Now we want to determine all arc-transitive $\mathbb{Z}_p^2$-covering projections of GP(8, 3) where $p \geq 13$ is a prime . For this purpose, it suffices to find those which are K_1- or K_2-admissible. By Proposition 2.1, this is equivalent to finding all invariant 2-dimensional subspaces of the representations $M_{K_1}^t$ or $M_{K_2}^t$. Now a precise study [13] reveals that there are exactly six types of such subspaces which are either $M_{K_1}^t$- or/and $M_{K_2}^t$-invariant, namely:

1. There is one $M_{K_1}^t$- and $M_{K_2}^t$-invariant 2-dimensional subspace where $p \equiv 5$ or $-1 \pmod{12}$, namely $U_2 = \langle v_2, v_3 \rangle$ where $v_2 = (-4, 1, -3, 1, 1, -3, 1, 1, -3)^t$ and $v_3 = (0, 1, -1, 1, -1, 1, -1, 1, -1)^t$. Note that the subspace U_2 is indeed $M_{\langle H,\tau,\eta\rangle}^t$-invariant, where $\langle H, \tau, \eta\rangle$ is the subgroup of Aut(GP(8, 3)) of order 96.
2. There are two $M_{K_1}^t$-invariant 2-dimensional subspaces where $p \equiv 1 \pmod 4$, namely $U(-1)$ and $U(i)$ with the bases $\{(0, -1, 1, -i, i, 1, -1, i, -i)^t, (0, 1, i, -i, 1, -1, -i, i, -1)^t\}$ and $\{(0, i, 1, -1, i, -i, -1, 1, -i)^t, (0, i, -i, 1, -1, -1, -i, i, -1, 1)^t\}$ respectively, where $i^2 = -1$ in $\mathbb{Z}_p$.
3. There are three $M_{K_2}^t$-invariant 2-dimensional subspaces.

   - Two $M_{K_2}^t$-invariant 2-dimensional subspaces where $p \equiv 1 \pmod 8$, namely $U(v)$ and $U(-v)$ with the bases $\{(0, v, 1, iv, i, -v, -1, -iv, -i)^t, (0, 1+(1+v)i, -1-v-iv, 1+v-i, -v+(1+v)i, -1-(1+v)i, 1+v+iv, -1-v+i, v-(1+v)i)^t\}$ and $\{(0, -v, 1, -iv, i, v, -1, iv, -i)^t, (0, 1+(1-v)i, -1+v+iv, 1-v-i, v+(1-v)i, -1-(1-v)i, 1-v-iv, -1+v+i, -v-(1-v)i)^t\}$ respectively, where $i^2 = -1$ and $v^2 = -i$ in $\mathbb{Z}_p$.
   - One $M_{K_2}^t$-invariant 2-dimensional subspace where $p \equiv 3 \pmod 8$, namely $U_{-1,\sqrt{-2}} = \langle (0, 0, -1, \sqrt{-2}, 1, 0, 1, -\sqrt{-2}, -1)^t, (0, \sqrt{-2}, 1, 0, 1, -\sqrt{-2}, -1, 0, -1)^t \rangle$.

We denote the corresponding voltage assignments of the subspaces $U_2, U(-1)$, $U(i), U(v)$, $U(-v)$ and $U_{-1,\sqrt{-2}}$ with ξ_i $(i = 1, \ldots, 6)$ respectively. Let T be the spanning tree of GP(8, 3) given in the beginning of this section. We denote with $\tilde{X}_i$ $(i = 1, \ldots, 6)$ the graph derived from a T-reduced voltage assignment ξ_i with respect to the spanning tree T. Now, by Proposition 2.1, the corresponding voltage assignments ξ_i are as follows:

The corresponding voltage assignments ξ_1:

$\xi_1(x)$	$\xi_1(x_1)$	$\xi_1(x_2)$	$\xi_1(x_3)$	$\xi_1(x_4)$	$\xi_1(x_5)$	$\xi_1(x_6)$	$\xi_1(x_7)$	$\xi_1(x_8)$
$\binom{-4}{0}$	$\binom{1}{1}$	$\binom{-3}{-1}$	$\binom{1}{1}$	$\binom{1}{-1}$	$\binom{-3}{1}$	$\binom{1}{-1}$	$\binom{1}{1}$	$\binom{-3}{-1}$

The corresponding voltage assignments ξ_2:

