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Analysis of a malaria model with mosquito-dependent transmission coefficient for humans

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Abstract. In this paper, we discuss an ordinary differential equation mathematical model for the spread of malaria in human and mosquito population. We suppose the human population to act as a reservoir. Both the species follow a logistic population model. The transmission coefficient or the interaction coefficient of humans is considered to be dependent on the mosquito population. It is seen that as the factors governing the transmission coefficient of humans increase, so does the number of infected humans. Further, it is observed that as the immigration constant increases, it leads to a rise in infected humans, giving an endemic shape to the disease.

Keywords. Transmission coefficient for humans; immigration; reservoir class of humans.

1. Introduction

The epidemiology of malaria is an example of an intricate biological system in which the parasite and its vertebrate and the anthropod hosts interact in close dependence on environmental conditions. Malaria is caused by four species of protozoa in the genus *Plasmodium*. The parasites are transmitted to humans, only by female mosquitoes belonging to certain species of the genus anopheles, each time the infected insect takes a blood meal. Conversely, the female mosquito can pick up the infection when they bite infected humans. Mosquito density largely depends on the longevity of the adult mosquitoes and on the availability of suitable breeding places.

In most of the tropical countries including India, the emergence of malaria has occurred and it has become endemic in the North-Eastern part of India, where this disease is mainly spread by the lethal parasite called *Plasmodium falciparum*. The environmental condition in the tropics is the prime factor for it being endemic. The moderate-to-warm temperatures, high humidity and water bodies allow mosquito and parasites to reproduce. The epidemiological patterns of malaria usually vary with season because of the dependence of transmission on mosquito. Every year 1–3 million deaths are attributed worldwide to malaria, out of which one-third are children.

Malaria can be prevented and controlled if detected early. It is therefore important to understand the main parameters in the transmission of the disease and to develop effective solution strategies for its prevention and control.

In our study, we consider the transmission coefficient of humans to be dependent on the dynamics of the vector population. Also the dynamics of the mosquito population plays

an important role. We propose to study the dynamics and transmission of the malaria parasites as well along with that of humans.

In highly endemic areas, people who are repeatedly infected by malaria develop a certain degree of immunity (which suppresses most clinical symptoms). They carry gametocytes in their blood that infect the mosquito biting them. They form a class of persons who are partially immune to the disease but who may be infectious Ngwa and Shu [11].

Mathematical models of the transmission of infectious agents in human communities have been studied by Hethcote [8], Gao and Hethcote [6], Lorca and Hethcote [10] and Becker [4] with a view to interpret observed trends of the disease and to design programmes for its control.

Modeling of malaria epidemic has been going on since 1911 with Ross's model by Bailey [2,3], Anderson and May [1], Koella [9], Ngwa and Shu [11], Ghosh *et al* [7] and Singh *et al* [12]. They have analysed models with two populations – one representing humans and another representing mosquitoes. Some of them have included density-dependent death rates, environmental factors and other factors. However, in these studies, the effect of transmission coefficient for humans, dependent on density of mosquitoes has not been considered.

In the present paper, an SIRS model for humans and an SI model for mosquitoes have been proposed which takes into account the above mentioned factors—a variable transmission coefficient and a class of persons who are immune to the disease but infectious.

2. Malaria model

We divide the human population N_1 into three classes: susceptible X_1, infected Y_1 and recovered Z_1. People enter the susceptible class either through immigration or through birth. When an infected mosquito bites a susceptible human, the sporozoites get passed off to the susceptible human and he enters the infected class. After sometime the infectious human recovers and moves to the recovered class. The recovered individuals have some immunity to the disease and do not get clinically ill, but they harbour low levels of parasites in their blood stream and can pass the infection to mosquitoes. After sometime they lose their immunity and return to the susceptible class. Humans leave the population through emigration from one class to the other or by natural death.

The mosquito population N_2 is divided into two classes: susceptible X_2 and infected Y_2. Mosquito enter the susceptible class through birth. The parasites (in the form of gametocytes) enter the mosquito, when the susceptible mosquito bites an infected human or recovered human (where the chances of transmission of infection from a recovered human is much lower than that from an infected human) and the mosquito then moves to the infected class. The mosquito remains infectious life-long until removed by death.

By considering the criss-cross interaction of the mosquito population with the human population, we write the equations that describe the spread of the disease in the form:

$$\frac{\mathrm{d}X}{\mathrm{d}t} = b_1 + A - d_1 X_1 - \beta_1 (N_2)\, X_1 Y_2 + \delta Z_1,$$

$$\frac{\mathrm{d}Y_1}{\mathrm{d}t} = \beta_1 (N_2)\, X_1 Y_2 - \gamma Y_1 - d_1 Y_1,$$

$$\frac{dZ_1}{dt} = \gamma Y_1 - d_1 Z_1 - \delta Z_1,$$

$$\frac{dN_1}{dt} = b_1 + A - d_1 N_1,$$

$$\frac{dX_2}{dt} = b_2 - d_2 X_2 - \beta_2 X_2 Y_1 - \beta_3 X_2 Z_1,$$

$$\frac{dY_2}{dt} = -d_2 Y_2 + \beta_2 X_2 Y_1 + \beta_3 X_2 Z_1,$$

$$\frac{dN_2}{dt} = b_2 - d_2 N_2, \tag{1.1}$$

where $N_1 = X_1 + Y_1 + Z_1$ and $N_2 = X_2 + Y_2$.

