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Principal bundles whose restrictions to a curve are isomorphic

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Abstract. Let X be a normal projective variety defined over an algebraically closed field k. Let $|O_X(1)|$ be a very ample invertible sheaf on X. Let G be an affine algebraic group defined over k. Let E_G and F_G be two principal G-bundles on X. Then there exists an integer $n \gg 0$ (depending on E_G and F_G) such that if the restrictions of E_G and F_G to a curve $C \in |O_X(n)|$ are isomorphic, then they are isomorphic on all of X.

Keywords. Vector bundle; very ample invertible sheaf; principal bundles; affine algebraic group; structure group; normal projective variety; Enrique–Severi; unipotent radical; reductive; parabolic subgroup.

1. Introduction

In a paper of Graber *et al.* the following is proved (Corollary 6.1 of [5]). Let B be a smooth variety defined over an algebraically closed field of characteristic zero and $A \longrightarrow B$ an abelian scheme over B (i.e., a family of abelian varieties over B). Let $\pi : T \to B$ be a torsor for $A \longrightarrow B$. Then π is trivial if and only if for every curve $C \subset B$, the restriction $T|_C \longrightarrow C$ is a trivial torsor for $A|_C \longrightarrow C$. They further ask whether such a restriction theorem holds for other (possibly noncommutative) group schemes (see the end of §6 of [5]).

In this direction, Biswas and Holla [3] proved the following theorem: Let k be an algebraically closed field and X be a smooth projective variety over k. Let G be an affine algebraic group defined over k. Let E_G be a principal G-bundle over X with the property that for every smooth curve $C \subset X$ the restriction of E_G to C is isomorphic to the trivial bundle over C. Then the G-bundle E_G over X is trivial. We prove the following:

Theorem 1.1. *Let k be an algebraically closed field and let X be a normal projective variety defined over k. Fix a very ample line bundle $|O_X(1)|$ on X. Let G be an affine algebraic group defined over k. Let E_G and F_G be two principal G-bundles over X. Let $n \gg 0$. Let $C \in |O_X(n)|$ be a general curve which is a complete intersection of ample hypersurfaces and having the property that the restrictions of the E_G and F_G to C are isomorphic. Then we show that E_G and F_G are isomorphic on all of X.*

The theorem will be proved later in §4.

Remark. In Lemma 5.16 of [1], Balaji has proved the above theorem assuming X to be smooth, G to be reductive and both E_G and F_G to be polystable bundles.

2. Vector bundles over a normal projective variety

Let G be an affine algebraic group defined over k. Let X be a normal variety defined over k. We first prove our main result for vector bundles i.e for $G = GL(n, k)$, then for affine algebraic groups G embedded in $Gl(n)$ such that the quotient $Gl(n)/G$ is affine and then for arbitrary affine algebraic groups.

Lemma 2.1. *Let X, k be as in Theorem* 1.1. *Let E and F be two vector bundles of rank n (equivalently principal $Gl(n)$ bundles) on X. Let C be a connected complete intersection curve of hypersurfaces on X such that the canonical restriction map* $\mathrm{Hom}(E, F) \longrightarrow \mathrm{Hom}(E|_C, F|_C)$ *is an isomorphism. (Such a curve can always be chosen, thanks to the Enrique–Severi lemma.)*

If there exists an isomorphism $\phi_C : E|_C \longrightarrow F|_C$, then it extends to an isomorphism $\phi : E \longrightarrow F$ such that $\phi|_C = \phi_C$.

Proof. Choose an isomorphism $\phi_C : E|_C \longrightarrow F|_C$. Lift ϕ_C to an element $\phi \in \mathrm{Hom}(E, F)$. We claim that ϕ is actually an isomorphism. Let

$$\det\phi : \bigwedge^n E \longrightarrow \bigwedge^n F$$

be the induced homomorphism between the determinant bundles. Since ϕ_C is an isomorphism, ϕ will continue to be an isomorphism on an open subset containing C. Thus ϕ is not identically zero. Let D be the effective divisor defined by $\det\phi$. Then clearly ϕ fails to be an isomorphism exactly over D. Since C is a complete intersection of ample hypersurfaces $D \cap C \neq \emptyset$, provided $D \neq \emptyset$. On the other hand, from the condition that ϕ is an isomorphism over C it follows that D does not intersect C. Thus we conclude that $D = \emptyset$, thereby proving that ϕ is an isomorphism over all of X. □

For later use we will need the following lemma.