$\xi_2(x)$	$\xi_2(x_1)$	$\xi_2(x_2)$	$\xi_2(x_3)$	$\xi_2(x_4)$	$\xi_2(x_5)$	$\xi_2(x_6)$	$\xi_2(x_7)$	$\xi_2(x_8)$
$\binom{0}{0}$	$\binom{-1}{1}$	$\binom{1}{i}$	$\binom{-i}{-i}$	$\binom{i}{1}$	$\binom{1}{-1}$	$\binom{-1}{-i}$	$\binom{i}{i}$	$\binom{-i}{-1}$

The corresponding voltage assignments ξ_3:

$\xi_3(x)$	$\xi_3(x_1)$	$\xi_3(x_2)$	$\xi_3(x_3)$	$\xi_3(x_4)$	$\xi_3(x_5)$	$\xi_3(x_6)$	$\xi_3(x_7)$	$\xi_3(x_8)$
$\binom{0}{0}$	$\binom{i}{i}$	$\binom{1}{-i}$	$\binom{-1}{1}$	$\binom{i}{-1}$	$\binom{-i}{-i}$	$\binom{-1}{i}$	$\binom{1}{-1}$	$\binom{-i}{1}$

The corresponding voltage assignments ξ_4:

$\xi_4(x)$	$\xi_4(x_1)$	$\xi_4(x_2)$	$\xi_4(x_3)$	$\xi_4(x_4)$
$\binom{0}{0}$	$\binom{v}{1+(1+v)i}$	$\binom{1}{-1-v-vi}$	$\binom{vi}{1+v-i}$	$\binom{i}{-v+(1+v)i}$

$\xi_4(x_5)$	$\xi_4(x_6)$	$\xi_4(x_7)$	$\xi_4(x_8)$
$\binom{v}{-1-(1+v)i}$	$\binom{-1}{1+v+vi}$	$\binom{vi}{-1-v+i}$	$\binom{-i}{v-(1+v)i}$

The corresponding voltage assignments ξ_5:

$\xi_5(x)$	$\xi_5(x_1)$	$\xi_5(x_2)$	$\xi_5(x_3)$	$\xi_5(x_4)$
$\binom{0}{0}$	$\binom{-v}{1+(1+s)i}$	$\binom{1}{-1+v+vi}$	$\binom{-vi}{1-v-i}$	$\binom{i}{v+(1-v)i}$

$\xi_5(x_5)$	$\xi_5(x_6)$	$\xi_5(x_7)$	$\xi_5(x_8)$
$\binom{-v}{-1-(1-v)i}$	$\binom{-1}{1-v-vi}$	$\binom{-vi}{-1+v+i}$	$\binom{-i}{-v-(1-v)i}$

The corresponding voltage assignments ξ_6:

$\xi_6(x)$	$\xi_6(x_1)$	$\xi_6(x_2)$	$\xi_6(x_3)$	$\xi_6(x_4)$	$\xi_6(x_5)$	$\xi_6(x_6)$	$\xi_6(x_7)$	$\xi_6(x_8)$
$\binom{0}{0}$	$\binom{0}{\sqrt{-2}}$	$\binom{-1}{1}$	$\binom{\sqrt{-2}}{0}$	$\binom{1}{1}$	$\binom{0}{-\sqrt{-2}}$	$\binom{1}{-1}$	$\binom{-\sqrt{-2}}{0}$	$\binom{-1}{-1}$

Note that $\tilde{X}_1$ as a $\mathbb{Z}_p^2$-covering of the generalized Petersen graph GP(8, 3) admits a lift of Aut(GP(8, 3)) and $\tilde{X}_i$ $(i = 2, \ldots, 6)$ admits a lift of K_1 or K_2 as a maximal one. It follows that the derived graph $\tilde{X}_1$ is 2-regular and the derived graphs $\tilde{X}_i$ $(i = 2, \ldots, 6)$ are 1-regular. So we have proved the following lemma.

Lemma 3.3. *Let* $p \geq 13$ *be a prime and* $\tilde{X}$ *be an arc-transitive* $\mathbb{Z}_p^2$*-covering of the generalized Petersen graph* GP(8, 3). *Then* $\tilde{X}$ *is* 1- *or* 2-*regular. Furthermore*,

(1) $\tilde{X}$ *is* 1-*regular if and only if* $\tilde{X}$ *is isomorphic to one of* $\tilde{X}_2$, $\tilde{X}_3$, $\tilde{X}_4$, $\tilde{X}_5$ *and* $\tilde{X}_6$.
(2) $\tilde{X}$ *is* 2-*regular if and only if* $\tilde{X}$ *is isomorphic to* $\tilde{X}_1$.

