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A note on existence and stability of solutions for semilinear Dirichlet problems

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Abstract. We provide existence and stability results for a fourth-order semilinear Dirichlet problem in the case when both the coefficients of the differential operator and the nonlinear term depend on the numerical parameter. We use a dual variational method.

Keywords. Dirichlet problem; dual variational method; existence of solutions; stability of solutions.

1. Introduction

Due to their importance in physics and mechanics, higher order problems with various types of boundary conditions (mainly fourth-order problems, see for example [1] and references therein) have been investigated recently. We refer to [5,9,10], to mention a few works. The dual least action principle, the mountain pass geometry, the usage of topological arguments, the Krasnosel'skij fixed point theorem, the strong maximum principle and other methods could be applied in order to tackle the existence of solutions for higher order O.D.E. For us the most convenient method seems to be the dual variational method mentioned in [7], which allows for investigation of the stability of solutions for Dirichlet problem with respect to a numerical parameter. In [3] we have obtained some results in this field using a dual variational approach and concerning nonlinear terms which are subject to some general local growth conditions in an interval and which have convex primitives. The aim of this note is to improve the dual methodology from [3] in order to consider such nonlinear terms whose primitive is the difference of two convex functions with local growth imposed on the minuend term. Using an abstract realization of the given problem we simplify the dual variational method from [3] and reach the existence result. The dual method of the calculus of variations originates from [7], however in the present paper we apply an apparently different and simplified methodology in obtaining the existence result. While we do not use stronger assumptions than in [3], we still reach the existence but with simpler arguments which do not involve complicated investigations concerning the relations between the primal and the dual functional. In fact, we do not need to define a dual functional at all. We also obtain some additional properties of the solutions. We further investigate the stability of solutions in such a case when both the differential operator and the nonlinear term depend on a parameter and with weaker assumptions when compared with [3]. We conclude with an example of nonlinear terms satisfying our assumptions.

2. Problem formulation and assumptions

We will consider the following family of Dirichlet problems

$$\beta_k \frac{\mathrm{d}^4}{\mathrm{d}t^4} x(t) + \gamma_k \frac{\mathrm{d}^2}{\mathrm{d}t^2} x(t) + \delta_k x(t) + G_x^k(t, x(t)) = F_x^k(t, x(t)),$$

$$x(0) = x(\pi) = \operatorname{sgn}(\beta_k) \frac{\mathrm{d}}{\mathrm{d}t} x(0) = \operatorname{sgn}(\beta_k) \frac{\mathrm{d}}{\mathrm{d}t} x(\pi) = 0, \tag{2.1}$$

where for all $k = 0, 1, 2, ...$ the constants β_k, γ_k, δ_k are such that $\beta_k - \gamma_k - |\delta_k| > 0$, $\beta_k \in [\beta', \beta'']$, $\gamma_k \in [\gamma', \gamma'']$, $\delta_k \in [\delta', \delta'']$, $\gamma'' < 0$, $\beta' \geq 0$; sequences $\{\beta_k\}_{k=1}^{\infty}$, $\{\gamma_k\}_{k=1}^{\infty}$, $\{\delta_k\}_{k=1}^{\infty}$ are convergent to β_0, γ_0, δ_0; $\operatorname{sgn}(\beta_k) = 0$ in case $\beta_k = 0$ and $\operatorname{sgn}(\beta_k) = 1$ in case $\beta_k > 0$.

F_x^k means the derivative of function F^k. We assume that $F^k, F_x^k, G^k, G_x^k : [0, \pi] \times R \to R$ are Caratheodory functions; F^k and G^k are continuously differentiable in x for a.e. $t \in [0, \pi]$; G_x^k is the L_2 Caratheodory function, i.e. for any $c > 0$, function $t \to G(t, x)$ is integrable with square for $x \in [-c, c]$; F^k, F_x^k, G^k are the L_1 Caratheodory function.

Let $\{d_k\}_{k=1}^{\infty}$ be a decreasing sequence of positive numbers bounded away from 0 and let $d_0 > d_k$ for all $k = 1, 2, ...$ We assume for all $k = 0, 1, 2, ...$ as follows.

H1: For a.e. $t \in [0, \pi]$, we have

$$\max_{x \in [-d_k, d_k]} \left| F_x^k(t, x) \right| \leq \frac{(\beta_k - \gamma_k - |\delta_k|)}{\pi} d_k; \tag{2.2}$$

H2: $F_x^k(t, 0) \neq 0$ for a.e. $t \in [0, \pi]$;

H3: G^k is convex with respect to the second variable in R for a.e. $t \in [0, \pi]$ and $G_x^k(t, x)x \geq 0$ for all $x \in R$ and for a.e. $t \in [0, \pi]$; $G_x^k(t, 0) = 0$ for a.e. $t \in [0, \pi]$;

H4: F^k is convex with respect to x in R for a.e. $t \in [0, \pi]$;

H5: F_x^k and G_x^k are locally Lipschitz in x uniformly in $t \in [0, \pi]$ and $k = 0, 1, 2....$

Remark 2.1. Assumptions H1, H3, H4 (convexity) are required in order to apply the dual variational method since it involves some calculations using the Fenchel–Young transform (see [2] for its definition and properties). Assumption H2 guarantees that the solutions which we obtain are not trivial. Assumption H5 is used in proving the stability of solutions. Stability is understood in the sense described in Theorem 4.2.

3. Auxiliary results

Let $k \in N \cup \{0\}$ be fixed. If $\beta_k > 0$ we define

$$X_k = \Bigg\{ u \in H_0^2(0,\pi) : \ \left\| \frac{\mathrm{d}}{\mathrm{d}t} u \right\|_{L^2(0,\pi)} \leq \left(\sqrt{\pi}\right)^{-1} d_k$$
$$\left\| \frac{\mathrm{d}^2}{\mathrm{d}t^2} u \right\|^2_{L^2(0,\pi)} \leq \frac{1}{\beta' \pi} (\beta'' - 2\gamma') d_0^2, \ u(t) \in [-d_k, d_k] \text{ on } [0,\pi] \Bigg\},$$

otherwise we set

$$X_k = \Bigg\{ u \in H_0^1(0,\pi) : \ \left\| \frac{\mathrm{d}}{\mathrm{d}t} u \right\|_{L^2(0,\pi)} \leq \left(\sqrt{\pi}\right)^{-1} d_k$$
$$u(t) \in [-d_k, d_k] \ \text{ on } \ [0,\pi] \Bigg\}.$$

The Euler action functional associated with (2.1), $J_k : X_k \to R$ is given by the formula

$$J_k(u) = \frac{\beta_k}{2} \int_0^\pi \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} u(t) \right)^2 \mathrm{d}t - \frac{\gamma_k}{2} \int_0^\pi \left(\frac{\mathrm{d}}{\mathrm{d}t} u(t) \right)^2 \mathrm{d}t + \frac{\delta_k}{2} \int_0^\pi u^2(t)\,\mathrm{d}t$$
$$+ \int_0^\pi G^k(t, u(t))\,\mathrm{d}t - \int_0^\pi F^k(t, u(t))\,\mathrm{d}t.$$

Note that J_k is well-defined for $u \in X_k$.

