Proc. Indian Acad. Sci. (Math. Sci.) Vol. 121, No. 2, May 2011, pp. 217–228.

The almost sure local central limit theorem for the product of partial sums

ZHICHAO WENG[1], ZUOXIANG PENG[2] and SARALEES NADARAJAH[3]

[1]Department of Mathematics, Ningde Normal University, Fujian 352100, Taiwan
[2]School of Mathematics and Statistics, Southwest University, Chongqing 400715, China
[3]School of Mathematics, University of Manchester, Manchester, UK
E-mail: mbbsssn2@manchester.ac.uk

MS received 19 March 2010; revised 16 December 2010

Abstract. We derive under some regular conditions an almost sure local central limit theorem for the product of partial sums of a sequence of independent identically distributed positive random variables.

Keywords. Almost sure local central limit theorem; partial sum; product.

1. Introduction

Let $\{Y_n\}_{n\geq 1}$ be a sequence of independent identically distributed (i.i.d.) positive random variables. Denote $\mu = \mathrm{E}(Y_1) > 0$ and the coefficient of variation $\gamma = \sigma/\mu$, where $\sigma^2 = \mathrm{Var}(Y_1)$. Let $S_k = Y_1 + \cdots + Y_k$, $k \geq 1$ denote the partial sums. Rempała and Wesołowski [13] showed under the assumption $\mathrm{E}(Y_1^2) < \infty$ that

$$\left(\frac{\prod_{k=1}^{n} S_k}{n!\mu^n}\right)^{\frac{1}{\gamma\sqrt{n}}} \stackrel{d}{\to} \exp\left(\sqrt{2}\mathcal{N}\right) \tag{1.1}$$

as $n \to \infty$, where $\mathcal{N}$ is a standard normal random variable. Qi [11] and Lu and Qi [8] extended (1.1) to the case that the underling distribution is in the domain of attraction of a stable law with exponent in [1, 2]. For more recent work on the weak convergence of the product of partial sums, see [6,9,16].

There are many papers on the global almost sure central limit theorem for partial sums (see [1,2,4,7,14,15] and references therein). On the almost sure central limit theorem for the product of partial sums of positive random variables, Gonchigdanzan and Rempała [5] and Miao [10] proved the following related to (1.1):

Theorem A. *Let $\{Y_n\}_{n\geq 1}$ be the i.i.d. positive random variables defined above. Then for any real x,*

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} I\left\{\left(\frac{\prod_{k=1}^{n} S_k}{n!\mu^n}\right)^{\frac{1}{\gamma\sqrt{n}}} \leq x\right\} = F(x) \qquad a.s.,$$

where $F(x)$ is the distribution function of random variable $\exp(\sqrt{2}\mathcal{N})$.

The objective of this paper is to prove the almost sure local central limit theorem of the product of partial sums of i.i.d. positive random variables. For the almost sure local central limit theorem of i.i.d. random variables, Csáki *et al* [3] proved

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{I\{a_k \le S_k < b_k\}}{k P\{a_k \le S_k < b_k\}} = 1, \qquad \text{a.s.,} \tag{1.2}$$

if

$$\sum_{k=1}^{n} \frac{\log k}{k^{3/2} P\{a_k \le S_k < b_k\}} = O(\log n)$$

as $n \to \infty$, where both $\{a_n\}_{n\ge 1}$ and $\{b_n\}_{n\ge 1}$ are real sequences such that $a_n \le 0 \le b_n$ for $n \ge 1$. Gonchigdanzan [4] extended (1.2) to the case of ρ-mixing sequences.

This paper is organized as follows: In §2, we give the main result. Some auxiliary results are provided in §3. Proofs are deferred to §4.

2. Main result

Let $\{a_k\}_{k\ge 1}$ and $\{b_k\}_{k\ge 1}$ be two sequences of real numbers satisfying

$$0 \le a_k \le 1 \le b_k \le \infty, \qquad k = 1, 2, \ldots. \tag{2.1}$$

Set

$$p_k = P\left\{ a_k \le \left(\frac{\prod_{i=1}^{k} S_i}{k! \mu^k} \right)^{\frac{1}{\gamma\sqrt{k}}} < b_k \right\}$$

and

$$\alpha_k = \begin{cases} \frac{1}{p_k} I\left\{ a_k \le \left(\frac{\prod_{i=1}^{k} S_i}{k! \mu^k} \right)^{\frac{1}{\gamma\sqrt{k}}} < b_k \right\}, & \text{if } p_k \ne 0; \\ 1, & \text{if } p_k = 0. \end{cases}$$

The main result is as follows.