The following theorem is the main result of this paper.

Theorem 3.4. *Let p be a prime and let X be a connected cubic symmetric graph of order* $16p^2$. *Then X is* 1- *or* 2-*regular. Furthermore,*

(1) *X is* 1-*regular if and only if X is isomorphic to one of the graphs* F_{64}, F_{144A}, F_{400A}, F_{784A}, F_{1936A}, $\tilde{X}_2$, $\tilde{X}_3$, $\tilde{X}_4$, $\tilde{X}_5$ *and* $\tilde{X}_6$.
(2) *X is* 2-*regular if and only if X is isomorphic to one of the graphs* F_{144B}, F_{400B}, F_{784B}, F_{1936B} *and* $\tilde{X}_1$.

Proof. Let X be a cubic symmetric graph of order $16p^2$. By Lemma 3.1, we can assume that $p \geq 13$. Let $A = \mathrm{Aut}(X)$ and let P be a Sylow p-subgroup of A. By Lemma 3.2, P is a normal subgroup of A. Since $|P| = p^2$, we have $P \cong \mathbb{Z}_{p^2}$ or $\mathbb{Z}_p^2$.

Let $P \cong \mathbb{Z}_{p^2}$. By Proposition 2.3, P is semiregular on $V(X)$ and the quotient graph X_P is a cubic symmetric graph and A/P is an arc-transitive subgroup of $\mathrm{Aut}(X_P)$. Let $C = C_A(P)$ be the centralizer of P in A. Clearly $P \leq C$. Suppose that $P = C$. Then by N/C-theorem (Theorem 6.11 of [18]), A/P is isomorphic to a subgroup of $\mathrm{Aut}(P) \cong \mathbb{Z}_{p(p-1)}$, which implies that A/P is abelian. Since A/P is transitive on $V(X_P)$, by Proposition 4.4 of [20], A/P is regular on $V(X_P)$. Consequently $|A| = 16p^2$, which contradicts the fact that X is symmetric. Hence $P < C$. Let T/P be a minimal normal subgroup of A/P contained in C/P. Then T/P is solvable, because A/P is a {2, 3}-group. By Proposition 2.3, T/P is semiregular on $V(X_P)$. It follows that $T/P \cong \mathbb{Z}_2^k$ where $1 \leq k \leq 4$ and since $T < C$, we have that $T \cong \mathbb{Z}_{2p^2}$ or $\mathbb{Z}_2^k \times \mathbb{Z}_{p^2}$ for $2 \leq k \leq 4$.

If $T \cong \mathbb{Z}_2^k \times \mathbb{Z}_{p^2}$ for $2 \leq k \leq 4$, then T has a normal subgroup isomorphic to $\mathbb{Z}_2^k$, say N, which is characteristic in T and hence is normal in A. By Proposition 2.3, the quotient graph X_N is a cubic symmetric graph of order $2^{4-k}p^2$ and A/N is an arc-transitive subgroup of $\mathrm{Aut}(X_N)$. If $N \cong \mathbb{Z}_2^2$ or $\mathbb{Z}_2^4$, then X_N is a cubic symmetric graph of order $4p^2$ or p^2. The first case is impossible because by Theorem 6.2 of [8] there is no symmetric cubic graph of order $4p^2$ for $p \geq 13$, while the last case is excluded since X_N be of odd order. If $N \cong \mathbb{Z}_2^3$, then X_N is a cubic symmetric graph of order $2p^2$. Note that the Sylow p-subgroups of $\mathrm{Aut}(X_N)$ are cyclic, because $A/N \leq \mathrm{Aut}(X_N)$. Since $p \geq 11$, by Lemma 3.4 of [3] and Theorem 3.5 of [7], $p - 1$ is divisible by 3 and X_N is a normal cubic 1-regular Cayley graph on dihedral group D_{2p^2}. Hence, $A/N = \mathrm{Aut}(X_N)$ and A has a normal subgroup G such that G/N acts regularly on $V(X_N)$. Consequently, G is regular on $V(X)$ and X is a normal cubic 1-regular Cayley graph on G. Let $X = \mathrm{Cay}(G, S)$. Since X has valency 3, S contains at least one involution. By Proposition 2.4, $\mathrm{Aut}(G, S)$ is transitive on S, which implies that S consists of three involutions and by the connectivity of X, G