For a fixed $x \in X_k$ we now consider the following functional:

$$\tilde{J}_k(f) = \frac{\beta_k}{2} \int_0^\pi \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} f(t) \right)^2 \mathrm{d}t - \frac{\gamma_k}{2} \int_0^\pi \left(\frac{\mathrm{d}}{\mathrm{d}t} f(t) \right)^2 \mathrm{d}t + \frac{\delta_k}{2} \int_0^\pi f^2(t)\mathrm{d}t$$
$$+ \int_0^\pi G^k(t, f(t))\,\mathrm{d}t - \int_0^\pi F_x^k(t, x(t))\, f(t)\mathrm{d}t.$$

In contrast to $\tilde{J}_k$, functional J_k can be investigated either on $H_0^2(0,\pi)$ with $\beta_k > 0$ or on $H_0^1(0,\pi)$ with $\beta_k = 0$. This happens since term $\int_0^\pi F_x^k(t, x(t)) f(t)\mathrm{d}t$ is linear in f and $t \to F_x^k(t, x(t))$ is a fixed function which belongs to $L^\infty(0,\pi)$. Functional $\tilde{J}_k$ serves as an Euler functional for the Dirichlet problem

$$\beta_k \frac{\mathrm{d}^4}{\mathrm{d}t^4} u(t) + \gamma_k \frac{\mathrm{d}^2}{\mathrm{d}t^2} u(t) + \delta_k u(t) + G_x^k(t, u(t)) = F_x^k(t, x(t))$$
$$u(0) = u(\pi) = \operatorname{sgn}(\beta_k) \frac{\mathrm{d}}{\mathrm{d}t} u(0) = \operatorname{sgn}(\beta_k) \frac{\mathrm{d}}{\mathrm{d}t} u(\pi) = 0. \tag{3.1}$$

By the assumptions made on the nonlinear term G^k, we see that $\tilde{J}_k$ is differentiable in the sense of Gâteaux.

The weak solutions to (3.1) are in fact the critical points to functional $\tilde{J}_k$.

Now we describe what we mean by the weak solution. Let $\beta_k > 0$. Function $u \in H_0^2(0,\pi)$ is a weak solution to (3.1) if for any $h \in H_0^2(0,\pi)$ we have

$$\beta_k \int_0^\pi \frac{d^2}{dt^2}u(t)\frac{d^2}{dt^2}h(t)dt - \gamma_k \int_0^\pi \frac{d}{dt}u(t)\frac{d}{dt}h(t)dt + \delta_k \int_0^\pi u(t)h(t)dt$$
$$+ \int_0^\pi G_x^k(t,u(t))\,h(t)dt - \int_0^\pi F_x^k(t,x(t))\,h(t)dt = 0.$$

Due to the higher order du Bois Reymond lemma (see [8]), we get immediately the fact that $u \in H_0^2(0,\pi) \cap H^4(0,\pi)$.

Let $\beta_k = 0$. Function $u \in H_0^1(0,\pi)$ is a weak solution to (3.1) if for any $h \in H_0^1(0,\pi)$ we have

$$-\gamma_k \int_0^\pi \frac{d}{dt}u(t)\frac{d}{dt}h(t)dt + \delta_k \int_0^\pi u(t)h(t)dt + \int_0^\pi G_x^k(t,u(t))\,h(t)dt$$
$$- \int_0^\pi F_x^k(t,x(t))\,h(t)dt = 0.$$

Due to the fundamental lemma of the calculus of variations (see [6]), we obtain that $u \in H_0^1(0,\pi) \cap H^2(0,\pi)$.

Now we provide some properties of the functional $\tilde{J}_k$ considering separately cases when $\beta_k > 0$ and when $\beta_k = 0$.

Lemma 3.1. *We assume* H1–H4. *Let $\beta_k > 0$. For each $x \in X_k$ there exists at least one solution $u \in X_k \cap H^4(0,\pi)$ to the Dirichlet problem* (3.1) *such that* $\tilde{J}_k(u) = \inf_{f \in H_0^2(0,\pi)} \tilde{J}_k(f)$.

Moreover, if the Dirichlet problem (3.1) *admits any weak solution $u \in H_0^2(0,\pi)$, then necessarily $u \in X_k \cap H^4(0,\pi)$.*

Proof. We will use a direct variational approach (see for example [6]), proving that functional $\tilde{J}_k$ is weakly lower semicontinuous and coercive.

Firstly we show that $\tilde{J}_k$ is coercive. By the convexity of G and by H3, we get for a.e. $t \in [0,\pi]$ and any $f \in H_0^2(0,\pi)$,

$$G^k(t,f(t)) - G^k(t,0) \geq G_x^k(t,0)f(t) = 0.$$

Hence

$$\int_0^\pi G^k(t,f(t))dt \geq \int_0^\pi G^k(t,0)dt. \tag{3.2}$$

Next, using (2.2) and by the Schwartz and Poincaré inequalities,

$$\int_0^\pi f^2(t)dt \leq \int_0^\pi \left(\frac{d}{dt}f(t)\right)^2 dt \leq \int_0^\pi \left(\frac{d^2}{dt^2}f(t)\right)^2 dt.$$

We see that for all $f \in H_0^2(0,\pi)$,

$$\begin{aligned}\int_0^\pi F_x^k(t,x(t))\, f(t)\mathrm{d}t &\le \frac{(\beta_k-\gamma_k-|\delta_k|)}{\pi} d_k\sqrt{\pi}\, \|f\|_{L^2(0,\pi)} \\ &\le \frac{(\beta_k-\gamma_k-|\delta_k|)}{\pi} d_k\sqrt{\pi}\left\|\frac{\mathrm{d}}{\mathrm{d}t}f\right\|_{L^2(0,\pi)} \\ &\le \frac{(\beta_k-\gamma_k-|\delta_k|)}{\pi} d_k\sqrt{\pi}\left\|\frac{\mathrm{d}^2}{\mathrm{d}t^2}f\right\|_{L^2(0,\pi)}. \end{aligned} \tag{3.3}$$