Lemma 2.2. *Let X, k be as in Theorem* 1.1. *Let P be a parabolic subgroup of $Gl(n)$ and let E_P and F_P be two principal P bundles on X. Let $n \gg 0$. Let $C \in |O_X(n)|$ be a general curve which is a complete intersection of ample hypersurfaces and having the property that the restrictions of E_P and F_P to C are isomorphic. Then we show that E_P and F_P are isomorphic on all of X.*

Proof. Principal P-bundles, where P is a parabolic in $Gl(n)$ correspond naturally to vector bundles with a filtration. Hence corresponding to the principal P bundles E_G and F_G, we get rank n vector bundles E and F together with a flag of subbundles

$$0 = E_{io} \subset E_{i1} \subset \cdots \subset E_{ik} = E$$

and similarly,

$$0 = F_{io} \subset F_{i1} \subset \cdots \subset F_{ik} = F.$$

Proving that E_P is isomorphic to F_P is equivalent to proving that there exists an isomorphism between E and F which carries E_i isomorphically onto F_i for all i.

By hypothesis, there exists an isomorphism $\phi_C : E|_C \longrightarrow F|_C$ which take the subbundles E_i of E isomorphically onto the subbundles F_i of F. Lift ϕ_C to an isomorphism, say $\phi : E \longrightarrow F$ which restricts to the isomorphism ϕ_C on C. Suppose for some i, ϕ does not carry E_i isomorphically onto F_i, then we get a nontrivial homomorphism, namely $f_i : E_i \longrightarrow F/F_i$ between the bundles E_i and F/F_i which restricts to the zero homomorphism on the curve C. Assuming the curve C to be of very high degree we get a contradiction, by Enrique–Severi. Hence we conclude that the principal P-bundles E_P and F_P are isomorphic on all of X. □

3. Principal bundles over a normal projective variety with certain algebraic groups as structure group

Let G be an affine algebraic group over k such that G admits an embedding $i : G \hookrightarrow Gl(n)$ such that the quotient $Gl(n)/G)$ is affine. This includes reductive algebraic groups in particular. We now appeal to a theorem of Chevalley stated below.

Theorem 3.1 (Chevalley). *Let H and K be reduced affine algebraic groups over k such that K is a subgroup of H, the quotient H/K is affine and the coordinate ring of H/K is a finitely generated k algebra. Then there exists a rational representation (V, ρ) of H and a vector $v \in V$ such that the scheme-theoretic stabiliser of v is precisely K.*

Proof. Let us suppose that the coordinate ring of H/K be generated by n elements $f_1, \ldots, f_n$. Thus $A(H/K) = k[f_1, \ldots, f_n]$, where $A(H/K)$ is the coordinate ring of H/K. Then each f_i is contained in a finite dimensional H sub-module of $A(H/K)$, say $V = \bigoplus_{j=1}^{r} kg_j$, for the standard action of H on $A(H/K)$. Thus the polynomial algebra $B = k[g_1 ... g_r]$ surjects onto $A(H/K)$. By taking induced map on the spectrum this implies that H/K becomes a closed embedded subscheme of V with the embedding clearly being H equivariant. (Here we are identifying the vector space V with the affine space corresponding to Spec B.) Thus V becomes a H module under a representation, say ρ. Let $v \in V$ be the image of the identity coset e in H/K. Consider the orbit map $\Phi : H \to V$ given by $\Phi(h) = \rho(h)(v)$. Clearly K is the scheme theoretic stabiliser of v. □

Applying this to our situation by taking $K = G$ and $H = Gl(n, k)$, we get a rational representation (V, ρ) of $Gl(n, k)$ and a vector v in V whose scheme-theoretic stabiliser is G. Let W be the $Gl(n, k)$ orbit of v. Let rank of W be m. Then W becomes a $Gl(n, k)$ submodule of V and let ρ' be the corresponding representation.

Define a map Θ: $Gl(n)/G \hookrightarrow W$ by sending the class of $[gG]$ to the element $\rho'(g)v \in V$ for any element $g \in Gl(n)$. Since $\Theta[gg'G] = \rho'(gg')(v) = \rho'(g)\rho'(g')(v)$, we see that with respect to the left action of $Gl(n, k)$ on $Gl(n, k)/G$, this mapping is $Gl(n, k)$ equivariant. Since the scheme-theoretic stabiliser of v is precisely G, this map is actually an embedding.