In the model (1.1), $b_1 = c_1 N_1$ is the total birth of the human population (c_1 is the birth rate of human population), A is the constant immigration rate and d_1 is the natural death rate of the human population. $\beta_1(N_2)$ is the transmission coefficient of a susceptible human with the infective mosquito population which depends upon the density of the mosquito population. γ is the recovery rate of the human population and δ is the rate at which recovered humans lose their immunity and join the susceptible class. $b_2 = c_2 N_2$ is the total birth of the mosquito population (c_2 is the birth rate of mosquito population) and d_2 is its death rate. β_2 and β_3 are the interaction rates of susceptible mosquitoes with the infective and recovered classes of the human population respectively ($\beta_2 > \beta_3$).

The model has been analysed in the following two cases:

(a) the transmission coefficient $\beta_1(N_2)$ of the susceptible humans with the infective mosquito population is a constant, and
(b) it depends upon the density of the mosquito population. For positive constants a_0 and a_1, $\beta_1(N_2)$ takes the form

$$\beta_1(N_2) = a_0 + a_1 N_2.$$

2.1 *Case* 1. *When* $\beta_1(N_2) = \beta_0$*;* β_0 *is a constant*

Since $X_1 + Y_1 + Z_1 = N_1$ and $X_2 + Y_2 = N_2$, the system (1.1) can be reduced to the form:

$$\frac{dY_1}{dt} = \beta_0 (N_1 - Y_1 - Z_1) Y_2 - (\gamma + d_1) Y_1,$$

$$\frac{dZ_1}{dt} = \gamma Y_1 - (d_1 + \delta) Z_1,$$

$$\frac{dN_1}{dt} = b_1 + A - d_1 N_1,$$

$$\frac{dY_2}{dt} = -d_2 Y_2 + \beta_2 (N_2 - Y_2) Y_1 + \beta_3 (N_2 - Y_2) Z_1,$$

$$\frac{dN_2}{dt} = b_2 - d_2 N_2. \tag{1.2}$$

The region of attraction of the above system is

$$T_1 = \{(Y_1, Z_1, N_1, Y_2, N_2) : 0 \le Y_1 + Z_1 \le N_1 \le \bar{N}_1, 0 \le Y_2 \le N_2 \le \bar{N}_2\},$$

where $\bar{N}_1 = \limsup_{t\to\infty} N_1 = \frac{b_1+A}{d_1}$ and $\bar{N}_2 = \limsup_{t\to\infty} N_2 = \frac{b_2}{d_2}$.

There exist the following three equilibria corresponding to the system (1.2), namely:

(1) $E_0\left(0, 0, \frac{b_1+A}{d_1}, 0, 0\right)$.
(2) $E_1\left(0, 0, \frac{b_1+A}{d_1}, 0, \frac{b_2}{d_2}\right)$ and
(3) $E_2\left(\tilde{Y}_1, \tilde{Z}_1, \tilde{N}_1, \tilde{Y}_2, \tilde{N}_2\right)$, where

$$\tilde{Y}_1 = \frac{\beta_0 \bar{N}_1 \bar{N}_2 \left(\beta_2 + \frac{\beta_3\gamma}{d_1+\delta}\right) - d_2(\gamma + d_1)}{\left(\beta_2 + \frac{\beta_3\gamma}{d_1+\delta}\right)\left\{\beta_0\left(1 + \frac{\gamma}{d_1+\delta}\right)\bar{N}_2 + (\gamma + d_1)\right\}},$$

$$\tilde{Y}_2 = \frac{\left(\beta_2 + \frac{\beta_3\gamma}{d_1+\delta}\right)\bar{N}_2\tilde{Y}_1}{\left(\beta_2 + \frac{\beta_3\gamma}{d_1+\delta}\right)\tilde{Y}_1 + d_2},$$

$$\tilde{Z}_1 = \frac{\gamma\tilde{Y}_1}{d_1+\delta}, \qquad \tilde{N}_1 = \frac{b_1+A}{d_1}, \qquad \tilde{N}_2 = \frac{b_2}{d_2}. \tag{1.3}$$

The equilibrium E_2 exists if

$$R_0 = \frac{\beta_0 \bar{N}_1 \bar{N}_2 \left(\beta_2 + \frac{\beta_3\gamma}{d_1+\delta}\right)}{d_2(\gamma + d_1)} > 1, \tag{1.4}$$

where R_0 is the threshold parameter for the system (1.2).

2.1a *Stability analysis*. We state the local stability of the three equilibria E_0, E_1, E_2 in the following theorem.

