Proc. Indian Acad. Sci. (Math. Sci.) Vol. 121, No. 2, May 2011, pp. 133–141.

Irreducible multivariate polynomials obtained from polynomials in fewer variables, II

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MS received 14 June 2010; revised 14 January 2011

Abstract. We provide several irreducibility criteria for multivariate polynomials and methods to construct irreducible polynomials starting from irreducible polynomials in fewer variables.

Keywords. Irreducibility; nonarchimedean absolute value.

1. Introduction

There are many irreducibility criteria for multivariate polynomials in the literature. For an excellent account on the techniques used in the study of reducibility of polynomials over arbitrary fields the reader is referred to Schinzel's book [9]. Some irreducibility results involving linear combinations, compositions, multiplicative convolutions and lacunary multivariate polynomials have been obtained in [1–3,5]. Inspired by some results of Ram Murty [8] and Girstmair [6] in connection with the irreducibility criterion of A. Cohn (see for instance Pólya and Szegö [7]), we studied in [4] the following problem:

Given a field K, under what hypotheses a polynomial $F(X,Y)\in K[X,Y]$ with the property that $F(X,h(X))$ is irreducible over K for some $h\in K[X]$, is necessarily irreducible over $K(X)$?

We provided some methods to construct irreducible multivariate polynomials over an arbitrary field, starting from arbitrary irreducible polynomials in fewer variables, of which we mention the following two results:

Theorem A. *If we write an irreducible polynomial $f\in K[X]$ as a sum of polynomials $a_0,\dots,a_n\in K[X]$ with $\deg a_0>\max_{1\le i\le n}\deg a_i$, then $F(X,Y)=\sum_{i=0}^{n}a_i(X)Y^i$ is irreducible over $K(X)$.*

Theorem B. *Let K be a field of characteristic 0 and let $f_1, f_2\in K[X]$ with $\deg f_1\ge 1$, $\deg f_2\ge 2$. If $f_1\circ f_2(X)$ is irreducible over K, then $f_1\circ(f_2(X)-X+Y)\in K[X,Y]$ is irreducible over $K(X)$.*

In this note we complement the results from [4], by allowing $F(X, h(X))$ to satisfy an equality of the form $F(X, h(X)) = f(X)^s \cdot g(X)$, with $f, g \in K[X]$, f irreducible over K, g not divisible by f, $s \geq 2$, and studying under what hypotheses F will still be irreducible over $K(X)$. The first result we prove gives irreducibility conditions for polynomials in two variables over an arbitrary field.

Theorem 1.1. *Let K be a field, $f, g, h \in K[X]$, f irreducible over K, $g \neq 0$, $\deg g < \deg h$, and assume that for an integer $s \geq 2$ the polynomial $f^s \cdot g$ is expressed in base h via the Euclidean algorithm as $f^s \cdot g = \sum_{i=0}^{n} a_i h^i$, with $a_0, a_1, \ldots, a_n \in K[X]$. If $\sum_{i=1}^{n} i a_i h^{i-1}$ is not divisible by f, then the polynomial $\sum_{i=0}^{n} a_i(X) Y^i \in K[X, Y]$ is irreducible over $K(X)$.*

A more efficient method to obtain irreducible multivariate polynomials starting from an irreducible univariate polynomial is the following analogue of Theorem A:

Theorem 1.2. *Let $f \in K[X]$ be an irreducible polynomial. If for an integer $s \geq 2$ we write f^s as a sum of polynomials $a_0, \ldots, a_n \in K[X]$ with $\deg a_0 > \max_{1 \leq i \leq n} \deg a_i$, and $a_1 + 2a_2 + \cdots + n a_n$ is not divisible by f, then the polynomial $F(X, Y) = \sum_{i=0}^{n} a_i(X) Y^i$ is irreducible over $K(X)$.*

The next two results provide other ways to produce irreducible multivariate polynomials.