Since $\gamma_k < 0$, we see that $-\frac{\gamma_k}{2}\int_0^\pi\left(\frac{\mathrm{d}}{\mathrm{d}t}f(t)\right)^2\mathrm{d}t > 0$. Therefore

$$-\frac{\gamma_k}{2}\int_0^\pi\left(\frac{\mathrm{d}}{\mathrm{d}t}f(t)\right)^2\mathrm{d}t \ge -\frac{\gamma_k}{2}\int_0^\pi f^2(t)\mathrm{d}t$$

and obviously

$$\frac{\delta_k}{2}\int_0^\pi f^2(t)\mathrm{d}t \ge \frac{-|\delta_k|}{2}\int_0^\pi f^2(t)\mathrm{d}t \ge \frac{-|\delta_k|}{2}\int_0^\pi\left(\frac{\mathrm{d}}{\mathrm{d}t}f(t)\right)^2\mathrm{d}t. \tag{3.4}$$

Hence for all $f \in H_0^2(0,\pi)$ it holds that

$$\begin{aligned}&\frac{\beta_k}{2}\int_0^\pi\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}f(t)\right)^2\mathrm{d}t - \frac{\gamma_k}{2}\int_0^\pi\left(\frac{\mathrm{d}}{\mathrm{d}t}f(t)\right)^2\mathrm{d}t + \frac{\delta_k}{2}\int_0^\pi f^2(t)\mathrm{d}t \\ &\quad\ge \frac{\beta_k}{2}\int_0^\pi\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}f(t)\right)^2\mathrm{d}t - \frac{\gamma_k}{2}\int_0^\pi f^2(t)\mathrm{d}t - \frac{|\delta_k|}{2}\int_0^\pi f^2(t)\mathrm{d}t. \end{aligned} \tag{3.5}$$

Therefore (3.2), the second inequality in (3.3) and (3.5) we see that

$$\begin{aligned}\tilde{J}_k(f) \ge{}& \frac{\beta_k}{2}\left\|\frac{\mathrm{d}^2}{\mathrm{d}t^2}f\right\|^2_{L^2(0,\pi)} - \frac{\gamma_k}{2}\int_0^\pi f^2(t)\mathrm{d}t - \frac{|\delta_k|}{2}\int_0^\pi f^2(t)\mathrm{d}t \\ &+ \int_0^\pi G^k(t,0)\mathrm{d}t - \frac{(\beta_k-\gamma_k-|\delta_k|)}{\pi} d_k\sqrt{\pi}\left\|\frac{\mathrm{d}^2}{\mathrm{d}t^2}f(t)\right\|_{L^2(0,\pi)} \end{aligned} \tag{3.6}$$

for all $f \in H_0^2(0,\pi)$.

Let sequence $\{f_n\}_{n=1}^\infty \subset H_0^2(0,\pi)$ satisfy $\left\|\frac{\mathrm{d}^2}{\mathrm{d}t^2}f_n\right\|_{L^2(0,\pi)} \to \infty$. Then either

(i) there exists a constant $M > 0$ such that $\|f_n\|_{L^2(0,\pi)} \le M$ for all $n \in N$ or
(ii) $\|f_n\|_{L^2(0,\pi)} \to \infty$.

In case (i) we observe that from (3.6) we have for some constant C_1,

$$\begin{aligned}\tilde{J}_k(f_n) \ge{}& \frac{\beta_k}{2}\left\|\frac{\mathrm{d}^2}{\mathrm{d}t^2}f_n\right\|^2_{L^2(0,\pi)} + C_1 + \int_0^\pi G^k(t,0)\mathrm{d}t \\ &- \frac{(\beta_k-\gamma_k-|\delta_k|)}{\pi} d_k\sqrt{\pi}\left\|\frac{\mathrm{d}^2}{\mathrm{d}t^2}f_n(t)\right\|_{L^2(0,\pi)} \to \infty \end{aligned}$$

as $\left\| \frac{d^2}{dt^2} f_n \right\|_{L^2(0,\pi)} \to \infty$. Therefore $\tilde{J}_k(f) \to \infty$ as $\left\| \frac{d^2}{dt^2} f \right\|^2_{L^2(0,\pi)} \to \infty$ and $\tilde{J}_k$ is coercive.

In case (ii) we obtain from (3.5), by the Poincaré inequality and from the first inequality in (3.3) that

$$\tilde{J}_k(f_n) \geq \frac{(\beta_k - \gamma_k - |\delta_k|)}{2} \|f_n\|^2_{L^2(0,\pi)} + \int_0^\pi G^k(t,0)\mathrm{d}t - \frac{(\beta_k - \gamma_k - |\delta_k|)}{\pi} d_k \sqrt{\pi} \|f_n(t)\|_{L^2(0,\pi)} \to \infty$$

as $\|f_n\|_{L^2(0,\pi)} \to \infty$. Therefore $\tilde{J}_k$ is coercive.

Now we prove that $\tilde{J}_k$ is weakly lower semicontinous. Let us take a sequence $\{f_n\}_{n=1}^\infty \subset H_0^2(0,\pi)$ weakly convergent to a certain $f \in H_0^2(0,\pi)$. Then the sequences $\left\{\frac{d}{dt} f_n\right\}_{n=1}^\infty$, $\{f_n\}_{n=1}^\infty$ are strongly convergent in $L^2(0,\pi)$ and $C(0,\pi)$ respectively – possibly up to a subsequence which we assume to be chosen. The proof of this assertion follows as in [6]. Denote this subsequence by $\{f_n\}_{n=1}^\infty$ again. Thus by the properties of the lower limit