Theorem 1. *The equilibrium E_0 is stable. The equilibrium E_1 is stable if $R_0 < 1$, otherwise if $R_0 > 1$, it is unstable and the equilibrium E_2 exists and is stable if $q_1q_2 - q_3 > 0$.*

Proof. The general variational matrix M corresponding to the system (1.2) is

$$M = \begin{pmatrix} -(\beta_0 Y_2 + \gamma + d_1) & -\beta_0 Y_2 & \beta_0 Y_2 & \beta_0(\bar{N}_1 - Y_1 - Z_1) & 0 \\ \gamma & -(d_1+\delta) & 0 & 0 & 0 \\ 0 & 0 & -d_1 & 0 & 0 \\ \beta_2(\bar{N}_2 - Y_2) & \beta_3(\bar{N}_2 - Y_2) & 0 & -(\beta_2 Y_1 + \beta_3 Z_1 + d_2) & (\beta_2 Y_1 + \beta_3 Z_1) \\ 0 & 0 & 0 & 0 & -d_2 \end{pmatrix}.$$

At the equilibrium point $E_0(0, 0, \frac{b_1+A}{d_1}, 0, 0)$, the variational matrix M_0 is given by

$$M_0 = \begin{pmatrix} -(\gamma + d_1) & 0 & 0 & \beta_0\left(\frac{b_1+A}{d_1}\right) & 0 \\ \gamma & -(d_1+\delta) & 0 & 0 & 0 \\ 0 & 0 & -d_1 & 0 & 0 \\ 0 & 0 & 0 & -d_2 & 0 \\ 0 & 0 & 0 & 0 & -d_2 \end{pmatrix}.$$

The characteristic polynomial corresponding to the above matrix is

$$(d_1 + \lambda)(d_2 + \lambda)^2(d_1 + \delta + \lambda)(d_1 + \gamma + \lambda) = 0$$

which gives all negative roots of λ.

Thus the equilibrium E_0 is stable.

At the equilibrium point $E_1(0, 0, \frac{b_1+A}{d_1}, 0, \frac{b_2}{d_2})$, the variational matrix M_1 is given by

$$M_1 = \begin{pmatrix} -(\gamma + d_1) & 0 & 0 & \beta_0\left(\frac{b_1+A}{d_1}\right) & 0 \\ \gamma & -(d_1+\delta) & 0 & 0 & 0 \\ 0 & 0 & -d_1 & 0 & 0 \\ \frac{\beta_2 b_2}{d_2} & \frac{\beta_3 b_2}{d_2} & 0 & -d_2 & 0 \\ 0 & 0 & 0 & 0 & -d_2 \end{pmatrix}.$$

The characteristic polynomial corresponding to the above matrix is given by

$$(d_1 + \lambda)(d_2 + \lambda)\{\lambda^3 + p_1\lambda^2 + p_2\lambda + p_3\} = 0 \tag{1.5}$$

where

$$p_1 = 2d_1 + d_2 + \delta + \gamma,$$

$$p_2 = -\beta_0\beta_2\bar{N}_2\bar{N}_1 + (\delta + d_1)(d_1 + \gamma) + d_2(2d_1 + \delta + \gamma),$$

$$p_3 = -\beta_0\beta_3\bar{N}_2\bar{N}_1\gamma - \beta_0\beta_2\bar{N}_2\bar{N}_1(d_1 + \delta) + d_2(\gamma + d_1)(\delta + d_1).$$

We find that the eigenvalues of (1.5) are $-d_1$, $-d_2$ and the roots of the polynomial $\lambda^3 + p_1\lambda^2 + p_2\lambda + p_3$.

By the Routh–Hurwitz criterion [5], the above polynomial has roots with negative real part if $p_1p_2 - p_3 > 0$. Here

$$p_1p_2 - p_3 = (d_1 + \gamma)^2(\delta + d_1) + d_2(2d_1 + \delta + \gamma)(d_1 + \gamma + d_2)$$

$$+ \beta_0\beta_3\frac{b_2}{d_2}\gamma\left(\frac{b_1 + A}{d_1}\right)$$

which is positive.

Therefore, the stability of E_1 is given by the sign of p_3, which is positive iff $R_0 < 1$. For $R_0 > 1$, E_1 is unstable and E_2 exists.

At the equilibrium point $E_2(\tilde{Y}_1, \tilde{Z}_1, \tilde{N}_1, \tilde{Y}_2, \tilde{N}_2)$, the variational matrix M_2 is given by

$$M_2 = \begin{pmatrix} -(\beta_0\tilde{Y}_2+\gamma+d_1) & -\beta_0\tilde{Y}_2 & \beta_0\tilde{Y}_2 & \beta_0(\tilde{N}_1-\tilde{Y}_1-\tilde{Z}_1) & 0 \\ \gamma & -(d_1+\delta) & 0 & 0 & 0 \\ 0 & 0 & -d_1 & 0 & 0 \\ \beta_2(\tilde{N}_2-\tilde{Y}_2) & \beta_3(\tilde{N}_2-\tilde{Y}_2) & 0 & -(\beta_2\tilde{Y}_1+\beta_3\tilde{Z}_1+d_2) & (\beta_2\tilde{Y}_1+\beta_3\tilde{Z}_1) \\ 0 & 0 & 0 & 0 & -d_2 \end{pmatrix}.$$

