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Real hypersurfaces of a complex projective space

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Abstract. In this paper, we classify real hypersurfaces in the complex projective space $CP^{\frac{n+1}{2}}$ whose structure vector field is a φ-analytic vector field (a notion similar to analytic vector fields on complex manifolds). We also define Jacobi-type vector fields on a Riemannian manifold and classify real hypersurfaces whose structure vector field is a Jacobi-type vector field.

Keywords. Real hypersurfaces; mean curvature; Ricci curvature; shape operator; φ-analytic vector fields; Jacobi-type vector fields.

1. Introduction

The study of real hypersurfaces in complex space forms $\bar{M}(c)$ (Kaehler manifolds of constant holomorphic sectional curvature c) is a very interesting and active area of research. The ambient space $\bar{M}(c)$, specially in the case $c \neq 0$ imposes quite significant restrictions on the geometry of its real hypersurfaces. For instance $\bar{M}(c)$, $c \neq 0$, does not admit totally umbilical or Einstein hypersurfaces as well as their geodesic spheres do not have constant curvature. Compact minimal real hypersurfaces, real hypersurfaces of constant mean curvature and Hopf hypersurfaces in a complex space form and in nearly Kaehler sphere S^6 are studied in [1, 4, 6–13, 15–17] and these hypersurfaces are fully characterized up to some dimensional constraints. Homogeneous real hypersurfaces in a complex space form are completely classified (cf. [2, 14, 19–21]). Also in [5], the authors have shown that for compact real hypersurfaces of positive Ricci curvature in a complex projective space $CP^{\frac{n+1}{2}}$ the holomorphic distribution is not integrable. In general, the study of real hypersurfaces in a complex space form is a difficult area as compared to the study of hypersurfaces in real space forms (Riemannian manifolds of constant sectional curvature). To overcome this difficulty, different restrictions on the structure vector field ξ of the real hypersurfaces in a complex space form are imposed for obtaining geometric informations of these hypersurfaces, such as the vector field ξ is a Killing vector field (equivalently the condition $A\varphi = \varphi A$) or a closed vector field (equivalently the condition $A\varphi + \varphi A = 0$) or the shape operator A is parallel in the direction of ξ or Ricci operator is parallel in the direction of ξ (cf. [15]). In this paper, taking a clue from the analytic vector fields on a complex manifold, we define φ-analytic vector fields on an almost contact manifold and use this notion to study complete real hypersurfaces of the complex projective space

$CP^{\frac{n+1}{2}}$ whose structure vector field ξ is φ-analytic and it is interesting to note that these hypersurfaces are of type-A in the Takagi–Montiel classification list of the real hypersurfaces (cf. [15]). It is well-known that a Killing vector field on a Riemannian manifold is a Jacobi vector field along any geodesic, however a smooth vector field that is a Jacobi vector field along each geodesic need not be a Killing vector field. We define a Jacobi-type vector field on a Riemannian manifold (which in particular implies that a Jacobi-type vector field is Jacobi field along each geodesic). This leads to the question of finding conditions under which Jacobi-type vector fields are Killing vector fields. In this paper, we also study the real hypersurface of the complex projective space $CP^{\frac{n+1}{2}}$ whose structure vector field ξ is Jacobi-type and shows that in this situation ξ is a Killing vector field and consequently, real hypersurfaces with Jacobi-type ξ are of type-A in the Takagi–Montiel list.

2. Preliminaries

Let J, g and $\bar{\nabla}$ be the complex structure, the Hermitian metric and the Riemannian connection on the complex projective space $CP^{\frac{n+1}{2}}$ of constant holomorphic sectional curvature 4. On an orientable real hypersurface M of $CP^{\frac{n+1}{2}}$, we denote by the same letter g the induced metric, by N, A and ∇ the unit normal vector field, the shape operator and the induced Riemannian connection on M respectively. There is an induced almost contact metric structure (φ, ξ, η, g) on the real hypersurface M, where the characteristic vector field ξ is defined by $J\xi = N$, η is 1-form dual to ξ with respect to the Riemannian metric g and $\varphi X = JX - \eta(X)N$, $X \in \mathfrak{X}(M)$, $\mathfrak{X}(M)$ being the Lie algebra of smooth vector fields on M (cf. [3]), that satisfies

$$\varphi^2 = -I + \eta \otimes \xi, \varphi\xi = 0, \eta \circ \varphi = 0, \eta(X) = g(X, \xi), \tag{2.1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad X, Y \in \mathfrak{X}(M). \tag{2.2}$$