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{2}\left(\beta_k \left\|\frac{d^2}{dt^2} f_n\right\|^2_{L^2(0,\pi)} - \gamma_k \left\|\frac{d}{dt} f_n\right\|^2_{L^2(0,\pi)} + \delta_k \|f_n\|^2_{L^2(0,\pi)}\right) \\
&\geq \liminf_{n\to\infty} \frac{1}{2}\beta_k \left\|\frac{d^2}{dt^2} f_n\right\|^2_{L^2(0,\pi)} \\
&\quad + \lim_{n\to\infty} \frac{1}{2}\left(-\gamma_k \left\|\frac{d}{dt} f_n\right\|^2_{L^2(0,\pi)} + \delta_k \|f_n\|^2_{L^2(0,\pi)}\right) \\
&\geq \frac{1}{2}\left(\beta_k \left\|\frac{d^2}{dt^2} f\right\|^2_{L^2(0,\pi)} - \gamma_k \left\|\frac{d}{dt} f\right\|^2_{L^2(0,\pi)} + \delta_k \|f\|^2_{L^2(0,\pi)}\right).
\end{aligned}$$

Since G^k is weakly lower semicontinuous we see

$$\liminf_{n\to\infty} \int_0^\pi G^k(t, f_n(t))\mathrm{d}t \geq \int_0^\pi G^k(t, f(t))\mathrm{d}t$$

and obviously

$$\lim_{n\to\infty} \int_0^\pi F_x^k(t, x(t))\, f_n(t)\mathrm{d}t = \int_0^\pi F_x^k(t, x(t)) f(t)\mathrm{d}t.$$

Hence $\tilde{J}_k$ weakly lower semicontinuous on $H_0^2(0,\pi)$.

Since $\tilde{J}_k$ is coercive, weakly lower semicontinuous and Gâteaux differentiable, it has a minimizer u over $H_0^2(0,\pi)$ which satisfies its Euler equation (3.1) in the weak sense. Therefore by the remarks preceding the formulation of the lemma, we see that $u \in H_0^2(0,\pi) \cap H^2(0,\pi)$.

Now we show that $u \in X_k$. Note that now the equation in (3.1) is satisfied for a.e. $t \in [0, \pi]$. We multiply (3.1) by u and take integrals. Then we have after integration by parts,

$$\beta_k \int_0^\pi \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} u(t)\right)^2 \mathrm{d}t - \gamma_k \int_0^\pi \left(\frac{\mathrm{d}}{\mathrm{d}t} u(t)\right)^2 \mathrm{d}t + \delta_k \int_0^\pi u^2(t)\mathrm{d}t + \int_0^\pi G_x^k(t, u(t))u(t)\mathrm{d}t = \int_0^\pi F_x^k(t, x(t))u(t)\mathrm{d}t. \tag{3.7}$$

The left-hand side of (3.7) is estimated as follows. By (3.5) and the relation (3.4) – both multiplied by 2 – using relation

$$\int_0^\pi G_x^k(t, u(t))u(t) \geq 0,$$

we get

$$\beta_k \int_0^\pi \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} u(t)\right)^2 \mathrm{d}t - \gamma_k \int_0^\pi \left(\frac{\mathrm{d}}{\mathrm{d}t} u(t)\right)^2 \mathrm{d}t + \delta_k \int_0^\pi u^2(t)\mathrm{d}t + \int_0^\pi G_x^k(t, u(t))u(t)\mathrm{d}t \geq (\beta_k - \gamma_k - |\delta_k|) \left\| \frac{\mathrm{d}}{\mathrm{d}t} u \right\|_{L^2(0,\pi)}^2.$$

Therefore we have by the above and the middle inequality in relation (3.3),

$$(\beta_k - \gamma_k - |\delta_k|) \left\| \frac{\mathrm{d}}{\mathrm{d}t} u \right\|_{L^2(0,\pi)}^2 \leq \int_0^\pi F_x^k(t, x(t))u(t)\mathrm{d}t \leq \frac{(\beta_k - \gamma_k - |\delta_k|)}{\sqrt{\pi}} d_k \left\| \frac{\mathrm{d}}{\mathrm{d}t} u \right\|_{L^2(0,\pi)}. \tag{3.8}$$

Hence $\left\| \frac{\mathrm{d}}{\mathrm{d}t} u \right\|_{L^2(0,\pi)} \leq \left(\sqrt{\pi}\right)^{-1} d_k$ and by Sobolev's inequality we now get

$$\max_{t\in[0,\pi]} |u(t)| \leq \sqrt{\pi} \left\| \frac{\mathrm{d}}{\mathrm{d}t} u \right\|_{L^2(0,\pi)} \leq d_k.$$

In order to obtain the second estimation we use $\left\| \frac{\mathrm{d}}{\mathrm{d}t} u \right\|_{L^2(0,\pi)} \leq \left(\sqrt{\pi}\right)^{-1} d_k$, (3.5) and (3.8) in order to calculate

$$\beta_k \left\| \frac{\mathrm{d}^2}{\mathrm{d}t^2} u \right\|_{L^2(0,\pi)}^2 \leq \frac{(\beta_k - \gamma_k - |\delta_k|)}{\pi} d_k^2 - \gamma_k \left\| \frac{\mathrm{d}}{\mathrm{d}t} u \right\|_{L^2(0,\pi)}^2 + |\delta_k| \, \|u\|_{L^2(0,\pi)}^2.$$

Hence

$$\left\| \frac{\mathrm{d}^2}{\mathrm{d}t^2} u \right\|_{L^2(0,\pi)}^2 \leq \frac{1}{\beta_k} \left(\frac{(\beta_k - \gamma_k - |\delta_k|)}{\pi} d_k^2 - \gamma_k \frac{d_k^2}{\pi} + |\delta_k| \frac{d_k^2}{\pi} \right) = \frac{1}{\beta_k \pi} (\beta_k - 2\gamma_k)\, d_k^2 \leq \frac{1}{\beta' \pi} \left(\beta'' - 2\gamma'\right) d_0^2.$$

Now suppose that $u \in H_0^2(0,\pi)$ is any weak solution to (3.1). Then reasoning as above we show that $u \in X_k \cap H^2(0,\pi)$. □

Lemma 3.2. *We assume* H1–H4. *Let* $\beta_k = 0$. *For each* $x \in X_k$ *there exists at least one solution* $u \in X_k \cap H^2(0,\pi)$ *to the Dirichlet problem*

$$\gamma_k \frac{d^2}{dt^2}u(t) + \delta_k u(t) + G_x^k(t,u(t)) = F_x^k(t,x(t)),$$

$$u(0) = u(\pi) = 0 \tag{3.9}$$

such that $\tilde{J}_k(u) = \inf_{f \in H_0^2(0,\pi)} \tilde{J}_k(f)$.