The characteristic polynomial corresponding to the above matrix is

$$(d_1+\lambda)(d_2+\lambda)\left\{\lambda^3+q_1\lambda^2+q_2\lambda+q_3\right\}=0$$

where

$$\begin{aligned} q_1 &= \beta_2\tilde{Y}_1+\beta_3\tilde{Z}_1+d_2+\beta_0\tilde{Y}_2+\gamma+2d_1+\delta, \\ q_2 &= (d_1+\delta)(\beta_0\tilde{Y}_2+\gamma+d_1)+(\beta_0\tilde{Y}_2+\gamma+2d_1+\delta) \\ &\quad\times(\beta_2\tilde{Y}_1+\beta_3\tilde{Z}_1+d_2)-\beta_0\beta_2(\tilde{N}_1-\tilde{Y}_1-\tilde{Z}_1)(\tilde{N}_2-\tilde{Y}_2)+\beta_0\gamma\tilde{Y}_2, \\ q_3 &= (d_1+\delta)(\beta_0\tilde{Y}_2+\gamma+d_1)(\beta_2\tilde{Y}_1+\beta_3\tilde{Z}_1+d_2) \\ &\quad-\gamma\beta_0\beta_3(\tilde{N}_1-\tilde{Y}_1-\tilde{Z}_1)(\tilde{N}_2-\tilde{Y}_2)-\beta_0\beta_2(\tilde{N}_1-\tilde{Y}_1-\tilde{Z}_1) \\ &\quad\times(\tilde{N}_2-\tilde{Y}_2)(d_1+\delta)+\beta_0\gamma\tilde{Y}_2(\beta_2\tilde{Y}_1+\beta_3\tilde{Z}_1+d_2). \end{aligned}$$

Now

$$\begin{aligned} q_1q_2-q_3 &= (\gamma+d_1+\beta_0\tilde{Y}_2)^2(d_1+d_2+\delta+\beta_2\tilde{Y}_1+\beta_3\tilde{Z}_1) \\ &\quad+\gamma\beta_0\beta_3(\tilde{N}_1-\tilde{Y}_1-\tilde{Z}_1)(\tilde{N}_2-\tilde{Y}_2). \end{aligned}$$

It is seen that $q_1q_2 - q_3 > 0$. Hence by Routh–Hurwitz criterion, the equilibrium point E_2 is locally asymptotically stable.

2.1b *Simulations*. The system (1.2) is integrated using the fourth-order Runge Kutta method. Since malaria takes into account the births and deaths of humans, so a time unit of months is chosen.

The following data set has been used:

$$\begin{aligned} &\beta_0 = 0.00000029, \quad \beta_2 = 0.00000021, \quad \beta_3 = 0.00000009, \\ &d_1 = 0.00012 = b_1, \quad d_2 = 0.0085 = b_2, \quad \delta = 0.00146, \\ &\gamma = 0.012, \quad A = 12. \end{aligned}$$

Here $\beta_3 < \beta_2$. This is because the chance of acquiring infection by mosquitoes from recovered humans is less than the chance of acquiring infection by mosquitoes from infected humans.

In the numerical simulations, the initial values are taken in the light of Ghosh *et al* [7], and are given by

$$Y_1(0) = 4181.8, \quad Z_1(0) = 209.2, \quad N_1(0) = 19771.8,$$
$$Y_2(0) = 14051.1, \quad N_2(0) = 960977.1.$$

2.2 *Case* 2. *When* $\beta_1 = a_0 + a_1 N_2$; a_0 *and* a_1 *are constants*

In this case using $X_1 + Y_1 + Z_1 = N_1$, $X_2 + Y_2 = N_2$ and $\beta_1 = a_0 + a_1 N_2$, the model (1.1) can be written as

$$\begin{aligned}
\frac{\mathrm{d}Y_1}{\mathrm{d}t} &= (a_0 + a_1 N_2)(N_1 - Y_1 - Z_1) Y_2 - (\gamma + d_1) Y_1, \\
\frac{\mathrm{d}Z_1}{\mathrm{d}t} &= \gamma Y_1 - (d_1 + \delta) Z_1, \\
\frac{\mathrm{d}N_1}{\mathrm{d}t} &= b_1 + A - d_1 N_1, \\
\frac{\mathrm{d}Y_2}{\mathrm{d}t} &= -d_2 Y_2 + \beta_2 (N_2 - Y_2) Y_1 + \beta_3 (N_2 - Y_2) Z_1, \\
\frac{\mathrm{d}N_2}{\mathrm{d}t} &= b_2 - d_2 N_2. \qquad (1.6)
\end{aligned}$$

The region of attraction in this case is

$$T_2 = \{(Y_1, Z_1, N_1, Y_2, N_2,) : 0 \le Y_1 + Z_1 \le N_1 \le \widehat{N}_1, 0 \le Y_2 \le N_2 \le \widehat{N}_2\},$$

where $\widehat{N}_1 = \limsup_{t\to\infty} N_1 = \frac{b_1+A}{d_1}$ and $\widehat{N}_2 = \limsup_{t\to\infty} N_2 = \frac{b_2}{d_2}$.