Theorem 2.1. *Suppose* $\{Y_n\}_{n\ge 1}$ *are i.i.d. positive random variables having finite third moments*, $\mathrm{E}(Y_1) = \mu$ *and* $\mathrm{Var}(Y_1) = \sigma^2$. *Denote* $\gamma = \sigma/\mu$ *the coefficient of variation. Let* a_k, b_k *satisfy* (2.1) *and assume for large* k,

$$p_k \ge \frac{1}{(\log k)^{\delta_1}} \tag{2.2}$$

for some $\delta_1 > 0$. *Then we have*

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{\alpha_k}{k} = 1 \qquad a.s.$$

3. Auxiliary results

In order to prove the main result, we first establish a certain triangular array of random variables. For convenience, we shall use the following notation. Set the positive absolute constant $\mathfrak{C}$ to vary from line to line. Let $b_{k,n} = \sum_{i=k}^{n} 1/i$ and $X_i = (Y_i - \mu)/\sigma$. We define a triangular array $Z_{1,n}, Z_{2,n}, \dots, Z_{n,n}$ as $Z_{k,n} = b_{k,n} X_k$ and put $S_{k,n} = Z_{1,n} + Z_{2,n} + \cdots + Z_{k,n}$ for $1 \le k \le n$. Let

$$U_k = \frac{1}{\gamma\sqrt{2k}} \sum_{i=1}^{k} \log \frac{S_i}{i\mu} = \frac{1}{\gamma\sqrt{2k}} \sum_{i=1}^{k} \left(\frac{S_i}{i\mu} - 1 \right) + T_k = \frac{1}{\sqrt{2k}} S_{k,k} + T_k,$$

where

$$T_k = \frac{1}{\gamma\sqrt{2k}} \sum_{i=1}^{k} \frac{(S_i/i\mu - 1)^2}{(1 + \theta\,(S_i/i\mu - 1))^2}, \qquad |\theta| \le 1.$$

Note that for $l > k$, we have

$$\begin{aligned} S_{l,l} - S_{k,k} &= \sum_{j=1}^{l} b_{j,l} X_j - \sum_{j=1}^{k} b_{j,k} X_j \\ &= b_{k+1,l}\,(X_1 + \cdots + X_k) + (b_{k+1,l} X_{k+1} + \cdots + b_{l,l} X_l) \\ &= b_{k+1,l} \tilde{S}_k + (b_{k+1,l} X_{k+1} + \cdots + b_{l,l} X_l). \end{aligned}$$

So, $S_{l,l} - S_{k,k} - b_{k+1,l}\tilde{S}_k$ and U_k are independent.

Now we obtain the central limit theorem for triangular arrays.

Lemma 3.1. *Let* $X_1, X_2, \dots, X_n$ *be i.i.d. random variables with* $\mathrm{E}(X_1) = 0$ *and* $\mathrm{Var}(X_1) = 1$. *Suppose that there exist constants* δ_2 *and* δ_3 *such that* $0 < \delta_2, \delta_3 < 1$. *If*

$$\log l > (\log n)^{\delta_2}, \quad k < \frac{l}{(\log l)^{2+\delta_3}} \tag{3.1}$$

for large n then

$$\frac{1}{\sqrt{2l - 2k}} \sum_{j=k+1}^{l} b_{j,l} X_j \xrightarrow{d} \varsigma$$

as $n \to \infty$, *where* ς *is a standard normal random variable.*

Proof. By elementary calculations, under condition (3.1) we have

$$\mathrm{Var}\left(\frac{1}{\sqrt{2l - 2k}} \sum_{j=k+1}^{l} b_{j,l} X_j \right) = \frac{1}{2l-2k} \left(2l - 2k - k \log^2 l \right) (1 + o(1)) \to 1.$$

By arguments similar to that of Lemma 1 in [13], we obtain the desired result. □

Next we estimate the convergence rates of the distributions of the weighted sums of $X_1, X_2, \dots, X_n$ to the standard normal distribution.