Moreover, *if the Dirichlet problem* (3.9) *admits a weak solution* $u \in H_0^1(0,\pi)$, *then* $u \in X_k \cap H^2(0,\pi)$.

Proof. We will only provide the main steps of the proof since it is similar to the proof of Lemma 3.1. Now functional $\tilde{J}_k$ has the following form:

$$\tilde{J}_k(f) = -\frac{\gamma_k}{2}\int_0^\pi \left(\frac{d}{dt}f(t)\right)^2 dt + \frac{\delta_k}{2}\int_0^\pi f^2(t)dt + \int_0^\pi G^k(t,f(t))dt - \int_0^\pi F_x^k(t,x(t))f(t)dt$$

and is considered on $H_0^1(0,\pi)$. Exactly as in the proof of Lemma 3.1, we show that $\tilde{J}_k$ is weakly lower semicontinuous on $H_0^1(0,\pi)$.

In order to demonstrate coercivity we note that by (3.2) and (3.3) and since $(-\gamma_k - |\delta_k|) > 0$ it follows that

$$\tilde{J}_k(f) \geq \frac{(-\gamma_k - |\delta_k|)}{2}\left\|\frac{d}{dt}f(t)\right\|^2_{L^2(0,\pi)} + \int_0^\pi G^k(t,0)dt - \frac{(-\gamma_k - |\delta_k|)}{\sqrt{\pi}}d_k\left\|\frac{d}{dt}f(t)\right\|_{L^2(0,\pi)} \tag{3.10}$$

for all $f \in H_0^1(0,\pi)$. Hence $\tilde{J}_k$ is coercive.

The remaining part of the proof follows exactly as in the proof of Lemma 3.1. □

4. Main results

Theorem 4.1. *Let us assume* H1–H4.

(A) *Let* $\beta_k > 0$. *There exists at least one solution* $x_k \in X_k \cap H^4(0,\pi)$ *to the Dirichlet problem* (2.1) *such that* $J_k(x_k) = \inf_{x \in X_k} J_k(x)$.

(B) *Let* $\beta_k = 0$. *There exists at least one solution* $x_k \in X_k \cap H^2(0,\pi)$ *to the Dirichlet problem* (2.1) *such that* $J_k(x_k) = \inf_{x \in X_k} J_k(x)$.

Theorem 4.1 asserts that the Dirichlet problem (2.1) has at least one solution which need not be unique. Problem (2.1) may still have solutions outside X_k.

Now, we turn to the stability results. In the first one, the limit problem has the same order as all problems in the sequence of Dirichlet problems, while in the second one, we observe that the limit problem can have lower order.

Theorem 4.2. *We assume* H1–H5 *and that* $\{\beta_k\}_{k=1}^{\infty}$ *is a decreasing sequence bounded away from* 0*. Let* $\{x_k\}_{k=1}^{\infty}$ *be the sequence of solutions to* (2.1) *for* $k = 1, 2, \ldots$ *. Assume that for each* $x \in X_0$ *there exists a subsequence* $\{k_i\}_{i=1}^{\infty}$ *such that*

$$F_x^{k_i}(t, x(t)) \underset{i\to\infty}{\to} F_x^0(t, x(t)) \text{ a.e. in } [0, \pi],$$

$$G_x^{k_i}(t, x(t)) \underset{i\to\infty}{\to} G_x^0(t, x(t)) \text{ a.e. in } [0, \pi]. \tag{4.1}$$

Then, there exists $\bar{x} \in X_0$ *and a subsequence* $\left\{x_{k_i}\right\}_{i=1}^{\infty}$ *of the sequence* $\{x_k\}_{k=1}^{\infty}$ *such that* $\lim_{i\to\infty} x_{k_i} = \bar{x}$ *(strongly in* $H_0^2(0,\pi)$*, weakly in* $H_0^2(0,\pi) \cap H^4(0,\pi)$*,* $x_{k_i} \rightrightarrows \bar{x}$*) and*

$$\beta_0 \frac{d^4}{dt^4}\bar{x}(t) + \gamma_0 \frac{d^2}{dt^2}\bar{x}(t) + \delta_0 \bar{x}(t) + G_x^0(t, \bar{x}(t)) = F_x^0(t, \bar{x}(t))$$

$$\bar{x}(0) = \bar{x}(\pi) = \dot{\bar{x}}(0) = \dot{\bar{x}}(\pi) = 0. \tag{4.2}$$

When $\beta_0 = 0$ and $\beta_k > 0$ the following theorem is obtained.

Theorem 4.3. *We assume* H1–H5 *and that sequence* $\{\beta_k\}_{k=1}^{\infty}$ *is decreasing and convergent to* $\beta_0 = 0$*. Let* $\{x_k\}_{k=1}^{\infty}$ *be the sequence of solutions to* (2.1) *for* $k = 1, 2, \ldots$ *. Assume that for each* $x \in X_0$ *there exists a subsequence* k_i *such that* (4.1) *holds. There exists* $\bar{x} \in X_0$ *and a subsequence* $\left\{x_{k_i}\right\}_{i=1}^{\infty}$ *of the sequence* $\{x_k\}_{k=1}^{\infty}$ *such that* $\lim_{i\to\infty} x_{k_i} = \bar{x}$ *strongly in* $H_0^1(0,\pi)$*, weakly in* $H_0^1(0,\pi) \cap H^2(0,\pi)$*,* $x_{k_i} \rightrightarrows \bar{x}$ *such that*

$$\gamma_0 \frac{d^2}{dt^2}\bar{x}(t) + \delta_0 \bar{x}(t) + G_x^0(t, \bar{x}(t)) = F_x^0(t, \bar{x}(t)),$$

$$\bar{x}(0) = \bar{x}(\pi) = 0.$$

5. The abstract existence result and proof of Theorem 4.1

As an immediate consequence of the proof of Lemma 3.1 we get the following:

Theorem 5.1. *We assume* H1–H4*. There exists* $x_k \in X_k$ *such that* $J_k(x_k) = \inf_{x\in X_k} J_k(x)$*.*

Proof. Using the same arguments as in the proof of Lemma 3.1 we get that J_k is weakly lower semicontinuous on X_k. We employ the Lebesgue dominated convergence theorem and (2.2) to the term with F^k. Since J_k is weakly lower semicontinuous on X_k and since X_k is weakly compact and convex in either $H_0^1(0,\pi)$ or $H_0^2(0,\pi)$ (depending on the sign of β_k) we get the assertion. □

Theorem 5.1 provides the existence of an $x \in X_k$ which is a candidate for a solution of (2.1). Since X_k is not open, we simply cannot set the derivative equal to 0 in order to show that x is a solution of (2.1). To prove this assertion we modify the abstract dual variational method from [3] providing some general abstract scheme in which higher order problems can be investigated.