There exist the following three equilibria corresponding to the system (1.6) namely:

(1) $P_0\left(\dot{Y}_1, 0, \frac{b_1+A}{d_1}, 0, 0\right)$,
(2) $P_1\left(Y_1', 0, \frac{b_1+A}{d_1}, 0, \frac{b_2}{d_2}\right)$,
(3) $P_2\left(\hat{Y}_2, \hat{Z}_1, \hat{N}_1, \hat{Y}_2, \hat{N}_2\right)$,

where

$$\dot{Y}_1 = \frac{-d_2}{\beta_2 + \frac{\beta_3\gamma}{d_1+\delta}},$$

$$Y_1' = \frac{\left(a_0 + a_1\frac{b_2}{d_2}\right)\frac{b_2}{d_2}\left(\frac{b_1+A}{d_1}\right)\left(\beta_2 + \frac{\beta_3\gamma}{d_1+\delta}\right) - d_2(\gamma + d_1)}{\left(\beta_2 + \frac{\beta_3\gamma}{d_1+\delta}\right)\left\{\left(a_0 + a_1\frac{b_2}{d_2}\right)\left(1 + \frac{\gamma}{d_1+\delta}\right)\frac{b_2}{d_2} + (\gamma + d_1)\right\}},$$

$$\hat{Y}_1 = \frac{(a_0 + a_1 N_2)\widehat{N}_1\widehat{N}_2\left(\beta_2 + \frac{\beta_3\gamma}{d_1+\delta}\right) - d_2(\gamma + d_1)}{\left(\beta_2 + \frac{\beta_3\gamma}{d_1+\delta}\right)\left\{(a_0 + a_1 N_2)\left(1 + \frac{\gamma}{d_1+\delta}\right)\widehat{N}_2 + (\gamma + d_1)\right\}}$$

and

$$\hat{Y}_2 = \frac{\left(\beta_2 + \frac{\beta_3\gamma}{d_1+\delta}\right)\widehat{N}_2\hat{Y}_1}{\left(\beta_2 + \frac{\beta_3\gamma}{d_1+\delta}\right)\hat{Y}_1 + d_2},$$

$$\hat{Z}_1 = \frac{\gamma\hat{Y}_1}{d_1+\delta}, \quad \hat{N}_1 = \frac{b_1+A}{d_1} \quad \text{and} \quad \hat{N}_2 = \frac{b_2}{d_2}.$$

The equilibrium P_3 exists if

$$\frac{(a_0 + a_1N_2)\,\widehat{N}_1\widehat{N}_2\left(\beta_2 + \frac{\beta_3\gamma}{d_1+\delta}\right)}{d_2\,(\gamma + d_1)} = R_0' > 1,$$

where R_0' is the threshold for the system (1.6).

2.2a *Stability analysis.* Here we state the stability analysis of the three equilibria and the local stability results of these equilibria are given in the following theorem:

Theorem 2. *The equilibrium P_0 is stable. The equilibria P_1 is stable if $R_0' < 1$, otherwise if $R_0' > 1$, it is unstable and the equilibrium P_2 exists, which is stable iff $s_1s_2 - s_3 > 0$.*

Proof. The variational matrix at $(Y_1, Z_1, N_1, Y_2, N_2)$ is given by

$$Q = \begin{pmatrix} -(a_0Y_2 + a_1N_2Y_2 + \gamma + d_1) & -(a_0Y_2 + a_1N_2Y_2) & a_0Y_2 + a_1N_2Y_2 \\ \gamma & -(d_1+\delta) & 0 \\ 0 & 0 & -d_1 \\ \beta_2\,(N_2 - Y_2) & \beta_3\,(N_2 - Y_2) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} (a_0 + a_1N_2)\,(N_1 - Y_1 - Z_1) & a_1N_1Y_2 - a_1Y_2Y_1 - a_1Z_1Y_2 \\ 0 & 0 \\ 0 & 0 \\ -(\beta_2Y_1 + \beta_3Z_1 + d_2) & \beta_2Y_1 + \beta_3Z_1 \\ 0 & -d_2 \end{pmatrix}.$$

At the equilibrium point $P_0(\dot{Y}_1, 0, \frac{b_1+A}{d_1}, 0, 0)$, the variational matrix Q_0 is given by

$$Q_0 = \begin{pmatrix} -(\gamma + d_1) & 0 & 0 & a_0\left(\frac{b_1+A}{d_1} - \dot{Y}_1\right) & 0 \\ \gamma & -(d_1+\delta) & 0 & 0 & 0 \\ 0 & 0 & -d_1 & 0 & 0 \\ 0 & 0 & 0 & -(\beta_2\dot{Y}_1 + d_2) & 0 \\ 0 & 0 & 0 & 0 & -d_2 \end{pmatrix}.$$

The characteristic polynomial corresponding to the above matrix is

$$(\gamma + d_1 + \lambda)\,(d_1 + \delta + \lambda)\,(d_1 + \lambda)\,(d_2 + \lambda)\left(\beta_2\dot{Y}_1 + d_2 + \lambda\right) = 0$$

which gives all negative roots of λ. Thus the equilibrium P_0 is stable.