Theorem 1.3. *Let K be a field, $f \in K[X]$ irreducible over K, and assume that for an integer $s \geq 2$ we have $f(X)^s = b_0 X^{n_0} + b_1 X^{n_1} + \cdots + b_k X^{n_k} \in K[X]$, $0 = n_0 < n_1 < \cdots < n_k$, $b_0 \cdots b_k \neq 0$. Let us construct from f^s the polynomial $F(X, Y) = b_0 X^{i_0} Y^{j_0} + b_1 X^{i_1} Y^{j_1} + \cdots + b_k X^{i_k} Y^{j_k} \in K[X, Y]$, with $i_l, j_l \geq 0$, $i_l + j_l = n_l$, $l = 0, \ldots, k$. If $\partial F / \partial Y(X, X)$ is not divisible by f and for an index $t \in \{0, \ldots, k\}$ we have*

$$\max_{j_v < j_t} \frac{i_v - i_t}{j_t - j_v} < 1 < \min_{j_v > j_t} \frac{i_v - i_t}{j_t - j_v},$$

then F is irreducible over $K(X)$.

Theorem 1.4. *Let K be a field, $f \in K[X]$ irreducible over K, and assume that for an integer $s \geq 2$ we have $f(X)^s = b_0 X^{n_0} + b_1 X^{n_1} + \cdots + b_k X^{n_k} \in K[X]$, $0 = n_0 < n_1 < \cdots < n_k$, $b_0 \cdots b_k \neq 0$. Then for every partition of the set $S = \{0, 1, \ldots, k\}$ into two disjoint, nonempty subsets S_1, S_2 with $k \in S_1$, the polynomial in two variables*

$$F(X, Y) = \sum_{i \in S_1} b_i X^{n_i} + \sum_{i \in S_2} b_i Y^{n_i} \in K[X, Y]$$

is irreducible over $K(X)$, if $\partial F / \partial Y(X, X)$ is not divisible by f.

The following results extend Theorem B by considering polynomials of the form $f_1 \circ f_2 + f_3$ instead of $f_1 \circ f_2$:

Theorem 1.5. *Let K be a field of characteristic 0 and let $f, f_1, f_2, f_3 \in K[X]$ with $\deg f_2 \geq 2$, $\deg f_3 < \deg f_1$ and f irreducible over K. If $f_1 \circ f_2 + f_3 = f^s$ for an integer $s \geq 2$ and $f_1' \circ f_2 + f_3'$ is not divisible by f, then $f_1 \circ (f_2(X) - X + Y) + f_3(Y) \in K[X, Y]$ is irreducible over $K(X)$.*

The corresponding result for $s = 1$ is the following:

Theorem 1.6. *Let K be a field of characteristic 0 and let $f_1, f_2, f_3 \in K[X]$ with $\deg f_2 \geq 2$, $\deg f_3 < \deg f_1$. If $f_1 \circ f_2 + f_3$ is irreducible over K, then $f_1 \circ (f_2(X) - X + Y) + f_3(Y) \in K[X, Y]$ is irreducible over $K(X)$.*

The above results are flexible, and may be useful in applications. On the one hand, they allow to test the irreducibility of various polynomials when other irreducibility criteria fail, and on the other hand, they allow to construct various classes of irreducible multivariate polynomials from arbitrary irreducible polynomials in a smaller number of variables. In this connection, we provide several examples in the last section of the paper. In the proofs of the above results we follow the same method employed in [4]. For the sake of completeness, we give detailed proofs of the results in §2 below.

2. Proofs of the main results

For the proof of our results we need the following lemma:

Lemma 2.1. *Let K be a field and let $F(X, Y) = \sum_{i=0}^{n} a_i(X)Y^i \in K[X, Y]$, with $a_0, a_1, \ldots, a_n \in K[X]$, $a_0 a_n \neq 0$. Let us assume that there exist three polynomials $f, g, h \in K[X]$ such that f is irreducible over K, $g \neq 0$, $F(X, h(X)) = f(X)^s \cdot g(X)$ for an integer $s \geq 2$, and $\partial F/\partial Y(X, h(X))$ is not divisible by f. Then F is irreducible over $K(X)$ if either $\deg g < \deg h$ and for an index $j \in \{1, \ldots, n\}$ with $a_j \neq 0$ we have*

$$\max_{k<j} \frac{\deg a_k - \deg a_j}{j - k} < \deg h < \min_{k>j} \frac{\deg a_k - \deg a_j}{j - k}, \tag{1}$$