can be generated by three involutions. Clearly $T < G$. Since $G/N \cong D_{2p^2}$, we conclude that G is one of the groups $\mathbb{Z}_2^3 \times D_{2p^2}$ and $\mathbb{Z}_2^2 \times D_{4p^2}$. In each case, one may easily check that G cannot be generated by involutions, a contradiction. Thus $T \cong \mathbb{Z}_{2p^2}$ and by Proposition 2.3, X is an arc-transitive $\mathbb{Z}_{2p^2}$-covering of the hypercube Q_3. But by Proposition 2.2, there is no such covering, a contradiction. Thus we can assume that $P \cong \mathbb{Z}_p^2$. Now we suppose that X is a $\mathbb{Z}_p^2$-covering of the graph GP(8, 3) with the corresponding covering projection as $\wp$. By the normality of P in A, we have that the arc-transitive subgroup A/P lifts along $\wp$ eventually to A. Therefore, $\wp$ is an arc-transitive covering projection. Now, by Lemma 3.3, the proof is complete. □

References

[1] Conder M, Trivalent (cubic) symmetric graphs on up to 2048 vertices, http://www.math.auckland.ac.nz/conder/ (2006)

[2] Conder M and Dobcsányi P, Trivalent symmetric graphs on up to 768 vertices, *J. Combin. Math. Combin. Comput.* **40** (2002) 41–63

[3] Du S F, Feng Y-Q, Kwak J H and Xu M Y, Cubic Cayley graphs on dihedral groups, Mathematical Analysis and Applications (New Delhi: Narosa Publishing House) (2004) pp. 224–235

[4] Feng Y Q, Kwak J H and Wang K, Classifying cubic symmetric graphs of order $8p$ or $8p^2$, *European J. Combin.* **26** (2005) 1033–1052

[5] Feng Y Q, Kwak J H and Xu M Y, Cubic s-regular graphs of order $2p^3$, *J. Graph Theory* **52** (2006) 341–352

[6] Feng Y Q and Kwak J H, Classifying cubic symmetric graphs of order $10p$ or $10p^2$, *Science in China* **A49(3)** (2006) 300–319

[7] Feng Y Q and Kwak J H, Cubic symmetric graphs of order twice an odd prime-power, *J. Austral. Math. Soc.* **81** (2006) 153–164

[8] Feng Y Q and Kwak J H, Cubic symmetric graphs of order a small number times a prime or a prime square, *J. Combin Theory* **B97** (2007) 627–646

[9] Feng Y Q and Wang K, s-Regular cyclic coverins of the three-dimensional hypercube Q_3, *European J. Combin.* **24** (2003) 719–731

[10] Gorenstein D, Finite Simple Groups (New York: Plenum Press) (1982)

[11] Gross J L and Tucker T W, Generating all graph covering by permutation voltages assignment, *Discrete Math.* **18** (1977) 273–283

[12] Lorimer P, Vertex-transitive graphs: Symmetric graphs of prime valency, *J. Graph Theory* **8** (1984) 55–68

[13] Malnič A, Marušič D, Miklavič S and Potočnik P, Semisymmetric elementary abelian covers of the Mobius-Kantor graph, *Discrete Math.* **307** (2007) 2156–2175

[14] Malnič A, Marušič D and Potočnik P, Elementary abelian covers of graphs, *J. Algebraic Combin.* **20** (2004) 71–97

[15] Oh J M, A classification of cubic s-regular graphs of order $14p$, *Discrete Math.* **309(9)** (2009) 2721–2726

[16] Oh J M, A classification of cubic s-regular graphs of order $16p$, *Discrete Math.* **309(10)** (2009) 3150–3155

[17] Sabidussi G, On a class of fixed-point-free graphs, *Proc. Amer. Math. Soc.* **9** (1958) 800–804

[18] Suzuki M, Group Theory I (New York: Springer) (1982)

[19] Tutte W T, A family of cubical graphs, *Proc. Cambridge Philos. Soc.* **43** (1947) 459–574

[20] Wieldant H, Finite Permutation Groups (New York: Academic Press) (1964)

[21] Xu M Y, Automorphism groups and isomorphisms of Cayley digraphs, *Discrete Math.* **182** (1998) 309–319