Let $E_{GL(n)}$ (resp. $F_{Gl(n)}$) denote the principal $Gl(n)$ bundles (equivalently vector bundles) obtained by extensions of structure group of E_G (resp. F_G) under the embedding i. Thus

$$E_{Gl(n)} \overset{\text{by defn}}{=} (E \times_G Gl(n, k))/\sim,$$

where a pair $(e, h) \sim$ a pair (e', h') if there exists an element $g \in G$ such that $(e', h') = (eg, i(g)^{-1}h)$. Similarly,

$$F_{Gl(n)} \overset{\text{by defn}}{=} (F \times_G Gl(n, k))/\sim,$$

where a pair $(e, h) \sim$ a pair (e', h') if there exists an element $g \in G$ such that $(e', h') = (eg, i(g)^{-1}h)$.

Similarly let $E_{Gl(n)/G}$ denote the associated fibre space defined by

$$E_{Gl(n)/G} = (E_{GL(n)} \times_{Gl(n)} Gl(n)/G)/\sim,$$

where a pair $(e, [gG]) \sim$ a pair $(e', [g'G])$ if there exists an $h \in Gl(n)$ such that $(e', [g'G]) = (eh, [h^{-1}gG])$. Similarly we can define $F_{Gl(n)/G}$, E_W (resp. F_W) as the vector bundles (equivalently principal $Gl(m)$ bundles) obtained by extension of structure group of the principal $Gl(n)$ bundle $E_{GL(n)}$ (resp. $F_{GL(n)}$) using the representation ρ'.

The $Gl(n)$-equivariant morphism Θ defines an embedding of the fiber bundles

$$\varphi : E_{Gl(n)/G} \text{ (resp. } F_{Gl(n)/G}) \hookrightarrow E_W \text{ (resp. } F_W).$$

We now state the main theorem of our paper for the special case when G is such that the quotient $Gl(n)/G$ is affine.

Theorem 3.2 (Main theorem (Special case)). *Let G be an affine algebraic group embedded in $Gl(n)$ such that the quotient $Gl(n)/G$ is affine. Let C be a curve such that C is an intersection of ample hypersurfaces and the restriction maps*

$$\mathrm{Hom}(E_{GL(n)}, F_{GL(n)}) \longrightarrow \mathrm{Hom}(E_{GL(n)}|_C, (F_{GL(n)}|_C)$$

and

$$\mathrm{Hom}(E_W, F_W) \longrightarrow \mathrm{Hom}(E_W|_C, F_W|_C))$$

are both isomorphisms. If the restricted bundles $E_G|_C$ and $F_G|_C$ are isomorphic on C, then E_G and F_G are isomorphic on all of X.

Proof. By assumption, $E_G|_C$ and $F_G|_C$ are isomorphic. Choose an isomorphism $\tau_c : E_G|_C \longrightarrow F_G|_C$ between the restricted bundles. By extension of structure group using the embedding i we get an isomorphism, say

$$\psi_c : E_{GL(n)}|_C \longrightarrow F_{GL(n)}|_C.$$

By Lemma 2.1 we can extend ψ_c to an isomorphism $\psi : E_{GL(n)} \longrightarrow F_{GL(n)}$. On extension of structure group via the homomorphism ρ' we get an isomorphism

$$\tilde{\psi} : E_W. \longrightarrow F_W.$$

Let $\tilde{\psi}_c$ denote the restriction of $\tilde{\psi}$ to C. The isomorphism $\tilde{\psi}$ restricts to give an isomorphism (again denoted by $\tilde{\psi}$) from $E_{Gl(n)/G}$ to $F_{Gl(n)/G}$. Thus it is easy to see that the following diagram is commutative.

$$\begin{array}{ccc} E_W & \xrightarrow{\tilde{\psi}} & F_W \\ \uparrow & & \uparrow \\ E_{Gl(n)/G} & \xrightarrow[\tilde{\psi}]{} & F_{Gl(n)/G} \end{array} .$$