or if

$$\min_{k>0} \frac{\deg a_0 - \deg a_k}{k} > \max\{\deg h,\ \deg g\}. \tag{2}$$

Proof. As in [4], we base our proof on the study of the location of the roots of F, regarded as a polynomial in Y with coefficients in $K[X]$. We first introduce a nonarchimedean absolute value $|\cdot|$ on $K(X)$, as follows. We fix an arbitrary real number $\rho > 1$, and for any polynomial $u(X) \in K[X]$ we define $|u(X)|$ by the equality

$$|u(X)| = \rho^{\deg u(X)}.$$

We then extend the absolute value $|\cdot|$ to $K(X)$ by multiplicativity. Thus for any $w(X) \in K(X)$, $w(X) = \frac{u(X)}{v(X)}$, with $u(X), v(X) \in K[X]$, $v(X) \neq 0$, and we let $|w(X)| = \frac{|u(X)|}{|v(X)|}$. Let us note that for any non-zero element u of $K[X]$ one has $|u| \geq 1$. The nonarchimedean absolute value $|\cdot|$ arises from the discrete valuation v on $K(X)$ given by $v(f) = -\deg f$ for $f \in K[X] \setminus \{0\}$. This v is the $1/X$-adic valuation on $K(X)$, i.e., the valuation associated to the discrete valuation ring obtained by localizing $K[1/X]$ at its principal ideal generated by $1/X$. Let now $\overline{K(X)}$ be a fixed algebraic closure of $K(X)$. The valuation ring of v has an extension to a valuation ring with quotient field $\overline{K(X)}$, and one can use that ring to extend v to a valuation w on $\overline{K(X)}$. This in turn provides us with a nonarchimedean absolute value which is an extension of our absolute value $|\cdot|$ to $\overline{K(X)}$, and which we will also denote by $|\cdot|$.

Next, assume contrarily that our polynomial F factors as $F(X,Y) = F_1(X,Y)\cdot F_2(X,Y)$, with $F_1,\ F_2 \in K[X,Y]$, $\deg_Y F_1 = t \geq 1$ and $\deg_Y F_2 = s \geq 1$. Then, since

$$F(X,h(X)) = f(X)^s \cdot g(X) = F_1(X,h(X)) \cdot F_2(X,h(X)) \tag{3}$$

and

$$\begin{aligned}\partial F/\partial Y(X,h(X)) &= \partial F_1/\partial Y(X,h(X)) \cdot F_2(X,h(X)) \\ &\quad + \partial F_2/\partial Y(X,h(X)) \cdot F_1(X,h(X)),\end{aligned}$$

our condition that $\partial F/\partial Y(X,h(X))$ is not divisible by f prevents f to divide both $F_1(X,h(X))$ and $F_2(X,h(X))$. So by (3) it follows that one of the polynomials $F_1(X,h(X))$, $F_2(X,h(X))$ must divide $g(X)$, say $F_1(X,h(X))\,|\,g(X)$. In particular, one has

$$\deg F_1(X,h(X)) \leq \deg g(X). \tag{4}$$

We now consider the factorization of the polynomial $F(X,Y)$ over $\overline{K(X)}$, say $F(X,Y) = a_n(X)(Y-\theta_1)\cdots(Y-\theta_n)$, with $\theta_1,\ldots,\theta_n \in \overline{K(X)}$. Since $a_0 \neq 0$ we must have $|\theta_i| \neq 0$, $i = 1,\ldots,n$. Let us denote

$$A = \max_{k<j} \frac{\deg a_k - \deg a_j}{j-k} \quad \text{and} \quad B = \min_{k>j} \frac{\deg a_k - \deg a_j}{j-k},$$

and notice that by (1) A is strictly smaller than B. Then for each $i = 1,\ldots,n$ we must either have $|\theta_i| \leq \rho^A$, or $|\theta_i| \geq \rho^B$. In order to prove this, let us assume contrarily that for some index $i \in \{1,\ldots,n\}$ we have $\rho^A < |\theta_i| < \rho^B$. Since $a_j \neq 0$ we deduce from $\rho^A < |\theta_i|$ that $|a_j|\cdot|\theta_i|^j > |a_k|\cdot|\theta_i|^k$ for each $k < j$, while from $|\theta_i| < \rho^B$ we find that $|a_j|\cdot|\theta_i|^j > |a_k|\cdot|\theta_i|^k$ for each $k > j$. By taking the maximum with respect to k in these inequalities, we obtain