On the real hypersurface M, we have (cf. [15])

$$\bar{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \bar{\nabla}_X N = -AX, \tag{2.3}$$

$$\nabla_X \xi = \varphi AX, \quad (\nabla\varphi)(X, Y) = \eta(Y)AX - g(AX, Y)\xi, \tag{2.4}$$

$X, Y \in \mathfrak{X}(M)$, where $(\nabla\varphi)(X, Y) = \nabla_X \varphi Y - \varphi(\nabla_X Y)$. The curvature tensor R, the Ricci tensor 'Ric' and the scalar curvature S of the real hypersurface M are given by (cf. [15])

$$\begin{aligned} R(X, Y)Z = {} & g(Y, Z)X - g(X, Z)Y + g(\varphi Y, Z)\varphi X - g(\varphi X, Z)\varphi Y \\ & + 2g(X, \varphi Y)\varphi Z + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{2.5}$$

$$\mathrm{Ric}(X, Y) = (n + 2)g(X, Y) - 3\eta(X)\eta(Y) + n\alpha g(AX, Y) - g(AX, AY), \tag{2.6}$$

where $X, Y, Z \in \mathfrak{X}(M)$.

$$S = (n + 3)(n - 1) + n^2\alpha^2 - \|A\|^2, \tag{2.7}$$

where α is the mean curvature of the hypersurface M. The Codazzi equation for the real hypersurface M is given by (cf. [15])

$$(\nabla A)(X, Y) - (\nabla A)(Y, X) = \eta(X)\,\phi Y - \eta(Y)\,\phi X + 2g(X, \phi Y)\,\xi, \quad (2.8)$$

$X, Y \in \mathfrak{X}(M)$, where $(\nabla A)(X, Y) = \nabla_X AY - A(\nabla_X Y)$. For the real hypersurface M, throughout this paper by an adapted local orthonormal frame, we mean $\{e_1,\ldots, e_m, \varphi e_1,\ldots, \varphi e_m, \xi\}$, where $n = 2m + 1$.

Define a vector field u by $u = \nabla_\xi \xi$, then u is orthogonal to ξ and using eqs (2.1)–(2.4) it is straight forward to arrive at

$$\nabla_X u = R(X, \xi)\xi + g(AX, \xi)A\xi - g(AX, A\xi)\xi + \varphi(\nabla A)(\xi, X) + (\varphi A)^2 X. \quad (2.9)$$

Also, define another smooth vector field v orthogonal to ξ by setting $A\xi = v + f\xi$, where $f = g(A\xi, \xi)$ is a smooth function on the real hypersurface M. It follows from the definitions of the vector fields u and v that

$$u = \varphi(v), \quad v = -\varphi(u) \quad (2.10)$$

and consequently, $\{u, v, \xi\}$ is an orthogonal set.

Lemma 2.1. *Let M be a real hypersurface of the complex projective space $CP^{\frac{n+1}{2}}$. Then*

$$\sum_{i=1}^{n} (\nabla A)(e_i, e_i) = n\nabla\alpha,$$

where $\{e_1,\ldots, e_n\}$ is a local orthonormal frame on M and $\nabla\alpha$ is the gradient of the mean curvature α.

Proof. For $X \in \mathfrak{X}(M)$, using eqs (2.1), (2.7) and symmetry of the shape operator A, we compute

$$\begin{aligned} nX(\alpha) &= \sum_{i=1}^{n} g((\nabla A)(X, e_i), e_i) \\ &= \sum_{i=1}^{n} g((\nabla A)(e_i, X) + \eta(X)\varphi e_i - \eta(e_i)\varphi X + 2g(X, \varphi e_i)\xi, e_i) \\ &= \sum_{i=1}^{n} g((\nabla A)(e_i, e_i), X) \end{aligned}$$

which proves the result.