At the equilibrium point $P_1(Y_1', 0, \frac{b_1+A}{d_1}, 0, \frac{b_2}{d_2})$, the variational matrix Q_1 is given by

$$Q_1 = \begin{pmatrix} -(\gamma+d_1) & 0 & 0 & \left(a_0+\frac{a_1b_2}{d_2}\right)\left(\frac{b_1+A}{d_1}-Y_1'\right) & 0 \\ \gamma & -(d_1+\delta) & 0 & 0 & 0 \\ 0 & 0 & -d_1 & 0 & 0 \\ \frac{\beta_2 b_2}{d_2} & \frac{\beta_3 b_2}{d_2} & 0 & -\left(\beta_2 Y_1'+d_2\right) & \beta_2 Y_1' \\ 0 & 0 & 0 & 0 & -d_2 \end{pmatrix}.$$

The characteristic polynomial corresponding to the above matrix is given by

$$(d_1+\lambda)(d_2+\lambda)\left\{\lambda^3+o_1\lambda^2+o_2\lambda+o_3\right\}=0,$$

where

$$o_1 = \gamma + 2d_1 + \delta + \beta_2 Y_1' + d_2,$$

$$o_2 = \gamma d_1 + \gamma\delta + d_1^2 + d_1\delta + (\gamma + 2d_1 + \delta)(\beta_2 Y_1' + d_2) + \left(a_0 + \frac{a_1 b_2}{d_2}\right)\left(\frac{b_1+A}{d_1} - Y_1'\right)\frac{\beta_2 b_2}{d_2},$$

$$o_3 = (\beta_2 Y_1' + d_2)(\gamma d_1 + \gamma\delta + d_1^2 + d_1\delta) + \left(a_0 + \frac{a_1 b_2}{d_2}\right)\left(\frac{b_1+A}{d_1} - Y_1'\right)\left(\frac{\gamma\beta_3 b_2}{d_2} + (d_1+\delta)\frac{\beta_2 b_2}{d_2}\right).$$

Two roots of the above polynomial are negative, while the other roots are given by the cubic equation. For no positive root to exist, we must have o_3 to be positive, which gives $R_0' < 1$. Further,

$$o_1 o_2 - o_3 = (\gamma + d_1)(\gamma d_1 + \gamma\delta + d_1^2 + d_1\delta) + d_2(\gamma + d_1 + d_2)(\gamma + 2d_1 + \delta) + (\beta_2 Y_1')^2(\gamma + 2d_1 + \delta) + \left\{(2d_2 + \gamma + d_1)(2d_1 + \gamma + \delta) + \left(a_0 + \frac{a_1 b_2}{d_2}\right)\gamma\frac{\beta_3 b_2}{d_2}\right\}\beta_2 Y_1' - \left(a_0 + \frac{a_1 b_2}{d_2}\right)\left(\frac{b_1+A}{d_1}\right)\frac{\beta_3 b_2}{d_2}\gamma$$

is positive.

So by Routh–Hurwitz criterion, the equilibrium point P_1 is stable if $R_0' < 1$ and if $R_0' > 1$, it is unstable and P_2 exists.

At the equilibrium point $P_2(\hat{Y}_1, \hat{Z}_1, \hat{N}_1, \hat{Y}_2, \hat{N}_2)$, the variational matrix is

$$Q_2 = \begin{pmatrix} -(a_0\hat{Y}_2 + a_1\hat{N}_2\hat{Y}_2 + \gamma + d_1) & -(a_0\hat{Y}_2 + a_1\hat{N}_2\hat{Y}_2) & a_0\hat{Y}_2 + a_1\hat{N}_2\hat{Y}_2 \\ \gamma & -(d_1+\delta) & 0 \\ 0 & 0 & -d_1 \\ \beta_2(\hat{N}_2 - \hat{Y}_2) & \beta_3(\hat{N}_2 - \hat{Y}_2) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} (a_0 + a_1\hat{N}_2)(\hat{N}_1 - \hat{Y}_1 - \hat{Z}_1) & a_1\hat{N}_1\hat{Y}_2 - a_1\hat{Y}_2\hat{Y}_1 - a_1\hat{Z}_1\hat{Y}_2 \\ 0 & 0 \\ 0 & 0 \\ -(\beta_2\hat{Y}_1 + \beta_3\hat{Z}_1 + d_2) & \beta_2\hat{Y}_1 + \beta_3\hat{Z}_1 \\ 0 & -d_2 \end{pmatrix}.$$