$$|a_j|\cdot|\theta_i|^j > \max_{k\neq j} |a_k|\cdot|\theta_i|^k. \tag{5}$$

On the other hand, since $F(X,\theta_i) = 0$, we must have

$$0 \geq |a_j\theta_i^j| - \Big|\sum_{k\neq j} a_k\theta_i^k\Big| \geq |a_j|\cdot|\theta_i|^j - \max_{k\neq j}|a_k|\cdot|\theta_i|^k,$$

which contradicts (5).

Now, since $F_1(X,Y)$ is a factor of our polynomial $F(X,Y)$, it will factor over $\overline{K(X)}$ as $F_1(X,Y) = b_t(X)(Y-\theta_1)\cdots(Y-\theta_t)$, say, with $b_t(X) \in K[X]$, $b_t(X) \neq 0$. In particular, we have

$$|b_t(X)| \geq 1. \tag{6}$$

Recalling the definition of our absolute value and using (4) and (6), we deduce that

$$\rho^{\deg g} \geq \rho^{\deg F_1(X,h(X))} = |F_1(X, h(X))|$$
$$= |b_t(X)| \cdot \prod_{i=1}^{t} |h(X) - \theta_i| \geq \prod_{i=1}^{t} |h(X) - \theta_i|.$$

Now, making use of the fact that the absolute value $|\cdot|$ is nonarchimedean, for any index $i \in \{1, \ldots, t\}$ we either have

$$|h(X) - \theta_i| = |h(X)| = \rho^{\deg h}, \quad \text{if} \quad |\theta_i| \leq \rho^A (< |h|),$$

or

$$|h(X) - \theta_i| = |\theta_i| \geq \rho^B, \quad \text{if} \quad |\theta_i| \geq \rho^B (> |h|).$$

Since $A < \deg h < B$ it follows that we must have

$$\rho^{\deg g} \geq \min\{\rho^{\deg h},\ \rho^B\},$$

since $t \geq 1$. On the other hand, by our assumption that $A < \deg h < B$ and $\deg g < \deg h$, both inequalities $\rho^{\deg g} \geq \rho^{\deg h}$ and $\rho^{\deg g} \geq \rho^B$ fail, and this completes the proof of the first part of the lemma.

Assume now that (2) holds. In this case all the θ_i's satisfy $|\theta_i| \geq \rho^B$ with $B = \min\limits_{k>0} \frac{\deg a_0 - \deg a_k}{k}$. Using again the fact that the absolute value $|\cdot|$ is nonarchimedean we have $|h(X) - \theta_i| \geq \rho^B$, for each $i \in \{1, \ldots, n\}$. This implies that we must have $\rho^{\deg g} \geq \rho^B$. On the other hand, this inequality cannot hold since $B > \deg g$, and this completes the proof. □

The conditions in the statement of Lemma 2.1 are in some sense best possible. First, let us note that the condition $\deg g < \deg h$ in the statement of Lemma 2.1 cannot be replaced by $\deg g \leq \deg h$. To see this, choose for instance $K = \mathbb{Q}$, $F(X, Y) = (X + 3)Y^2 + (4X + 4)Y + 3X + 1$, $f(X) = X + 1$, $g(X) = X^2 + 1$ and $h(X) = X^2$. Here we have $F(X, h(X)) = f(X)^3 \cdot g(X)$, f is irreducible over $\mathbb{Q}$, $\partial F/\partial Y(X, h(X)) = 2X^3 + 6X^2 + 4X + 4$, which is not divisible by f, and (1) is satisfied for $j = n = 2$, but our polynomial F is reducible over $\mathbb{Q}(X)$, since $F(X, Y) = (Y + 1)((X + 3)Y + 3X + 1)$.