3. Real hypersurfaces with analytic structure vector

Recall that a smooth vector field u on a complex manifold (M, J) with complex structure J is said to be an analytic vector field if the local flow $\{\psi_t\}$ of u consists of almost complex local diffeomorphisms ψ_t of M, that is the differential $\mathrm{d}\psi_t$ of ψ_t commutes with the complex structure J. The set of analytic vector fields on a complex manifold forms a Lie algebra which is isomorphic to the Lie algebra of holomorphic vector fields on M. Moreover, it follows that a smooth vector field u on M is analytic if and only if the following condition holds:

$$J[X, u] = [JX, u], \quad X \in \mathfrak{X}(M).$$

Since an almost contact manifold (M, φ, ξ, η) is an odd dimensional analogue of an almost complex manifold, motivated by the above definition, we define φ-analytic vector fields on an almost contact manifold (M, φ, ξ, η). We say a smooth vector field $u \in \mathfrak{X}(M)$ to be φ-analytic if the local flow $\{\psi_t\}$ of u consists of local diffeomorphisms ψ_t of M, such that the differential $d\psi_t$ of ψ_t commutes with φ.

Since a real hypersurface M of the complex projective space $CP^{\frac{n+1}{2}}$ has almost contact metric structure, in this section, we are interested in studying real hypersurfaces whose structure vector field ξ is φ-analytic. It turns out that if the structure vector field ξ is a φ-analytic vector field, then the real hypersurfaces are of type-A in the classification list of Takagi–Montiel.

Lemma 3.1. *Let M be a real hypersurface of the complex projective space $CP^{\frac{n+1}{2}}$. If the structure vector field ξ is φ-analytic, then the following hold*:

$$\operatorname{Tr}(\varphi A)^2 = \|A\xi\|^2 - \|A\|^2, \quad \|\varphi A - A\varphi\|^2 = 2\|u\|^2.$$

Proof. Since ξ is φ-analytic, for $X \in \mathfrak{X}(M)$ we have

$$[\varphi X, \xi] = \varphi[X, \xi]$$

which together with eqs (2.1) and (2.4) give

$$\varphi A\varphi X + AX = \eta(X)A\xi. \tag{3.1}$$

Replacing X by AX in the above equation, we arrive at

$$\varphi A\varphi AX + A^2X = \eta(AX)A\xi$$

and taking trace of the above equation, we get the first equation in the lemma. Also, operating φ on eq. (3.1) and using eqs (2.1) and (2.10), we have

$$\varphi AX - A\varphi X = \eta(X)u + g(X, u)\xi$$

which gives

$$\begin{aligned}\|\varphi A - A\varphi\|^2 &= \sum_{i=1}^{n} g(\eta(e_i)u + g(e_i, u)\xi, \eta(e_i)u + g(e_i, u)\xi)\\ &= \|u\|^2 \sum_{i=1}^{n} \eta(e_i)^2 + \sum_{i=1}^{n} g(e_i, u)^2 = 2\|u\|^2.\end{aligned}$$

We have the following result proved in [16].

Theorem (Okumura). *Let M be a real hypersurface of $CP^{\frac{n+1}{2}}$, $n \geq 3$. Then the following are equivalent*:

(1) *M is locally congruent to either a geodesic hypersphere or a tube over totally geodesic CP^k, $0 < k < n-1$.*
(2) *$\varphi A = A\varphi$.*

Now we prove the main result of this section.

Theorem 3.2. *Let M be a real hypersurface of $CP^{\frac{n+1}{2}}$, $n \geq 3$. If the structure vector field ξ is φ-analytic, then M is locally congruent to either a geodesic hypersphere or a tube over totally geodesic CP^k, $0 < k < n-1$.*

Proof. Suppose ξ is φ-analytic. Since $\varphi A - A\varphi$ is a symmetric operator, we have

$$\|\varphi A - A\varphi\|^2 = \mathrm{Tr}\,(\varphi A - A\varphi)^2 = 2\,\mathrm{Tr}(\varphi A)^2 + 2\,\mathrm{Tr}\,A^2 - 2\,\|A\xi\|^2 .$$

Using Lemma 3.1, we get

$$\|u\|^2 = \|A\xi\|^2 - \|A\|^2 + \|A\|^2 - \|A\xi\|^2 = 0$$

which proves $\varphi A - A\varphi = 0$. Using Okumura's theorem we get the result.