The characteristic polynomial corresponding to the above matrix is given by

$$(d_1 + \lambda)(d_2 + \lambda)\left\{\lambda^3 + s_1\lambda^2 + s_2\lambda + s_3\right\} = 0,$$

where

$$\begin{aligned} s_1 &= \beta_2\hat{Y}_1 + \beta_3\hat{Z}_1 + 2d_1 + d_2 + a_0\hat{Y}_2 + a_1\hat{N}_2\hat{Y}_2 + 2\gamma + \delta, \\ s_2 &= (a_0\hat{Y}_2 + a_1\hat{N}_2\hat{Y}_2 + \gamma + d_1)(d_1 + \delta) + (a\hat{Y}_2 + a_1N_2\hat{Y}_2 + \gamma + 2d_1 + \delta) \\ &\quad \times (\beta_2\hat{Y}_1 + \beta_3\hat{Z}_1 + d_2) + (a_0\hat{Y}_2 + a_1\hat{N}_2\hat{Y}_2 + \beta_2\hat{Y}_1 + \beta_3\hat{Z}_1 + d_2)\gamma \\ &\quad - (a_0 + a_1\hat{N}_2)(\hat{N}_1 - \hat{Y}_1 - \hat{Z}_1)(\hat{N}_2 - \hat{Y}_2)\beta_2, \\ s_3 &= (a_0\hat{Y}_2 + a_1\hat{N}_2\hat{Y}_2 + \gamma + d_1)(d_1 + \delta)(\beta_2\hat{Y}_1 + \beta_3\hat{Z}_1 + d_2) \\ &\quad + \gamma(a_0\hat{Y}_2 + a\hat{N}_2\hat{Y}_2)(\beta_2\hat{Y}_1 + \beta_3\hat{Z}_1 + d_2) - (a_0 + a_1\hat{N}_2) \\ &\quad \times (\hat{N}_1 - \hat{Y}_1 - \hat{Z}_1)(\hat{N}_2 - \hat{Y}_2)(\gamma\beta_3 + \beta_2(d_1 + \delta)). \end{aligned}$$

We find that the eigenvalues of the above characteristic polynomial are $-d_1$, $-d_2$ and the roots of the polynomial $\lambda_3 + s_1\lambda_2 + s_2\lambda + s_3$.

Here

$$\begin{aligned} s_1s_2 - s_3 &= (\beta_2\hat{Y}_1 + \beta_3\hat{Z}_1 + d_2 + a_0\hat{Y}_2 + a_1\hat{N}_2\hat{Y}_2 + 2\gamma + d_1) \\ &\quad \times (a_0\hat{Y}_2 + a_1\hat{N}_2\hat{Y}_2 + \gamma + d_1)(\beta_2\hat{Y}_1 + \beta_3\hat{Z}_1 + d_2) \\ &\quad + (a_0\hat{Y}_2 + a_1\hat{N}_2\hat{Y}_2 + 2\gamma + d_1)(\beta_2\hat{Y}_1 + \beta_3\hat{Z}_1 + d_2) \\ &\quad \times (d_1 + \delta + \gamma) + (\beta_2\hat{Y}_1 + \beta_3\hat{Z}_1 + d_2)^2(d_1 + \delta + \gamma) \\ &\quad + (a_0\hat{Y}_2 + a_1\hat{N}_2\hat{Y}_2 + \gamma + d_1)(a_0\hat{Y}_2 + a_1\hat{N}_2\hat{Y}_2 + 2\gamma + d_1) \\ &\quad \times (d_1 + \delta) + \gamma(a_0\hat{Y}_2 + a_1\hat{N}_2\hat{Y}_2)(a_0\hat{Y}_2 + a_1\hat{N}_2\hat{Y}_2 + 2\gamma + d_1) \\ &\quad + \gamma(a_0 + a_1\hat{N}_2)(\hat{N}_1 - \hat{Y}_1 - \hat{Z}_1)\beta_3(\hat{N}_2 - \hat{Y}_2). \end{aligned}$$

Using the Routh–Hurwitz criterion, we see that this equilibrium point P_2 is locally asymptotically stable since $s_1 s_2 - s_3 > 0$. □

2.2b *Simulations*. The system (1.6) is integrated using fourth-order Runge Kutta method. The following dataset has been used:

$$b_1 = d_1 = 0.00012, \quad b_2 = d_2 = 0.00085, \quad A = 10, \quad \delta = 0.000146,$$
$$\gamma = 0.012, \quad a_0 = 0.000021, \quad a_1 = 0.00000006,$$
$$\beta_2 = 0.000024, \quad \beta_3 = 0.9\text{e} - 8.$$

In the numerical simulations, the initial values are

$$Y_1(0) = 101.5, \quad Z_1(0) = 10.9, \quad N_1(0) = 402.8,$$
$$Y_2(0) = 602.3, \quad N_2(0) = 888.8$$

which are based on Ghosh *et al* [7].

3. Results and discussion

We have discussed two cases of malaria model in §§ 2.1 and 2.2 in this paper.