Lemma 3.2. *Assume the conditions of Lemma* 3.1. *Assume also* $\mathrm{E}\,|X_1|^3 < \infty$. *Let*

$$F_n(x) = P\left(\frac{\sum_{j=1}^n b_{j,n} X_j}{\sqrt{2n}} < x\right), \qquad F_{k,l}(x) = P\left(\frac{\sum_{j=k+1}^l b_{j,l} X_j}{\sqrt{2l-2k}} < x\right)$$

for $1 \le k < l \le n$. *Then*

$$\sup_x |F_n(x) - \Phi(x)| \le \mathfrak{C}\frac{1}{\sqrt{n}}$$

and

$$\sup_x \left|F_{k,l}(x) - \Phi(x)\right| \le \mathfrak{C}\frac{l}{(l-k)^{3/2}}.$$

Proof. Denote

$$L_n = (2n)^{-3/2}\sum_{j=1}^n \mathrm{E}\left|b_{j,n}X_j\right|^3, \quad L_{k,l} = (2l-2k)^{-3/2}\sum_{j=k+1}^l \mathrm{E}\left|b_{j,l}X_j\right|^3.$$

By arguments similar to that of Theorem 5.4 of [12], we obtain the desired results. □

The following result is due to [4].

Lemma 3.3. *Assume that* $\xi_1, \xi_2, \ldots, \xi_n$ *are random variables such that* $\mathrm{E}\,\xi_k = 1$ *for* $k = 1, 2, \ldots, n$. *Then*

$$\lim_{n\to\infty} \frac{1}{\log n}\mathrm{E}\left(\sum_{k=1}^n \frac{\xi_k}{k}\right) = 1.$$

Furthermore, if $\xi_k \ge 0$ *for* $k \ge 1$ *and*

$$\mathrm{Var}\left(\sum_{k=1}^n \frac{\xi_k}{k}\right) \le \mathfrak{C}(\log n)^{2-\epsilon} \tag{3.2}$$

for some $\epsilon > 0$ *and large* n, *then*

$$\lim_{n\to\infty} \frac{1}{\log n}\sum_{k=1}^n \frac{\xi_k}{k} = 1 \qquad a.s. \tag{3.3}$$

Proof. We only prove that (3.3) holds. Let $\eta_j = \xi_j - E\xi_j$ for $j \ge 1$ and

$$\mu_n = \frac{1}{\log n}\sum_{j=1}^n \frac{1}{j}\eta_j.$$

Similar to the proof of Lemma 4.2 in [4], we firstly prove that

$$\mu_{n_k} \to 0 \tag{3.4}$$

almost surely, where $n_k = \exp(\mathrm{e}^{k^v})$ with $1/(1+\epsilon) < v < 1$. For every large n, there exist n_k and n_{k+1} such that $n_k \le n \le n_{k+1}$. So,

$$|\mu_n| \le |\mu_{n_k}| + \frac{1}{\log n_k} \sum_{j=n_k+1}^{n_{k+1}} \frac{1}{j} |\eta_j|.$$

Note that $|X| = X^+ + X^-$ and $X = X^+ - X^-$. We have

$$\begin{aligned}
&\frac{1}{\log n_k} \sum_{j=n_k+1}^{n_{k+1}} \frac{1}{j} |\eta_j| \\
&\quad = \frac{1}{\log n_k} \sum_{j=n_k+1}^{n_{k+1}} \frac{1}{j} \{(\xi_j - E\xi_j) I(\xi_j \ge E\xi_j) - (E\xi_j - \xi_j) I(\xi_j < E\xi_j) \\
&\qquad\qquad + 2(E\xi_j - \xi_j) I(\xi_j < E\xi_j)\} \\
&\quad = \frac{1}{\log n_k} \sum_{j=n_k+1}^{n_{k+1}} \frac{1}{j} \{\eta_j + 2(E\xi_j - \xi_j) I(\xi_j < E\xi_j)\} \\
&\quad \le \frac{1}{\log n_k} \sum_{j=n_k+1}^{n_{k+1}} \frac{1}{j} (\eta_j + 2E\xi_j) \\
&\quad \le \frac{\log n_{k+1}}{\log n_k} \left|\mu_{n_{k+1}}\right| + \left|\mu_{n_k}\right| + 2\frac{\log n_{k+1} - \log n_k}{\log n_k}.
\end{aligned}$$

The last two inequalities are based on $\xi_j \ge 0$ and $E\xi_j = 1$ for $j \ge 1$. So,

$$|\mu_n| \le 2|\mu_{n_k}| + \frac{\log n_{k+1}}{\log n_k} |\mu_{n_{k+1}}| + 2\frac{\log n_{k+1} - \log n_k}{\log n_k}. \tag{3.5}$$

For $v < 1$, one can check that $(k+1)^v - k^v \to 0$ as $k \to \infty$, implying

$$\frac{1}{\log n_k} (\log n_{k+1} - \log n_k) \to 0. \tag{3.6}$$

We derive the desired result (3.3) by combining with (3.4), (3.5) and (3.6). □

The following result is obvious.