Let Y be a separable real Hilbert space with scalar product $\langle \cdot\,,\cdot\rangle$. Let $L : D(L) \subset Y \to Y$ be a densely defined linear operator, i.e. $D(L)$ is a dense subspace of Y and L is a self-adjoint and positive definite linear operator. There exists a densely defined self-adjoint square root operator $S : D(S) \to Y$. We observe that $Sx \in D(S)$ for any $x \in D(L)$ and $S^2 = L$ (see [4]). On $D(S)$ one uses norm $\|x\|_{D(S)} = \sqrt{\|Sx\|_Y + \|x\|_Y}$ which makes it into a complete space.

By $F^* : Y \to R$ we mean the Fenchel–Young conjugate of a functional $F : Y \to R$ (see [2]), given by

$$F^*(u) = \sup_{x\in Y} \{\langle u, x\rangle - F(x)\} .$$

Likewise we can define F^{**}. It is known that $F = F^{**}$ when F is a convex and weakly lower semicontinuous functional. For other functionals there may exist points $y \in Y$, for example point of unique global minima, for which $F(y) = F^{**}(y)$, while $F \neq F^{**}$ in the whole domain.

We assume that

A1: Any bounded sequence in $D(S)$ contains a subsequence convergent in Y;

A2: $G : Y \to R$ is convex lower semicontinuous and Gâteaux differentiable functional bounded on bounded sets; $F : Y \to R$ is a Gâteaux differentiable continuous functional; $F_x(0) \neq 0$.

Under the above assumptions we will investigate the existence of solution to equation

$$Lx + G_x(x) = F_x(x). \tag{5.1}$$

We call $u \in D(S)$ a (weak) solution to (5.1) when for all $f \in D(S)$,

$$\langle Su, Sf\rangle + \langle G_x(u), f\rangle - \langle F_x(u), f\rangle = 0$$

holds.

Using the action functional $J : D(S) \to R$ defined by

$$J(x) = \frac{1}{2}\langle Sx, Sx\rangle + G(x) - F(x),$$

we are able to reach weak solutions. We also give conditions which guarantee that solutions are strong ones, i.e. when $u \in D(L)$, we obtain

$$\langle Lu + G_x(u) - F_x(u), f\rangle = 0 \text{ for all } f \in D(S)$$

and therefore $Lu + G_x(u) - F_x(u) = 0$.

Theorem 5.2. *We assume* A1–A2. *Let $X \subset D(S)$ be a weakly compact convex set. There exists $x \in X$ such that $J(x) = \inf_{u\in X} J(u)$. Let us further assume that any* (*weak*) *solution $\tilde{x}$ to*

$$L\tilde{x} + G_x(\tilde{x}) = F_x(x) \tag{5.2}$$

belongs to $X \cap D(L)$ *and that* $F^{**}(x) = F(x)$. *Then* x *satisfies equation* (5.1). *Moreover,* $x \in D(L)$.

Proof. Firstly we show that J is weakly lower semicontinuous on X. Let us take any sequence $\{x_n\}_{n=1}^{\infty} \subset X$. Since X is weakly compact $\{x_n\}_{n=1}^{\infty}$ contains a weakly convergent subsequence whose limit, due to the convexity, belongs to X. By A1 this sequence converges strongly in Y, possibly up to a subsequence. Let us denote the resulting subsequence by $\{x_n\}_{n=1}^{\infty}$ and its limit by $\bar{x}$. Since $x \to \frac{1}{2}\langle Sx, Sx\rangle$ is weakly lower semicontinuous in $D(S)$ and since a convex weakly lower semicontinuous functional G on a bounded set is necessarily continuous (see [2]) we observe by continuity of F that

$$\begin{aligned}
&\liminf_{n\to\infty}\left(\frac{1}{2}\langle Sx_n, Sx_n\rangle + G(x_n) - F(x_n)\right)\\
&\quad\geq \liminf_{n\to\infty}\left(\frac{1}{2}\langle Sx_n, Sx_n\rangle + G(x_n)\right) - \lim_{n\to\infty} F(x_n)\\
&\quad\geq \frac{1}{2}\langle S\bar{x}, S\bar{x}\rangle + G(\bar{x}) - F(\bar{x}).
\end{aligned}$$

Thus J is weakly lower-semicontinuous on X. Now, since X is weakly compact in $D(S)$ and since J is weakly lower semicontinuous there exists $x \in X$ such that $J(x) = \inf_{u\in X} J(u)$.

Let us set $p = S\tilde{x}$ and $q = G_x(\tilde{x})$ and denote

$$J_D(p, q) = -\frac{1}{2}\langle p, p\rangle + F^*(Sp + q) - G^*(q).$$

Recalling that $\tilde{x} \in D(L) \cap X$ we obtain

$$Sp + q = F_x(x).$$

Since $F^{**}(x) = F(x)$ by the properties of the Fenchel–Young transform we get

$$\langle Sp + q, x\rangle = F(x) + F^*(Sp + q). \tag{5.3}$$

By the Fenchel–Young equality

$$G(\tilde{x}) + G^*(q) = \langle q, \tilde{x}\rangle \tag{5.4}$$

and by relation $p = S\tilde{x}$ we calculate

$$\begin{aligned}
&\inf_{u\in X}\left\{-\langle u, Sp\rangle + \frac{1}{2}\langle Su, Su\rangle - \langle q, u\rangle + G(u)\right\}\\
&\quad\leq -\langle \tilde{x}, Sp\rangle + \frac{1}{2}\langle S\tilde{x}, S\tilde{x}\rangle - \langle q, \tilde{x}\rangle + G(\tilde{x})\\
&\quad= -\frac{1}{2}\langle p, p\rangle - G^*(q).
\end{aligned} \tag{5.5}$$