Since the principal $Gl(n)$ bundles $E_{Gl(n)}$ (resp. $F_{Gl(n)}$) admit reduction of structure group to E_G (resp. F_G) we find that the associated fiber spaces $E_{Gl(n)/G}$ (resp. $F_{Gl(n)/G}$) canonically admit sections σ_1 (resp. σ_2) corresponding to these reductions. These sections can also be thought of as sections of E_W (resp. F_W). Since the isomorphisms ψ_c and $\tilde{\psi}_c$ come from the isomorphism τ_c by extension of structure group, the isomorphism $\tilde{\psi}_c$ takes σ_1 to σ_2, i.e., on C,

$$\tilde{\psi} \circ \sigma_1 = \sigma_2.$$

On X the isomorphism ψ_c may not carry σ_1 to σ_2. If it did, we would be through since it would imply that E_G is isomorphic to F_G on all of X. But $\tilde{\psi} \circ \sigma_1$ and σ_2 are two sections of F_W which agree on the curve C. Since the curve C is of high enough degree, we can conclude that $\tilde{\psi} \circ \sigma_1 = \sigma_2$ on all of X (Enrique–Severi). Thus E_G is isomorphic to F_G on all of X and this completes the proof of the main theorem in the special case. □

Remark 3.3. In fact we have proved the following more general fact: Let G, H be affine algebraic groups with H being a subgroup of G with G/H affine. Suppose G has the property that whenever two G bundles, say E_G and F_G on X are isomorphic on a high degree curve (high enough, depending on E_G and F_G), they are isomorphic on all of X, and the same property holds for any algebraic subgroup H in G such that G/H is affine. This follows immediately from the proof of Theorem 3.2 (for the only thing we need for Theorem 3.2 to work is that the result should hold for principal bundles with G as structure group and that we should be able to get a closed G-equivariantly embedding of G/H inside a finite dimensional G-module V).

4. Principal bundles over a projective variety with an arbitrary affine algebraic group as structure group

For a general affine algebraic group G, we would like to prove the following theorem:

Theorem 4.1 (Main theorem (General case)). *Let G be an affine algebraic group. Let E_G and F_G be two principal G-bundles on X. Let $n \gg 0$. Let $C \in |O_X(n)|$ be any curve which is a complete intersection of ample hypersurfaces. If the restricted bundles $E_G|_C$ and $F_G|_C$ are isomorphic on C, then E_G and F_G are isomorphic on all of X.*

We proceed as follows.

Proof. Let R_uG denote the unipotent radical of G. Then we have the following exact sequence of algebraic groups

$$0 \longrightarrow R_u(G) \longrightarrow G \longrightarrow L(G) \longrightarrow 0,$$

where $L(G)$ is the Levi quotient of G. R_uG is a connected, normal, closed subgroup of G and $L(G)$ is a reductive affine algebraic group defined over k. We begin by stating the following well-known group theoretic fact (Lemma 3.2 of [4]).

Lemma 4.2. *There is an injective homomorphism of algebraic groups*

$$\iota : G \hookrightarrow GL(V),$$

where V is a finite dimensional vector space over k, and a closed subgroup $Q \subseteq Gl(V)$ with $\iota(G) \subseteq Q$ such that either Q is a parabolic in $Gl(V)$, or $Q = Gl(V)$, and the following two conditions hold:

(1) $\iota(R_u(G)) \subseteq (R_u(Q))$.
(2) *The homomorphism*

$$\Psi : L(G) \longmapsto L(Q) := Q/R_u(Q)$$

induced by ι is injective.

(*Since $\iota(R_u(G)) \subseteq (R_u(Q))$, it induces a homomorphism on the quotients.*)

Continuing with the proof of Theorem 4.1, choose (as we may, by Lemma 4.2) an inclusion $\iota(G) \subseteq Q$, with Q being a parabolic subgroup of $Gl(n)$ (not neccessarily proper) satisfying conditions of Lemma 4.2. Consider the map

$$\mu : Q/G \longrightarrow L(Q)/L(G).$$

This is an étale locally trivial fibration with fiber isomorphic to $R_uQ/(R_uQ \cap G) \simeq R_uQ/R_uG$. Since $L(Q)/L(G)$ is a quotient of an algebraic group by a reductive group, the quotient is affine. Also, since R_uQ/R_uG is a quotient of an unipotent group by a unipotent subgroup, it is in fact isomorphic to affine space. This forces Q/G to be affine. Since we know that the main theorem holds for Q as structure group (see Lemmas 2.1 and 2.2), by Remark 3.3 in §3, the main theorem holds for G as well. This completes the proof of the main theorem for all affine algebraic groups G. □

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