Lemma 3.4. *Assume that the non-negative random sequence* $\{\xi_k\}_{k\ge1}$ *satisfies*

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{\xi_k}{k} = 1 \qquad a.s.$$

and the sequence $\{\eta_k\}_{k\ge1}$ *is such that, for any* $\varepsilon > 0$, *there exists a* $k_0 = k_0(\varepsilon)$ *for which*

$$(1-\varepsilon)\xi_k \le \eta_k \le (1+\varepsilon)\xi_k, \qquad k \ge k_0.$$

Then

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{\eta_k}{k} = 1 \qquad a.s.$$

Lemma 3.5. *Under the conditions of Theorem* 2.1, *assume that there exists* δ_4 *such that* $0 < \delta_1 < \delta_4 < 1/4$. *Let* $\varepsilon_l = 1/(\log l)^{\delta_4}$, *where* $l = 3, 4, \ldots, n$. *Then we have*

$$\sum_{l=1}^{n}\sum_{k=1}^{l-1}\frac{1}{k(l-k)^{3/2}p_l} \le \mathfrak{C}(\log n)^{2-\epsilon},$$

$$\sum_{l=1}^{n}\sum_{k=1}^{l-1}\frac{1}{l^{3/2}\sqrt{l-k}\,p_l} \le \mathfrak{C}(\log n)^{2-\epsilon},$$

$$\sum_{l=1}^{n}\sum_{k=1}^{l-1}\frac{\varepsilon_l}{k\sqrt{l}\sqrt{l-k}\,p_l} \le \mathfrak{C}(\log n)^{2-\epsilon},$$

$$\sum_{l=1}^{n}\sum_{k=1}^{l-1}\frac{1}{klp_kp_l}P\left\{\left|\frac{1}{\sqrt{2l}}S_{k,k}\right| \ge \varepsilon_l\right\} \le \mathfrak{C}(\log n)^{2-\epsilon},$$

$$\sum_{l=1}^{n}\sum_{k=1}^{l-1}\frac{1}{klp_kp_l}P\left\{\left|\frac{1}{\sqrt{2l}}b_{k+1,l}\tilde{S}_k\right| \ge \varepsilon_l\right\} \le \mathfrak{C}(\log n)^{2-\epsilon},$$

$$\sum_{l=1}^{n}\sum_{k=1}^{l-1}\frac{1}{klp_kp_l}P\left\{|T_l| \ge \varepsilon_l\right\} \le \mathfrak{C}(\log n)^{2-\epsilon},$$

where $0 < \epsilon < 1 - 2(\delta_1 + \delta_4)$.

Proof. By using Markov inequality, we have

$$P\left\{\left|\frac{1}{\sqrt{2l}}S_{k,k}\right| \ge \varepsilon_l\right\} \le \frac{\sum_{j=1}^{k} b_{j,k}^2}{2l\varepsilon_l^2} \le \frac{k}{l\varepsilon_l^2},$$

$$P\left\{\left|\frac{1}{\sqrt{2l}}b_{k+1,l}\tilde{S}_k\right| \ge \varepsilon_l\right\} \le \frac{k\left(\sum_{i=k+1}^{l}\frac{1}{i}\right)^2}{2l\varepsilon_l^2}$$

and

$$P\left\{|T_l| \ge \varepsilon_l\right\} \le P\left\{\left|\frac{4}{\gamma\sqrt{2l}}\sum_{i=1}^{l}\left(\frac{S_i}{i\mu}-1\right)^2\right| \ge \varepsilon_l\right\} \le \frac{2\sqrt{2}\gamma\sum_{i=1}^{l}\frac{1}{i}}{\sqrt{l}\varepsilon_l}. \quad (3.7)$$

The first inequality of eq. (3.7) follows from $x^2/(1+\theta x)^2 \le 4x^2$ for $|x| < 1/2$ and $\theta \in (0, 1)$. The rest of the proof follows by estimating the bounds. The details are omitted here.