Next we note that for any $u \in Y$ we have the following inequality

$$-F(u) \leq -\langle Sp + q, u\rangle + F^*(Sp + q). \tag{5.6}$$

By (5.5), (5.6) and by (5.4) we get

$$J(x) = \inf_{u\in X} J(u) = \inf_{u\in X}\left\{\frac{1}{2}\langle Su, Su\rangle - F(u) + G(u)\right\}$$
$$\leq \inf_{u\in X}\left\{-\langle p, Su\rangle + \frac{1}{2}\langle Su, Su\rangle - \langle q, u\rangle + G(u)\right\}$$
$$+\ F^*(Sp+q) \leq J_D(p,q).$$

Next using (5.3) and Fenchel-Young inequality in (5.4) with $\tilde{x}$ replaced by x, we have

$$J(x) = \frac{1}{2}\langle Sx, Sx\rangle - \langle Sp+q, x\rangle + F^*(Sp+q) + G(x) \geq J_D(p,q).$$

Thus $J(x) = J_D(p,q)$ and again by (5.3) we calculate that

$$\frac{1}{2}\langle Sx, Sx\rangle + \frac{1}{2}\langle p, p\rangle - \langle p, Sx\rangle + G^*(q) + G(x) - \langle q, x\rangle = 0. \tag{5.7}$$

Since (5.7) holds and by the inequalities

$$\frac{1}{2}\langle Sx, Sx\rangle + \frac{1}{2}\langle p, p\rangle - \langle p, Sx\rangle \geq 0,$$
$$G^*(q) + G(x) - \langle q, x\rangle \geq 0,$$

we actually have equalities in both the above relations. The first equality obviously leads to $Sx = p$, and it shows that $x \in D(L)$, while the second, by the properties of the Fenchel–Young transformation provides that $q = G_x(x)$. Now inserting p and q into (5.2) we see that (5.1) is satisfied. □

Now we provide the application of Theorem 5.2 to the problem under consideration.

Proof. We fix k in what follows. Let $Lx = \beta_k \frac{d^4}{dt^4}x + \gamma_k \frac{d^2}{dt^2}x + \delta_k x$ with $D(L) = H_0^2(0,\pi) \cap H^4(0,\pi)$ in case $\beta_k \neq 0$ and $D(L) = H^2(0,\pi) \cap H_0^1(0,\pi)$ otherwise for $k = 0, 1, 2, \ldots$. We put $Y = L^2(0,\pi)$ so that A1 is obviously satisfied. Sets X_k are, due to their definitions, weakly compact and convex. Hence we may put $X = X_k$ in Theorem 5.2.

By F, G we mean the terms corresponding to F^k, G^k in the action functional for eq. (2.1) while F_x, G_x are their Gâteaux derivatives which may be viewed as the Niemytskij operators defined by the LHS and the RHS nonlinear terms in (2.1). Functional G is convex while F is continuous on Y. Hence conditions A1–A2 are satisfied. Condition H4 shows that $F^{**} = F$, so we have relation (5.3). Lemma 3.1 provides relation (5.2). Using Theorem 5.2 we see that all assertions of Theorem 4.1 hold. □

6. Stability of solutions for (2.1) and an example

Now we may prove Theorem 4.2. Note that Theorem 4.3 is proved along the same lines.

Proof. By Theorem 4.1 for each $k = 1, 2, \ldots$ there exists a solution x_k to (2.1) satisfying relation $x_k \in X_k \subset X_0$ for all $k = 1, 2, \ldots$. Thus we may choose a subsequence, denoted by $\{x_k\}_{1=1}^{\infty}$, which is weakly convergent in $H_0^2(0,\pi)$ and therefore strongly convergent in $H_0^1(0,\pi)$ to a certain $\bar{x} \in X_0$. This subsequence may be assumed to be convergent uniformly.

We employ assumption (4.1) for $x = \bar{x}$ taking another suitable subsequence, still denoted by $\{x_k\}_{k=1}^{\infty}$, in order to prove that

$$\lim_{k\to\infty} F_x^k(t, x_k(t)) = F_x^0(t, \bar{x}(t)),$$
$$\lim_{k\to\infty} G_x^k(t, x_k(t)) = G_x^0(t, \bar{x}(t)) \text{ a.e.} \tag{6.1}$$

We demonstrate only the first relation. We write

$$F_x^k(t, x_k(t)) - F_x^0(t, \bar{x}(t))$$
$$= F_x^k(t, x_k(t)) - F_x^k(t, \bar{x}(t)) + F_x^k(t, \bar{x}(t)) - F_x^0(t, \bar{x}(t)). \tag{6.2}$$

Since $x_k(t) \in [-d_0, d_0]$ assumption H5 provides that each F_x^k is Lipschitz in $[-d_0, d_0]$. Thus there exists a constant a_1 such that

$$|F_x^k(t, x_k(t)) - F_x^k(t, \bar{x}(t))| \le a_1 |x_k(t) - \bar{x}(t)|.$$

Since $\{x_k\}_{k=1}^{\infty}$ is uniformly convergent it follows that

$$\lim_{k\to\infty} (F_x^k(t, x_k(t)) - F_x^k(t, \bar{x}(t))) = 0.$$

Thus from (6.2) using the above and (4.1) we obtain (6.1).

Since $G_x^k(t,0) = 0$, there exists a constant $a_2 > 0$ such that $\left|G_x^k(t, x_k(t))\right| \le a_2 |x_k(t)|$. Therefore by (2.2) there exists a constant $a_3 > 0$ such that

$$|F_x^k(t, x_k(t)) - G_x^k(t, x_k(t))| \le a_3$$

for a.e. $t \in [0,\pi]$. Hence the sequence $\left\{\beta_k \frac{d^4}{dt^4} x_k + \gamma_k \frac{d^2}{dt^2} x_k + \delta_k x_k\right\}_{k=1}^{\infty}$ is bounded in $L^2(0,\pi)$. Since $\left\{\frac{d^2}{dt^2} x_k\right\}_{k=1}^{\infty}$ is bounded in $L^2(0,\pi)$, it follows that $\left\{\frac{d^4}{dt^4} x_k\right\}_{k=1}^{\infty}$ is bounded in $L^2(0,\pi)$ and a weakly convergent subsequence can be chosen. We denote this subsequence again by $\{x_k\}_{k=1}^{\infty}$.

