On diophantine equations of the form
\((x - a_1)(x - a_2) \cdots (x - a_k) + r = y^n\)

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Abstract. Erdős and Selfridge [3] proved that a product of consecutive integers can never be a perfect power. That is, the equation
\(x(x + 1)(x + 2)\cdots(x + (m - 1)) = y^n\)
has no solutions in positive integers \(x, m, n\) where \(m, n > 1\) and \(y \in \mathbb{Q}\). We consider the equation
\[(x - a_1)(x - a_2) \cdots (x - a_k) + r = y^n\]
where \(0 \leq a_1 < a_2 < \cdots < a_k\) are integers and, with \(r \in \mathbb{Q}\), \(n \geq 3\) and we prove a finiteness theorem for the number of solutions \(x, m, n \in \mathbb{Z}\) and \(y \in \mathbb{Q}\). Following that, we show that, more interestingly, for every nonzero integer \(n > 2\) and for any nonzero integer \(r\) which is not a perfect \(n\)-th power for which the equation admits solutions, \(k\) is bounded by an effective bound.

Keywords. Diophantine equations; Erdős–Selfridge.

Erdős and Selfridge [3] proved that a product of consecutive integers can never be a perfect power. That is, the equation \(x(x + 1)(x + 2)\cdots(x + (m - 1)) = y^n\) has no solutions in positive integers \(x, m, n\) where \(m, n > 1\). After this, a natural question is to study \(x(x + 1)(x + 2)\cdots(x + (m - 1)) + r = y^n\) with a nonzero integral or rational parameter \(r\). However, this equation is not symmetric like the Erdős–Selfridge equation and requires different methods. In [1], we have proved that in this case there are effective finiteness results for \(x, m, n \in \mathbb{Z}\) and \(y \in \mathbb{Q}\). We shall also prove finiteness results if we delete many terms from the product involving consecutive integers. We consider the equation
\[(x - a_1)(x - a_2) \cdots (x - a_k) + r = y^n\]
where \(0 \leq a_1 < a_2 < \cdots < a_k\) are integers and, with \(r \in \mathbb{Q}\), \(n \geq 3\). Our first aim is to prove a finiteness theorem for the number of solutions \(x \in \mathbb{Z}, y \in \mathbb{Q}\). Following that, we show that, more interestingly, for every nonzero integer \(n > 2\) and for any nonzero integer \(r\) which is not a perfect \(n\)-th power for which the equation admits solutions, \(k\) is bounded by an effective bound. We recall that the height \(H(\alpha)\) of an algebraic number \(\alpha\) is the maximum of the absolute values of the integer coefficients in its minimal defining
polynomial. In particular, if \( \alpha \) is a rational integer, then \( H(\alpha) = |\alpha| \) and if \( \alpha \) is a rational number \( \frac{p}{q} \neq 0 \), then \( H(\alpha) = \max(|p|, |q|) \).

Our first result is as follows.

**Theorem 1.** Let \( r \in \mathbb{Q} \), let \( 0 \leq a_1 < a_2 < \cdots < a_k \) be integers where \( k > 2 \). Further, let \( n > 2 \) and assume that we are not in the case when \( n = k = 4 \). Then, there are only finitely many solutions \( x \in \mathbb{Z} \), \( y \in \mathbb{Q} \) to the equation

\[
(x - a_1)(x - a_2) \cdots (x - a_k) + r = y^n
\]

and, all the solutions satisfy

\[
\max\{H(x), H(y)\} < C,
\]

where \( C \) is an effectively computable constant depending only on \( n, r \) and the \( a_i \)'s.

When \( r \) is an integer and not a perfect \( n \)-th power, we bound \( k \) in the following result.

**Theorem 2.** Let \( n \) be a fixed positive integer \( > 2 \) and let \( r \) be a nonzero integer which is not a perfect \( n \)-th power. Let \( \{t_m\}_m \) be a sequence of positive integers such that \( m/t_m \to \infty \) as \( m \to \infty \). There exists an effectively computable number \( C \) depending only on \( n \) and \( r \) such that if \( (x - a_1)(x - a_2) \cdots (x - a_{m-t_m}) + r = y^n \) with \( 0 \leq a_1 < a_2 < \cdots < a_{m-t_m} \) has a solution, then \( m/(t_m + 1) < C \).

To prove Theorem 1, we use a theorem of Brindza [2].

Let \( K \) be an algebraic number field, \( R \subset K \) be a finitely generated subring and \( g \in R[X] \). Write \( g = a \prod_{i=1}^{s} (X - \beta_i)^{r_i} \) over an extension of \( K \), where \( a \neq 0 \) and \( \beta_i \neq \beta_j \) for \( i \neq j \). Let \( R_1 \) be the ring generated by \( R \) along with the denominators of the \( \beta_i \)'s. For an integer \( n > 1 \), consider the equation \( g(x) = y^n \) with \( x, y \in R_1 \). Then, Brindza’s theorem [2] asserts:

**Theorem [2].** With the above notations, put \( t_i = \left(\frac{n}{r_i(x_i)}\right) \), \( i = 1, 2, \ldots, s \). Suppose that \( (t_1, t_2, \ldots, t_s) \) is not a permutation of any of the \( s \)-tuples. Then

(i) \( (t, 1, 1, 1, \ldots, 1) \) for some \( t \), or
(ii) \( (2, 2, 1, 1, 1, \ldots, 1) \).