4. The proofs

Proof of Theorem 2.1. By Lemma 3.3, we prove that inequality (3.2) holds for (α_k). Let

$$\hat{a}_k = \frac{1}{\sqrt{2}}\log a_k, \qquad \hat{b}_k = \frac{1}{\sqrt{2}}\log b_k, \quad k \ge 1.$$

So, $-\infty \le \hat{a}_k \le 0 \le \hat{b}_k \le \infty$ by (2.1). Noting $p_k = P\{\hat{a}_k \le U_k < \hat{b}_k\}$ and

$$\alpha_k = \begin{cases} \frac{1}{p_k} I\{\hat{a}_k \le U_k < \hat{b}_k\}, & \text{if } \ p_k \ne 0; \\ 1, & \text{if } \ p_k = 0, \end{cases}$$

we consider the following case first:

$$\hat{b}_k - \hat{a}_k \le \mathfrak{C}, \qquad k = 1, 2, \dots. \tag{4.1}$$

Note that

$$\begin{aligned} \operatorname{Var}\left(\sum_{k=1}^{n} \frac{\alpha_k}{k}\right) &= \sum_{k=1}^{n} \frac{1}{k^2} \operatorname{Var}(\alpha_k) + 2 \sum_{1 \le k < l \le n} \frac{1}{kl} \operatorname{Cov}(\alpha_k, \alpha_l) \\ &= \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4, \end{aligned}$$

where

$$\Sigma_1 = \sum_{k=1}^{n} \frac{1}{k^2} \operatorname{Var}(\alpha_k),$$

$$\Sigma_2 = 2 \sum_{\substack{1 \le k < l \le n \\ \log l \le (\log n)^{\delta_2}}} \frac{1}{kl} \operatorname{Cov}(\alpha_k, \alpha_l),$$

$$\Sigma_3 = 2 \sum_{\substack{1 \le k < l \le n \\ \log l > (\log n)^{\delta_2} \\ k > l/(\log l)^{2+\delta_3}}} \frac{1}{kl} \operatorname{Cov}(\alpha_k, \alpha_l),$$

$$\Sigma_4 = 2 \sum_{\substack{1 \le k < l \le n \\ \log l > (\log n)^{\delta_2} \\ k < l/(\log l)^{2+\delta_3}}} \frac{1}{kl} \operatorname{Cov}(\alpha_k, \alpha_l).$$

Note also that $\operatorname{Var}(\alpha_k) = 0$ if $p_k = 0$ and

$$\operatorname{Var}(\alpha_k) = \frac{1 - p_k}{p_k} \le \frac{1}{p_k} \le (\log k)^{\delta_1},$$

if $p_k \ne 0$. So,

$$\Sigma_1 \le \mathfrak{C}(\log n)^{2-\epsilon}. \tag{4.2}$$

For Σ_2, by (2.2), we have

$$\begin{aligned} \Sigma_2 &= 2 \sum_{\substack{1 \le k < l \le n \\ \log l \le (\log n)^{\delta_2}}} \frac{1}{kl} \frac{P\{\hat{a}_k \le U_k < \hat{b}_k, \hat{a}_l \le U_l < \hat{b}_l\} - p_k p_l}{p_k p_l} \\ &\le 2 \sum_{\substack{1 \le k < l \le n \\ \log l \le (\log n)^{\delta_2}}} \frac{1}{kl} \frac{1 - p_k}{p_k} \end{aligned}$$

$$
\begin{aligned}
&\le 2 \sum_{\substack{1\le k<l\le n \\ \log l \le (\log n)^{\delta_2}}} \frac{1}{kl}\frac{1}{p_k} \\
&\le 2(\log n)^{\delta_1+2\delta_2} \\
&\le \mathfrak{C}(\log n)^{2-\epsilon} \qquad (4.3)
\end{aligned}
$$

for $\delta_2 < 7/8$. Now we estimate the bound of Σ_3. Let A be an integer such that $\log A \sim (\log n)^{\delta_2}$ for large n. Then

$$
\begin{aligned}
\Sigma_3 &\le 2\sum_{l=A}^{n} \sum_{k=l/(\log l)^{2+\delta_3}}^{l-1} \frac{1}{kl}\frac{1}{p_k} \\
&\le 2(\log n)^{\delta_1}\left[\sum_{l=A}^{n}\frac{1}{l}\frac{(\log l)^{2+\delta_3}}{l} + \sum_{l=A}^{n}\frac{1}{l}\sum_{k=1+l/(\log l)^{2+\delta_3}}^{l}\frac{1}{k}\right] \\
&\le \mathfrak{C}(\log n)^{\delta_1}\sum_{l=A}^{n}\frac{1}{l}\log(\log l)^{2+\delta_3} \\
&\le \mathfrak{C}(\log n)^{2-\epsilon}. \qquad (4.4)
\end{aligned}
$$