Remark. In the above proof, we see that for a real hypersurface M of $CP^{\frac{n+1}{2}}$, if the structure vector field ξ is φ-analytic, then $\varphi A - A\varphi = 0$, which in particular shows that the vector field ξ satisfies

$$\left(\pounds_\xi g\right)(X, Y) = g\left(\nabla_X \xi, Y\right) + g\left(\nabla_Y \xi, X\right) = g\left(\left(\varphi A - A\varphi\right)(X), Y\right) = 0,$$

that is, ξ is Killing. Thus combining this fact with Rong's result (cf. [18]), we have the following:

COROLLARY 3.3

Let M be a compact positively curved real hypersurface of $CP^{\frac{n+1}{2}}$. If the structure vector field ξ is φ-analytic, then the fundamental group of M has a cyclic subgroup with index $w(n)$, a constant depending only on n.

4. Real hypersurfaces with Jacobi-type structure vector

It is well-known that a Killing vector field on a Riemannian manifold (M, g) is a Jacobi field along each geodesic of M. However, the converse is not true as for example the position vector field on the Euclidean space R^n is a Jacobi field along each geodesic of R^n which is not a Killing vector field. Motivated by the definition of a Jacobi field along a geodesic, we define a Jacobi-type vector field u on a Riemannian manifold (M, g) which satisfies

$$\nabla_X \nabla_X u + R(u, X)X = 0, \quad X \in \mathfrak{X}(M),$$

where ∇ is the Riemannian connection and R is the curvature tensor field of the Riemannian manifold (M, g). Naturally a Jacobi-type vector field is a Jacobi field along each geodesic of M. It is an interesting question to obtain conditions under which a Jacobi-type vector field on a Riemannian manifold is Killing. In this section, we study compact real hypersurfaces of the complex projective space $CP^{\frac{n+1}{2}}$ whose structure vector field ξ is a Jacobi-type vector field and show that it is Killing.

Lemma 4.1. *Let M be a real hypersurface of the complex projective space $CP^{\frac{n+1}{2}}$. If the structure vector field ξ is a Jacobi-type vector field on M, then*

$$n\varphi(\nabla\alpha) = (\|A\|^2 - (n-1))\xi - n\alpha A\xi,$$

where $\nabla\alpha$ is the gradient of the mean curvature α.

Proof. Since ξ is a Jacobi-type vector field, for $X \in \mathfrak{X}(M)$ we have

$$\nabla_X\nabla_X\xi + R(\xi, X)X = 0, \quad X \in \mathfrak{X}(M)$$

which together with eqs (2.4) and (2.5) give

$$\nabla_X\varphi AX + \|X\|^2\xi - \eta(X)X + g(AX, X)A\xi - \eta(AX)AX = 0.$$

The above equation can be rearranged as

$$\begin{aligned}(\nabla\varphi)(X, AX) &+ \varphi(\nabla A)(X, X) - \varphi A(\nabla_X X)\\ &+ \|X\|^2\xi - \eta(X)X + g(AX, X)A\xi - \eta(AX)AX = 0\end{aligned}$$

which together with eq. (2.4) gives

$$\begin{aligned}-g(AX, AX)\xi &+ \varphi(\nabla A)(X, X) - \varphi A(\nabla_X X)\\ &+ \|X\|^2\xi - \eta(X)X + g(AX, X)A\xi = 0.\end{aligned} \tag{4.1}$$

Let $\{e_1,\ldots, e_n\}$ be a geodesic local orthonormal frame on M. Take $X = e_i$ in the above equation and add these n equations. We arrive at

$$-\|A\|^2\xi + n\varphi(\nabla\alpha) + (n-1)\xi + n\alpha A\xi = 0$$

which proves the lemma.