Since $\left\{\beta_k \frac{d^4}{dt^4} x_k + \gamma_k \frac{d^2}{dt^2} x_k + \delta_k x_k\right\}_{k=1}^{\infty}$ is weakly convergent in $L^2(0,\pi)$ there exists an element $f \in L^2(0,\pi)$ such that for all $h \in L^2(0,\pi)$ we have

$$\int_0^\pi \left(\beta_k \frac{d^4}{dt^4} x_k(t) + \gamma_k \frac{d^2}{dt^2} x_k(t) + \delta_k x_k(t)\right) h(t)dt \to \int_0^\pi f(t)h(t)dt.$$

Since $C_0^\infty(0,\pi)$ is dense in $L^2(0,\pi)$, we can assume $h \in C_0^\infty(0,\pi)$. Further, integrating by parts, we see that for all $h \in C_0^\infty(0,\pi)$,

$$\beta_k \int_0^\pi x_k(t) \frac{d^4}{dt^4} h(t)dt + \gamma_k \int_0^\pi x_k(t) \frac{d^2}{dt^2} h(t)dt + \int_0^\pi \delta_k x_k(t)h(t)dt$$
$$\to \beta_0 \int_0^\pi \bar{x}(t) \frac{d^4}{dt^4} h(t)dt + \gamma_0 \int_0^\pi \bar{x}(t) \frac{d^2}{dt^2} h(t)dt + \int_0^\pi \delta_0 \bar{x}(t)h(t)dt.$$

Therefore for all $h \in C_0^\infty(0,\pi)$,

$$\int_0^\pi f(t)h(t)\mathrm{d}t = \beta_0 \int_0^\pi \bar{x}(t)\frac{\mathrm{d}^4}{\mathrm{d}t^4}h(t)\mathrm{d}t + \gamma_0 \int_0^\pi \bar{x}(t)\frac{\mathrm{d}^2}{\mathrm{d}t^2}h(t)\mathrm{d}t + \int_0^\pi \delta_0 \bar{x}(t)h(t)\mathrm{d}t.$$

Since a weak limit is equal to an almost everywhere limit when both exist, we get for all $h \in C_0^\infty(0,\pi)$,

$$\beta_0 \int_0^\pi \bar{x}(t)\frac{\mathrm{d}^4}{\mathrm{d}t^4}h(t)\mathrm{d}t + \gamma_0 \int_0^\pi \bar{x}(t)\frac{\mathrm{d}^2}{\mathrm{d}t^2}h(t)\mathrm{d}t + \int_0^\pi \delta_0 \bar{x}(t)h(t)\mathrm{d}t = \int_0^\pi \left(F_x^0(t,\bar{x}(t)(t)) - G_x^0(t,\bar{x}(t))\right)h(t)\mathrm{d}t. \tag{6.3}$$

Using higher order version of the du Bois–Reymond lemma (see [8]), we observe that $\frac{\mathrm{d}^4}{\mathrm{d}t^4}\bar{x}$ exists and belongs to $L^2(0,\pi)$. Therefore integrating (6.3) by parts, we see that (4.2) is satisfied.

Now we show that $\{x_k\}_{1=1}^\infty$ is strongly convergent in $H_0^2(0,\pi)$. We multiply (2.1) and (4.2) by x_k which provides after subtraction

$$\beta_k \int_0^\pi \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}x_k(t)\right)^2 \mathrm{d}t - \beta_0 \int_0^\pi \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bar{x}(t)\right)^2 \mathrm{d}t$$
$$= \gamma_k \int_0^\pi \left(\frac{\mathrm{d}}{\mathrm{d}t}x_k(t)\right)^2 \mathrm{d}t - \gamma_0 \int_0^\pi \left(\frac{\mathrm{d}}{\mathrm{d}t}\bar{x}(t)\right)^2 \mathrm{d}t - \delta_k \int_0^\pi x_k^2(t)\mathrm{d}t$$
$$+ \delta_0 \int_0^\pi \bar{x}^2(t)\mathrm{d}t - \int_0^\pi G_x^k(t,x_k(t))x_k(t) + \int_0^\pi G_x^0(t,\bar{x}(t))\bar{x}(t)$$
$$+ \int_0^\pi F_x^k(t,x_k(t))x_k(t) - \int_0^\pi F_x^0(t,\bar{x}(t))\bar{x}(t).$$

From the previous part of the proof it follows that the right-hand side of the above equality converges to 0. We see that $\int_0^\pi \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}x_k(t)\right)^2 \mathrm{d}t \to \int_0^\pi \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bar{x}(t)\right)^2 \mathrm{d}t$. □

We now provide an example of a function for which our growth conditions H1–H5 are satisfied.

Example 6.1. Let $\{d_k\}_{k=0}^\infty$ be the sequence of decreasing positive numbers bounded away from 0. Let us set another numerical sequence $\alpha_k = \frac{((\beta_k-\gamma_k-|\delta_k|))d_k}{\pi\left(\frac{1}{2}\mathrm{e}^{d_k}+\frac{1}{2}d_k^3+2\right)}$ for $k = 0,1,2,\ldots$ Let v be any odd number. Let us consider the sequences of functions $\{f_k\}_{k=0}^\infty$, $\{g_k\}_{k=0}^\infty \subset L^\infty(0,\pi)$ such that $\operatorname{ess\,sup}_{t\in(0,\pi)} f_k(t) \in [1,2]$, $\lim_{k\to\infty} f_k(t) = f_0(t)$,

$\lim_{k\to\infty} g_k(t) = g_0(t)$ for a.e. $t \in (0,\pi)$; and $g_k(t) > \frac{1}{v}$ for a.e. $t \in (0,\pi)$. We define for $k = 0, 1, 2, \ldots$

$$F^k(t,x) = \alpha_k \left(\frac{1}{2}e^x + \frac{1}{8}x^4 + f_k(t)x\right)$$

and

$$G^k(t,x) = \frac{1}{2}x^2 + g_k(t)x^v.$$

We need to show that assumptions H1–H5 are satisfied. Both F and G are convex. We see that $F_x^k(t,0) \neq 0$ and that $G_x^k(t,0) = 0$. Moreover, $G_x^k(t,x)x = x^2 + \frac{1}{v}g_k(t)x^v \geq 0$. We see that

$$\max_{x\in[-d_k,d_k]} \alpha_k \left(\frac{1}{2}e^x + \frac{1}{2}x^3 + f_k(t)\right)$$

$$\leq \alpha_k \left(\frac{1}{2}\mathrm{e}^{d_k} + \frac{1}{2}d_k^3 + 2\right) \leq \frac{((\beta_k - \gamma_k - |\delta_k|))\, d_k}{\pi}.$$

Hence (2.2) is satisfied. Other assumptions are obviously satisfied.

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