So, it remains to estimate the bound of Σ_4. For $1 \le k \le l$, we have

$$
\begin{aligned}
\mathrm{Cov}(\alpha_k,\alpha_l) &= \frac{1}{p_k p_l}\mathrm{Cov}(I\{\hat{a}_k \le U_k < \hat{b}_k\}, I\{\hat{a}_l \le U_l < \hat{b}_l\}) \\
&= \frac{1}{p_k p_l}\left[P\left\{\hat{a}_k \le U_k < \hat{b}_k, \hat{a}_l \le \frac{1}{\sqrt{2l}}S_{l,l} + T_l < \hat{b}_l\right\}\right. \\
&\qquad \left. - P\{\hat{a}_k \le U_k < \hat{b}_k\}P\left\{\hat{a}_l \le \frac{1}{\sqrt{2l}}S_{l,l} + T_l < \hat{b}_l\right\}\right] \\
&\le \frac{1}{p_k p_l}\left\{P\left\{\hat{a}_k \le U_k < \hat{b}_k, \hat{a}_l - 3\varepsilon_l \le \frac{1}{\sqrt{2l}}(S_{l,l} - S_{k,k}\right.\right. \\
&\qquad \left. - b_{k+1,l}\tilde{S}_k) < \hat{b}_l + 3\varepsilon_l\right\} + 2P\left\{\left|\frac{1}{\sqrt{2l}}S_{k,k}\right| \ge \varepsilon_l\right\} \\
&\qquad + 2P\left\{\left|\frac{1}{\sqrt{2l}}b_{k+1,l}\tilde{S}_k\right| \ge \varepsilon_l\right\} + 2P\{|T_l| \ge \varepsilon_l\} \\
&\qquad - P\{\hat{a}_k \le U_k < \hat{b}_k\}\left[P\left\{\hat{a}_l + \varepsilon_l \le \frac{1}{\sqrt{2l}}S_{l,l} < \hat{b}_l - \varepsilon_l\right\}\right. \\
&\qquad \left.\left. - 2P\{|T_l| \ge \varepsilon_l\}\right]\right\} \\
&\le \frac{1}{p_l}B_1 + B_2,
\end{aligned}
$$

where

$$B_1 = P\left\{\hat{a}_l - 3\varepsilon_l \le \sqrt{1-\frac{k}{l}}\frac{S_{l,l} - S_{k,k} - b_{k+1,l}\tilde{S}_k}{\sqrt{2l-2k}} < \hat{b}_l + 3\varepsilon_l\right\} - P\left\{\hat{a}_l + \varepsilon_l \le \frac{1}{\sqrt{2l}}S_{l,l} < \hat{b}_l - \varepsilon_l\right\}$$

and

$$B_2 = \frac{1}{p_k p_l}\left[2P\left\{\left|\frac{1}{\sqrt{2l}}S_{k,k}\right| \ge \varepsilon_l\right\} + 2P\left\{\left|\frac{1}{\sqrt{2l}}b_{k+1,l}\tilde{S}_k\right| \ge \varepsilon_l\right\} + 4P\left\{|T_l| \ge \varepsilon_l\right\}\right].$$

So, by (3.1), Lemma 3.2 and the inequality $|\Phi(x) - \Phi(y)| \le \mathfrak{C}|x-y|$, $x, y \in R$, we obtain

$$\begin{aligned} B_1 \le& \left[F_{k,l}\left(\frac{\hat{b}_l + 3\varepsilon_l}{\sqrt{1-k/l}}\right) - \Phi\left(\frac{\hat{b}_l + 3\varepsilon_l}{\sqrt{1-k/l}}\right)\right] \\ &- \left[F_{k,l}\left(\frac{\hat{a}_l - 3\varepsilon_l}{\sqrt{1-k/l}}\right) - \Phi\left(\frac{\hat{a}_l - 3\varepsilon_l}{\sqrt{1-k/l}}\right)\right] \\ &+ \left[\Phi\left(\frac{\hat{b}_l + 3\varepsilon_l}{\sqrt{1-k/l}}\right) - \Phi\left(\frac{\hat{a}_l - 3\varepsilon_l}{\sqrt{1-k/l}}\right)\right] \\ &- [F_l(\hat{b}_l - \varepsilon_l) - \Phi(\hat{b}_l - \varepsilon_l)] \\ &+ [F_l(\hat{a}_l + \varepsilon_l) - \Phi(\hat{a}_l + \varepsilon_l)] - [\Phi(\hat{b}_l - \varepsilon_l) - \Phi(\hat{a}_l + \varepsilon_l)] \\ \le& \frac{\mathfrak{C}l}{(l-k)^{3/2}} + \Phi\left(\frac{\hat{b}_l + 3\varepsilon_l}{\sqrt{1-k/l}}\right) - \Phi\left(\frac{\hat{a}_l - 3\varepsilon_l}{\sqrt{1-k/l}}\right) \\ &+ \frac{\mathfrak{C}}{\sqrt{l}} - \Phi(\hat{b}_l - \varepsilon_l) + \Phi(\hat{a}_l + \varepsilon_l) \\ \le& \frac{\mathfrak{C}l}{(l-k)^{3/2}} + \left(\frac{\sqrt{l}}{\sqrt{l-k}} - 1\right)(\hat{b}_l - \hat{a}_l) + 6\varepsilon_l\frac{\sqrt{l}}{\sqrt{l-k}} + 2\varepsilon_l \\ \le& \mathfrak{C}\left(\frac{l}{(l-k)^{3/2}} + \frac{k}{\sqrt{l}\sqrt{l-k}} + \varepsilon_l\frac{\sqrt{l}}{\sqrt{l-k}}\right). \end{aligned}$$