Theorem 4.2. *Let M be a compact real hypersurface of the complex projective space $CP^{\frac{n+1}{2}}$. If the structure vector field ξ is a Jacobi-type vector field on M, then ξ is a Killing vector field.*

Proof. Let the structure vector field ξ be Jacobi-type. Then operating φ on eq. (4.1), we arrive at

$$\begin{aligned}(\nabla A)(X, X) = A(\nabla_X X) &- \eta(\nabla_X X)\xi - \eta(X)\varphi X\\ &+ \eta((\nabla A)(X, X))\xi + g(AX, X)u.\end{aligned}$$

Summing the above equation over a local orthonormal frame and using Lemma 2.1, we get

$$n\nabla\alpha = n\xi(\alpha)\xi + n\alpha u.$$

Operating φ on the above equation and using eq. (2.10), we conclude $n\varphi(\nabla\alpha) = -n\alpha v$ and inserting this value in Lemma 2.1 and using $A\xi = v + f\xi$, we arrive at

$$\|A\|^2 = (n-1) + n\alpha f. \tag{4.2}$$

Equation (2.6) gives

$$\mathrm{Ric}(\xi,\xi) = (n-1) + n\alpha f - \|A\xi\|^2 . \tag{4.3}$$

Next, we use eq. (2.9) to find the divergence of the vector field u as follows:

$$\operatorname{div} u = \mathrm{Ric}(\xi,\xi) - \sum_{i=1}^{n} g((\nabla A)(\xi, e_i), \varphi e_i) + \mathrm{Tr}\,(\varphi A)^2 ,$$

where $\{e_1,..., e_n\}$ is a local orthonormal frame on M. The above equation together with the symmetry of the shape operator A and eq. (2.8) give

$$\operatorname{div} u = \mathrm{Ric}(\xi,\xi) - (n-1) - \sum_{i=1}^{n} g((\nabla A)(e_i, \varphi e_i), \xi) + \mathrm{Tr}\,(\varphi A)^2 . \tag{4.4}$$

Next, using a local adapted frame $\{e_1,..., e_m, \varphi e_1,..., \varphi e_m, \xi\}$, where $n = 2m+1$ and eq. (2.8), we compute the sum

$$\begin{aligned}\sum_{i=1}^{n} g((\nabla A)(e_i, \varphi e_i), \xi) &= \sum_{i=1}^{m} g((\nabla A)(e_i, \varphi e_i) - (\nabla A)(\varphi e_i, e_i), \xi) \\ &= -2m = -(n-1).\end{aligned}$$

Consequently, inserting this value in eq. (4.4), we conclude that

$$\int_M \left\{\mathrm{Ric}(\xi,\xi) + \mathrm{Tr}\,(\varphi A)^2 \right\}\mathrm{dV} = 0. \tag{4.5}$$

Since $\varphi A - A\varphi$ is a symmetric operator, we have

$$\|\varphi A - A\varphi\|^2 = \mathrm{Tr}\,(\varphi A - A\varphi)^2 = 2\,\mathrm{Tr}(\varphi A)^2 + 2\,\mathrm{Tr}\,A^2 - 2\,\|A\xi\|^2$$

and consequently eq. (4.5) takes the form

$$\int_M \left\{\frac{1}{2}\,\|\varphi A - A\varphi\|^2 + \mathrm{Ric}(\xi,\xi) + \|A\xi\|^2 - \|A\|^2\right\} \mathrm{d}V = 0.$$

Using eqs (4.2) and (4.3) in the above equation, we arrive at

$$\frac{1}{2}\int_M \|\varphi A - A\varphi\|^2 \,\mathrm{d}V = 0$$

which proves that

$$\left(\pounds_\xi g\right)(X,Y) = g((\varphi A - A\varphi)(X), Y) = 0, \quad X, Y \in \mathfrak{X}(M),$$

that is, ξ is Killing.

As a direct consequence of the above theorem and the results in [16] and [18], we have the following:

COROLLARY 4.3

Let M be a compact real hypersurface of $CP^{\frac{n+1}{2}}$, $n \geq 3$. If the structure vector field ξ is a Jacobi-type vector field, then M is locally congruent to either a geodesic hypersphere or a tube over totally geodesic CP^k, $0 < k < n - 1$.

COROLLARY 4.4

Let M be a compact positively curved real hypersurface of $CP^{\frac{n+1}{2}}$, $n \geq 3$. If the structure vector field ξ is a Jacobi-type vector field, then the fundamental group of M has a cyclic subgroup with index $w(n)$, a constant depending only on n.

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