Then, all the solutions of the equation \( g(x) = y^n \) with \( x, y \in R_1 \) satisfy

\[
\max\{H(x), H(y)\} < C,
\]

where \( C \) is an effectively computable constant depending on \( K, n \) and \( g \).

Let us prove Theorem 1 using this now.

**Proof of Theorem 1.** Let us write \( f \) for the polynomial \( (X - a_1)(X - a_2) \cdots (X - a_k) \). Suppose \( f + r = a \prod_{i=1}^{s} (X - \beta_i)^{r_i} \) with \( a \neq 0 \) and \( \beta_i \neq \beta_j \) for \( i \neq j \) algebraic integers. We take \( R \) to be the subring \( \mathbb{Z}[r] \) of \( \mathbb{Q} \) and \( K = \mathbb{Q}(\beta_1, \beta_2, \ldots, \beta_s) \). We consider solutions \( x, y \in O_K[r] \). We show that \( r_i = 1 \) or 2 and then use Brindza’s theorem to get the result.

**Claim.** The multiplicity of a root of \( f(x) + r \) is at most 2.
Proof. Note that \( f + r \) is a polynomial of degree \( k \) and hence, its derivative \( f' \) is a polynomial of degree \( k - 1 \). Now, by Rolle’s theorem, it has zeroes in the intervals \((a_1, a_2), (a_2, a_3), \ldots, (a_{k-1}, a_k)\). Thus, the roots of \( f' \) are distinct. Therefore, if \( f + r \) has a multiple root then its multiplicity can be at most two which proves the claim.

Thus,

\[
 f + r = a \prod_{i=1}^{s} (X - \alpha_i)^{r_i},
\]

where each \( r_i = 1 \) or \( 2 \). Also note that \( s > 1 \) since \( k > 2 \).

Let \( t_i = \frac{n}{(n, x)} = \frac{n}{(n, 1)} \) or \( \frac{n}{(n, 2)} \). This implies \( t_i = n \) or \( \frac{n}{2} \). As \( n > 2 \), note that \( t_i > 1 \) for each \( i \). So, the \( s \)-tuple \((t_1, t_2, \ldots, t_s)\) can never look like \((1, 1, \ldots, 1)\) for any \( t \). If this \( s \)-tuple looks like \((2, 2, 1, \ldots, 1)\), then it must be \((2, 2)\) which gives \( k = 4 = n \) which is excluded by assumption. So, by Brindza’s theorem we get the result.

Proof of Theorem 2. Since \( r \) is not a perfect \( n \)-th power we can write \( r = r_1^{h_1} r_2^{h_2} \ldots p_i^{h_i} \), where \( p_i \)'s are primes in \( \mathbb{Z} \) and \( r_i \)'s are such that not all of them are zeroes. Choose the smallest \( p_i \) for which \( r_i \) is not zero; so, the exact power of \( p_i \) dividing \( r \) is \( h_i n + r_i \). Take \( C = (h_i n + r_i + 1) p_i \) and suppose, if possible, \( m/(tm + 1) > C \). Let us write \( k := m - t_m \) for simplicity. Then we claim that \((x - a_1)(x - a_2) \ldots (x - a_k)\) is divisible by \( p_i^{h_i n + r_i + 1} \). Indeed, look at the number of terms of the product \((x - 1)(x - 2) \ldots (x - m)\) which are missing in the product \((x - a_1)(x - a_2) \ldots (x - a_k)\); this number is \( m - k = t_m \). We claim that there is a string of consecutive integers of length at least \((h_i n + r_i + 1) p_i \) in the product \((x - a_1)(x - a_2) \ldots (x - a_k)\). Indeed, if each consecutive string of integers occurring in the last product is of length at most \((h_i n + r_i + 1) p_i - 1 \), then we would have \( k = m - t_m < (tm + 1)((h_i n + r_i + 1) p_i - 1) \) which means \( m < (tm + 1)(h_i n + r_i + 1) p_i \). Thus, \( m/(tm + 1) < C \). In other words, if \( m \) is so large that \( m/(tm + 1) \geq C \), then there is a string of consecutive integers of length at least \((h_i n + r_i + 1) p_i \) in the product \((x - a_1)(x - a_2) \ldots (x - a_k)\). Hence the power of \( p_i \) in \((x - a_1)(x - a_2) \ldots (x - a_k)\) is at least \( h_i n + r_i + 1 \). Thus the power of \( p_i \) in \((x - a_1)(x - a_2) \ldots (x - a_k) + r \) is exactly \( h_i n + r_i \not\equiv 0 \) (mod \( n \)) since \( 0 < r_i < n \). This is a contradiction to the equation under consideration.

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References

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