So, by using Lemma 3.5, we have

$$\Sigma_4 \le 2\sum_{\substack{1\le k<l\le n \\ \log l > (\log n)^{\delta_2} \\ k < l/(\log l)^{2+\delta_3}}} \frac{1}{kl}\left(\frac{1}{p_l}B_1 + B_2\right) \le \mathfrak{C}(\log n)^{2-\epsilon}. \tag{4.5}$$

Combining (4.2)–(4.5) yields

$$\mathrm{Var}\left(\sum_{k=1}^{n}\frac{\alpha_k}{k}\right) \le \mathfrak{C}(\log n)^{2-\epsilon}.$$

So, our theorem is proved if (4.1) holds.

Now we drop the condition (4.1). Fix $x > 0$ and define

$$\tilde{a}_k = \max(\hat{a}_k, -x), \quad \tilde{b}_k = \min(\hat{b}_k, x), \quad \tilde{p}_k = P\{\tilde{a}_k \le U_k \le \tilde{b}_k\}.$$

Clearly, $\tilde{b}_k - \tilde{a}_k \le \min(2x, \mathfrak{C})$ and $\tilde{p}_k \le p_k$. Assume $\tilde{p}_k \ne 0$. Then

$$\begin{aligned}
&\frac{1}{p_k} I\left\{a_k \le \left(\frac{\prod_{i=1}^k S_i}{k!\mu^k}\right)^{\frac{1}{\gamma\sqrt{k}}} < b_k\right\} \\
&\le \frac{1}{\tilde{p}_k} I\{\tilde{a}_k \le U_k < \tilde{b}_k\} + \frac{1}{p_k}[I\{\hat{a}_k \le U_k < \tilde{a}_k\} + I\{\tilde{b}_k \le U_k < \hat{b}_k\}] \\
&\le \frac{1}{\tilde{p}_k} I\{\tilde{a}_k \le U_k < \tilde{b}_k\} + \frac{I\{U_k < -x\}}{P\{-x \le U_k < 0\}} + \frac{I\{U_k > x\}}{P\{0 \le U_k < x\}}.
\end{aligned}$$

Note that $T_k \xrightarrow{P} 0$ by (3.7) and $S_{k,k}/\sqrt{2k} \xrightarrow{d} \mathcal{N}$ by Lemma 1 of [13]. So, by Slutsky theorem, we have

$$U_k = T_k + \frac{S_{k,k}}{\sqrt{2k}} \xrightarrow{d} \mathcal{N}. \tag{4.6}$$

So,

$$\begin{aligned}
\lim_{k\to\infty} P\{-x \le U_k < 0\} &= \Phi(0) - \Phi(-x), \\
\lim_{k\to\infty} P\{0 \le U_k < x\} &= \Phi(x) - \Phi(0).
\end{aligned}$$

Applying Theorem A and Lemma 3.4, both

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^n \frac{I\{U_k < -x\}}{kP\{-x \le U_k < 0\}} = \frac{\Phi(-x)}{\Phi(0) - \Phi(-x)}, \qquad \text{a.s.}$$

and

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^n \frac{I\{U_k > x\}}{kP\{0 \le U_k < x\}} = \frac{1 - \Phi(x)}{\Phi(x) - \Phi(0)}, \qquad \text{a.s.}$$

hold. Note that

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^n \frac{\tilde{\alpha}_k}{k} = 1, \qquad \text{a.s.}$$

as $\tilde{b}_k - \tilde{a}_k \le \min(2x, \mathfrak{C})$, where

$$\tilde{\alpha}_k = \begin{cases} \frac{1}{\tilde{p}_k} I\{\tilde{a}_k \le U_k < \tilde{b}_k\}, & \text{if } \tilde{p}_k \ne 0; \\ 1, & \text{if } \tilde{p}_k = 0. \end{cases}$$