In condition (1) we must have strict inequalities too, since there exist polynomials F, f, g and h as in Lemma 2.1, but with equality in at least one side of (1) for each index $j \in \{1, \ldots, n\}$, and such that F is reducible over $K(X)$. Take now $K = \mathbb{Q}$, $F(X, Y) = (X - 1)Y^2 - (X^2 - 3X + 1)Y + X - X^2$, $f(X) = X$, $g(X) = 1$ and $h(X) = X$. Here $F(X, h(X)) = f(X)^2 \cdot g(X)$, $\partial F/\partial Y(X, h(X)) = X^2 + X - 1$, which is not divisible by f, $\deg g < \deg h$, and we have equality in at least one side of (1) for each $j = 1, 2$, while F is reducible over $\mathbb{Q}(X)$, since $F(X, Y) = (Y - X + 1)((X - 1)Y + X)$.

Finally, for condition (2) let $F(X, Y) = (1 - X)Y^2 + (3X^2 - 3X + 1)Y - 2X^3 + 3X^2 - X$, $f(X) = h(X) = X$ and $g(X) = 1$. Here $F(X, h(X)) = f(X)^2 \cdot g(X)$ and $\partial F/\partial Y(X, h(X)) = X^2 - X + 1$, which is not divisible by f, and in this case we have equality in (2) and F is reducible over $\mathbb{Q}(X)$, since $F(X, Y) = (Y - X + 1)((1 - X)Y + 2X^2 - X)$.

2.1 *Proof of Theorem* 1.1

Note that in particular, for $j = n$ one obtains from Lemma 2.1 the following irreducibility criterion.

COROLLARY 2.2

Let K be a field and let $F(X,Y) = \sum_{i=0}^{n} a_i(X)Y^i \in K[X,Y]$, with $a_0, \dots, a_n \in K[X]$, $a_0 a_n \neq 0$. Suppose there exist polynomials $f, g, h \in K[X]$ with f irreducible over K, $g \neq 0$, $\deg g < \deg h$, $\deg a_i < \deg h + \deg a_n$, $i = 0, \dots, n-1$, and such that $F(X, h(X)) = f(X)^s \cdot g(X)$ for an integer $s \geq 2$. If $\partial F/\partial Y(X, h(X))$ is not divisible by f, then F is irreducible over $K(X)$.

Theorem 1.1 now follows from Corollary 2.2, since in our case $\deg a_i < \deg h$, $i = 0, \dots, n-1$.

2.2 *Proof of Theorem* 1.2

This follows immediately by Lemma 1.2 using (2) with $f(X) = \sum_{i=0}^{n} a_i(X)$ and $g(X) = h(X) = 1$. Thus, for an arbitrary irreducible polynomial $f \in K[X]$ and an arbitrary integer $s \geq 2$, by writing $f^s(X) = \sum_{i=0}^{n} a_i(X)$ with $\deg a_0 > \max_{1 \leq i \leq n} \deg a_i$ and $a_1 + 2a_2 + \cdots + na_n$ not divisible by f, one may construct polynomials $F(X,Y) = \sum_{i=0}^{n} a_i(X)Y^i$ of arbitrarily large degrees with respect to Y, and which are irreducible over $K(X)$.

2.3 *Proof of Theorem* 1.3

The conclusion follows by Lemma 2.1 with $h(X) = X$ and $g(X) = 1$. To see this, we notice that the polynomial F may contain different monomials having the same power of the indeterminate Y, say $b_u X^{i_u} Y^{j_u}$ and $b_t X^{i_t} Y^{j_t}$ with $j_u = j_t$ and $i_u < i_t$. In this case we have

$$\frac{i_v - i_u}{j_t - j_v} > \frac{i_v - i_t}{j_t - j_v} \quad \text{for } j_v < j_t \quad \text{and} \quad \frac{i_v - i_u}{j_t - j_v} < \frac{i_v - i_t}{j_t - j_v} \quad \text{for } j_v > j_t.$$

Therefore, grouping together all the monomials containing the same power of Y and writing F as a polynomial in Y with coefficients $a_i \in K[X]$, one obtains

$$\max_{j_v < j_t} \tfrac{\deg a_{j_v} - \deg a_{j_t}}{j_t - j_v} \leq \max_{j_v < j_t} \tfrac{i_v - i_t}{j_t - j_v} \quad \text{and} \quad \min_{j_v > j_t} \tfrac{\deg a_{j_v} - \deg a_{j_t}}{j_t - j_v} \geq \min_{j_v > j_t} \tfrac{i_v - i_t}{j_t - j_v},$$

so our condition

$$\max_{j_v < j_t} \frac{i_v - i_t}{j_t - j_v} < 1 < \min_{j_v > j_t} \frac{i_v - i_t}{j_t - j_v},$$

will imply (1).

2.4 *Proof of Theorem* 1.4

Let S_1 and S_2 be as in the statement of Theorem 1.4 and note that F remains unchanged if we move b_0 from one sum to another. So without loss of generality, one may assume that 0 and k belong to S_1. Let $S_2 = \{i_1, \ldots, i_t\}$, with $t \geq 1$ and $i_1 < i_2 < \ldots < i_t$. Then one may write the polynomial f^s as $f(X)^s = \sum_{j=0}^{n_{i_t}} a_j(X)X^j$, with $a_0(X) = \sum_{i \in S_1} b_i X^{n_i}$, $a_{n_i}(X) = b_i$, $i \in S_2$ and $a_j(X) = 0$ for any index $j \in \{1, \ldots, n_{i_t}\} \setminus \{n_i \mid i \in S_2\}$. Thus, $\deg a_0 = n_k$ and

$$\min_{j>0} \frac{\deg a_0 - \deg a_j}{j} = \frac{n_k}{n_{i_t}} > 1,$$

since $n_{i_t} \leq n_k - 1$. Therefore (2) holds for $h(X) = X$, $g(X) = 1$ and $n = n_{i_t}$, so by Lemma 2.1 the polynomial

$$F(X, Y) = \sum_{j=0}^{n_{i_t}} a_j(X)Y^j = \sum_{i \in S_1} b_i X^{n_i} + \sum_{i \in S_2} b_i Y^{n_i}$$

must be irreducible over $K(X)$.

2.5 *Proof of Theorem* 1.5

Let f, f_1, f_2, f_3 be as in the statement of Theorem 1.5, and let $F(X, Y) = f_1 \circ (f_2(X) - X + Y) + f_3(Y)$, $g(X) = 1$ and $h(X) = X$, so $F(X, h(X)) = f(X)^s \cdot g(X)$. Let us denote $\deg f_1$ by n. Then $\deg_Y F(X, Y) = n$, and using the Taylor expansion of $F(X, Y)$ with respect to Y we may write F as $\sum_{i=0}^{n} a_i(X)Y^i$, with

$$a_i(X) = \frac{1}{i!} \cdot [f_1^{(i)} \circ (f_2(X) - X) + f_3^{(i)}(0)], \quad i = 0, \ldots, n.$$

This shows us that $\deg a_i = (n - i) \deg f_2$ for each $i = 0, \ldots, n$, so

$$\min_{k>0} \frac{\deg a_0 - \deg a_k}{k} = \min_{k>0} \frac{n \deg f_2 - (n - k) \deg f_2}{k} = \deg f_2 \geq 2.$$

On the other hand $\max\{\deg g, \deg h\} = 1$, so (2) is satisfied, and we conclude by Lemma 2.1 that F must be irreducible over $K(X)$.

2.6 *Proof of Theorem* 1.6

Here one applies the proof of Lemma 2.1 for $s = 1$, by ignoring the condition that $\partial F/\partial Y(X, h(X))$ is not divisible by f, which is no longer necessary. The rest of the proof is similar to that given for Theorem 1.5.

3. Examples

(i) For each $n \geq 1$ the polynomial $(X^n - X^{n-1}) + (X^{n-1} - X^{n-2})Y + \cdots + (X - 1) Y^{n-1} + Y^n$ is irreducible over $\mathbb{Q}(X)$. Indeed, take $f(X) = X$, $s = n \geq 2$, and write $f^n = a_0 + a_1 + \cdots + a_n$ with $a_i(X) = X^{n-i} - X^{n-i-1}$ for $i = 0, \ldots, n - 1$

and $a_n(X) = 1$. Here $\deg a_0 > \max_{1\le i\le n} \deg a_i$, $f \nmid a_1 + 2a_2 + \cdots + na_n = 1 + X + \cdots + X^{n-1}$, and the conclusion follows from Theorem 1.2.