So,

$$\limsup_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^n \frac{\alpha_k}{k} \le 1 + 2\frac{1 - \Phi(x)}{\Phi(x) - \Phi(0)} \qquad \text{a.s.} \tag{4.7}$$

On the other hand, if $\tilde{p}_k \neq 0$,

$$\begin{aligned}
&\frac{1}{p_k} I \left\{ a_k \leq \left(\frac{\prod_{i=1}^{k} S_i}{k!\mu^k} \right)^{\frac{1}{\gamma\sqrt{k}}} < b_k \right\} \\
&\geq \frac{1}{\tilde{p}_k} I\{\tilde{a}_k \leq U_k < \tilde{b}_k\} \left(1 - \frac{p_k - \tilde{p}_k}{p_k}\right) \\
&\geq \frac{1}{\tilde{p}_k} I\{\tilde{a}_k \leq U_k < \tilde{b}_k\} \left(1 - \frac{P\{U_k < -x\} + P\{U_k > x\}}{\min(P\{0 \leq U_k < x\}, P\{-x \leq U_k < 0\})}\right)
\end{aligned}$$

and, by (4.6) and Lemma 3.4,

$$\lim_{k\to\infty} \frac{P\{U_k < -x\} + P\{U_k > x\}}{\min(P\{0 \leq U_k < x\}, P\{-x \leq U_k < 0\})} = \frac{2(1 - \Phi(x))}{\Phi(x) - \Phi(0)}.$$

So,

$$\liminf_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{\alpha_k}{k} \geq 1 - 2\frac{1 - \Phi(x)}{\Phi(x) - \Phi(0)}, \qquad \text{a.s.} \tag{4.8}$$

Now combining (4.7) and (4.8) and by the arbitrariness of x, we have

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{\alpha_k}{k} = 1 \qquad \text{a.s.}$$

The proof is complete. □

Acknowledgements

The authors would like to thank the referee for carefully reading the manuscript and for their comments which greatly improved the paper.

References

[1] Berkes I and Csáki E, A universal result in almost sure central limit theory, *Stoch. Process. Appl.* **94** (2001) 105–134

[2] Brosamler G A, An almost everywhere central limit theorem, *Math. Proc. Cambridge Philos. Soc.* **104** (1988) 561–574

[3] Csáki E, Földes A and Révész P, On almost sure local and global central limit theorems, *Probab. Theory Relat. Fields* **97** (1993) 321–337

[4] Gonchigdanzan K, Almost sure central limit theorems, Ph.D. Thesis (Cincinnati: University of Cincinnati) (2001)

[5] Gonchigdanzan K and Rempała G, A note on the almost sure limit theorem for the product of partial sums, *Appl. Math. Lett.* **19** (2006) 191–196

[6] Kosiński K, On the functional limits for sums of a function of partial sums, *Stat. Probab. Lett.* (2009) doi:10.1016/j.spl.2009.03.011

[7] Lacey M and Philipp W, A note on the almost sure central limit theorem, *Stat. Probab. Lett.* **9** (1990) 201–205

[8] Lu X and Qi Y, A note on asymptotic distribution of products of sums, *Stat. Probab. Lett.* **68** (2004) 407–413

[9] Matuła P and Stępień I, Weak convergence of products of sums of independent and non-identically distributed random variables, *J. Math. Anal. Appl.* **353** (2009) 49–54
[10] Miao Y, Central limit theorem and almost sure central limit theorem for the product of some partial sums, *Proc. Indian Acad. Sci. (Math. Sci.)* **118** (2008) 289–294
[11] Qi Y, Limit distributions for products of sums, *Stat. Probab. Lett.* **1** (2003) 93–100
[12] Petrov V V, Limit Theorems of Probability Theory: Sequences of Independent Random Variables (Oxford: Oxford University Press) (1995)
[13] Rempała G and Wesołowski J, Asymptotics for products of sums and U-statistics, *Electron. Commun. Probab.* **7** (2002) 47–54
[14] Schatte P, On strong versions of the central limit theorem, *Math. Nachr.* **137** (1988) 249–256
[15] Schatte P, On the central limit theorem with almost sure convergence, *Probab. Math. Stat.* **11** (1991) 237–246
[16] Zhang L and Huang W, A note on the invariance principle of the product of sums of random variables, *Electron. Commun. Probab.* **12** (2007) 51–56