(ii) Let K be a field, $f \in K[X]$ irreducible over K, and assume that for an integer $s \ge 2$ we have $f(X)^s = b_0X^{n_0} + b_1X^{n_1} + \cdots + b_kX^{n_k} \in K[X]$, $0 = n_0 < n_1 < \cdots < n_k$, $b_0 \cdots b_k \neq 0$. Construct as in Theorem 1.3 from f^s the polynomial $F(X,Y) = b_0X^{i_0}Y^{j_0} + b_1X^{i_1}Y^{j_1} + \cdots + b_kX^{i_k}Y^{j_k} \in K[X,Y]$, with $i_l, j_l \ge 0$, $i_l + j_l = n_l$, $l = 0,\dots,k$ and assume that $\partial F/\partial Y(X,X)$ is not divisible by f. Next, denote by S the set of those indices t for which $j_t = \max\{j_0,\dots,j_k\}$. If $\max_{\alpha\notin S} i_\alpha < 1 + \max_{\alpha\in S} i_\alpha$, then F must be irreducible over $K(X)$. Indeed, when we write F as $\sum_{i=0}^{n} a_i(X)Y^i$, we obtain $\deg a_n = \max_{\alpha\in S} i_\alpha$ and $\max_{0\le i\le n-1} \deg a_i = \max_{\alpha\notin S} i_\alpha$, and the conclusion follows from Corollary 2.2 with $h(X) = X$ and $g(X) = 1$.

For instance, taking $K = \mathbb{Q}$, $f(X) = X + 1$ and $s = 6$ we deduce that each of the polynomials

$$\begin{aligned} f_1(X,Y) &= 1 + 6Y + 15XY + 20X^2Y + 15X^3Y + 6X^2Y^3 + X^3Y^3,\\ f_2(X,Y) &= 1 + 6X + 15XY + 20XY^2 + 15X^2Y^2 + 6X^2Y^3 + X^2Y^4,\\ f_3(X,Y) &= 1 + 6Y + 15Y^2 + 20X^2Y + 15X^3Y + 6X^3Y^2 + X^3Y^3, \end{aligned}$$

is irreducible over $\mathbb{Q}(X)$.

(iii) Let $f(X) = 1 + X^2$, write f^5 as $f^5(X) = f_1 \circ f_2 + f_3$, with $f_1(X) = 10X^3 + 5X^4 + X^5$, $f_2(X) = X^2$ and $f_3(X) = 1 + 5X^2 + 10X^4$, and note that $f_1' \circ f_2(X) + f_3'(X) = 10X + 40X^3 + 30X^4 + 20X^6 + 5X^8$, which is not divisible by f. Then, in view of Theorem 1.5, the polynomial $1 + 5Y^2 + 10Y^4 + 10(X^2 - X + Y)^3 + 5(X^2 - X + Y)^4 + (X^2 - X + Y)^5$ is irreducible over $\mathbb{Q}(X)$.

(iv) An immediate consequence of Theorem 1.6 is the following irreducibility criterion:

COROLLARY 3.1

Let K be a field of characteristic 0, $f(X) = \sum_{i=0}^{n} a_iX^{di} \in K[X]$ an irreducible polynomial, $a_0 \cdots a_n \neq 0$, $d \ge 2$. Then for any fixed integer k such that $0 \le kd < n$, the polynomial $\sum_{i=0}^{k} a_iY^{di} + \sum_{i=k+1}^{n} a_i(X^d - X + Y)^i \in K[X,Y]$ is irreducible over $K(X)$.

Indeed, one may write f as $f = f_1 \circ f_2 + f_3$, with $f_1(X) = \sum_{i=k+1}^{n} a_iX^i$, $f_2(X) = X^d$, and $f_3(X) = \sum_{i=0}^{k} a_iX^{di}$, so by Theorem 1.6 the polynomial $\sum_{i=0}^{k} a_iY^{di} + \sum_{i=k+1}^{n} a_i(X^d - X + Y)^i$ must be irreducible over $K(X)$.

For instance, since 102 040 201 and 103 060 301 are prime numbers, by Cohn's theorem the polynomials $1+2X^2+4X^4+2X^6+X^8$ and $1+3X^2+6X^4+3X^6+X^8$ are irreducible over $\mathbb{Q}$, hence the polynomials

$$\begin{aligned} f_1(X,Y) &= 1 + 2Y^2 + 4(X^2 - X + Y)^2 + 2(X^2 - X + Y)^3\\ &\quad + (X^2 - X + Y)^4\\ f_2(X,Y) &= 1 + 3Y^2 + 6(X^2 - X + Y)^2 + 3(X^2 - X + Y)^3\\ &\quad + (X^2 - X + Y)^4 \end{aligned}$$

are irreducible over $\mathbb{Q}(X)$.

We end by noting that as another consequence of Lemma 2.1, one may formulate similar irreducibility criteria for polynomials in $r \geq 3$ variables $X_1, X_2, \ldots, X_r$ over K. For any polynomial $f \in K[X_1, \ldots, X_r]$ we denote by $\deg_r f$ the degree of f as a polynomial in X_r with coefficients in $K[X_1, \ldots, X_{r-1}]$. The next result follows from Lemma 2.1 by writing Y for X_r, X for X_{r-1} and by replacing K with the field generated by K and the variables $X_1, \ldots, X_{r-2}$.

COROLLARY 3.2

Let K be a field, $r \geq 3$, and let $F = \sum_{i=0}^{n} a_i X_r^i \in K[X_1, \ldots, X_r]$ with $a_0, \ldots, a_n \in K[X_1, \ldots, X_{r-1}]$, $a_0 a_n \neq 0$. Suppose there exist polynomials $f, g, h \in K[X_1, \ldots, X_{r-1}]$ such that $\sum_{i=0}^{n} a_i h^i = f^s \cdot g$ for an integer $s \geq 2$, f as a polynomial in X_{r-1} is irreducible over $K(X_1, \ldots, X_{r-2})$, $g \neq 0$, and $\partial F/\partial X_r$ $(X_1, \ldots, X_{r-1}, h(X_1, \ldots, X_{r-1}))$ is not divisible by f. Then F as a polynomial in X_r is irreducible over $K(X_1, \ldots, X_{r-1})$ if either $\deg_{r-1} g < \deg_{r-1} h$ and for an index $j \in \{1, \ldots, n\}$ with $a_j \neq 0$ we have

$$\max_{k<j} \frac{\deg_{r-1} a_k - \deg_{r-1} a_j}{j-k} < \deg_{r-1} h < \min_{k>j} \frac{\deg_{r-1} a_k - \deg_{r-1} a_j}{j-k},$$

or if

$$\min_{k>0} \frac{\deg_{r-1} a_0 - \deg_{r-1} a_k}{k} > \max\{\deg_{r-1} h,\ \deg_{r-1} g\}.$$

Acknowledgement

This work was supported by CNCSIS-UEFISCSU, project PNII-IDEI 51, code 304/2007.

References

[1] Bonciocat A I and Bonciocat N C, Some classes of irreducible polynomials, *Acta Arith.* **123(4)** (2006) 349–360

[2] Bonciocat A I and Bonciocat N C, A Capelli type theorem for multiplicative convolutions of polynomials, *Math. Nachr.* **281(9)** (2008) 1240–1253

[3] Bonciocat A I and Zaharescu A, Irreducibility results for compositions of polynomials in several variables, *Proc. Indian Acad. Sci. Math. Sci.* **115(2)** (2005) 117–126

[4] Bonciocat N C and Zaharescu A, Irreducible multivariate polynomials obtained from polynomials in fewer variables, *J. Pure Appl. Algebra* **212** (2008) 2338–2343

[5] Cavachi M, Vajaitu M and Zaharescu A, An irreducibility criterion for polynomials in several variables, *Acta Math. Univ. Ostrav.* **12(1)** (2004) 13–18

[6] Girstmair K, On an irreducibility criterion of M. Ram Murty, *Am. Math Mon.* **112(3)** (2005) 269–270

[7] Pólya G and Szegö G, Aufgaben und Lehrsätze aus der Analysis, vol. 2 (Berlin: Springer-Verlag) (1925)

[8] Ram Murty M, Prime numbers and irreducible polynomials, *Am. Math Mon.* **109(5)** (2002) 452–458

[9] Schinzel A, Polynomials with special regard to reducibility, Encycl. Math Appl., vol. 77 (Cambridge Univ. Press) (2000)