The results regarding the first case when β_1 is treated as a constant is discussed as follows:

Treatment of infected humans has been one of the most important control strategies in the fight against malaria. In the model, this is described by the parameter γ. The values of γ used in this simulation are $\gamma = 0.004$, 0.008 and 0.012. In figure 1, it is noted that

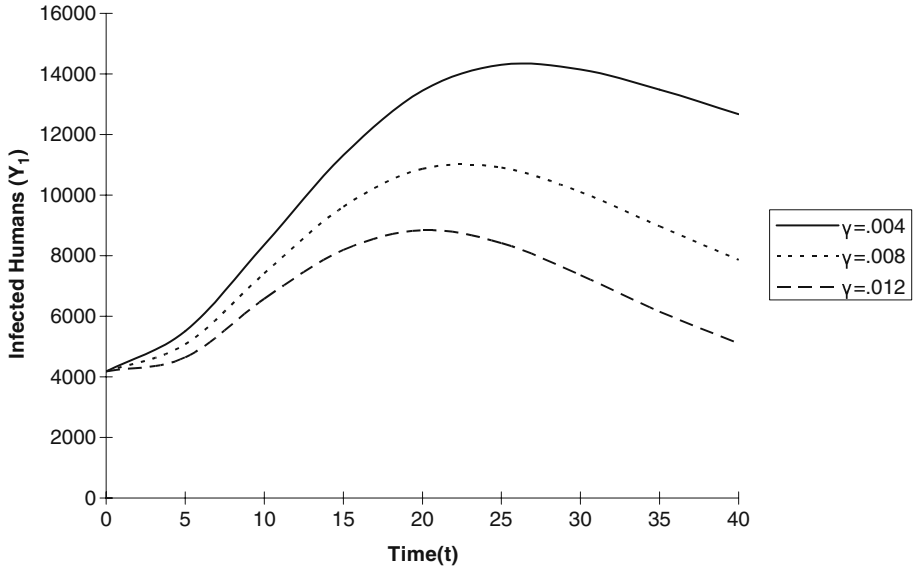

Figure 1. Variation in infective human population Y_1 with time for various values of recovery rate γ, when the transmission coefficient for humans is a constant.

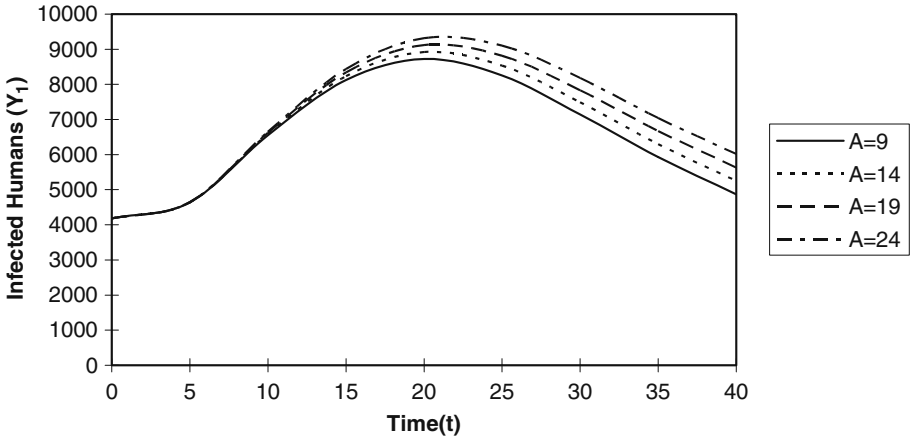

Figure 2. Variation in infective human population Y_1 with time for various values of immigration constant A, when the transmission coefficient for humans is a constant.

when the recovery rate is low, say, $\gamma = 0.004$, we have more infected humans than when the recovery rate is high, say $\gamma = 0.012$ (where there are less infected humans).

We explore various values of A. We take $A = 9$, 14, 19 and 24, which corresponds to the immigration constant being increased in number. Figure 2 illustrates the relation between A and Y_1. It is observed that the number of infected humans (Y_1) increase as the immigration constant (A) increases, which shows that the spread of malaria increases and becomes more endemic due to increased immigration.

In figure 3, the infective human population is plotted against time for different δ, which is the rate at which recovered humans lose immunity and move to the susceptible class. It is observed that as δ increases, recovered humans again get infected and move to the infected class and so the number of infected humans (Y_1) increase.

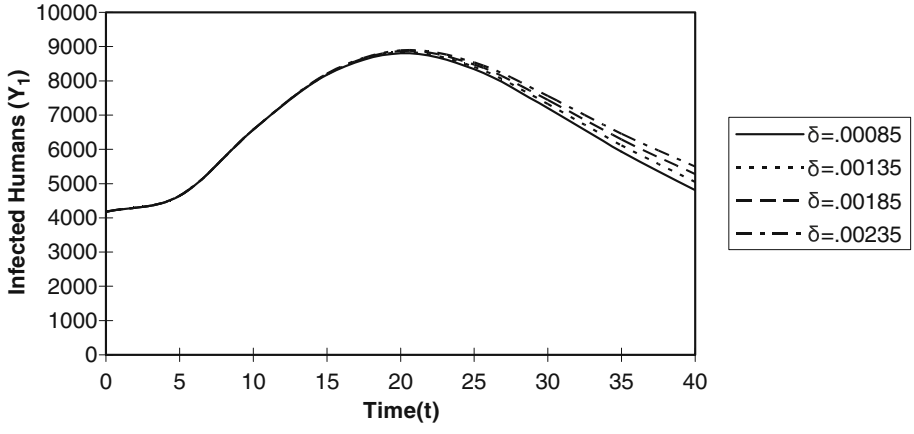

Figure 3. Variation in infective human population Y_1 with time for various values of the rate of immunity loss of recovered humans δ, when the transmission coefficient for humans is a constant.

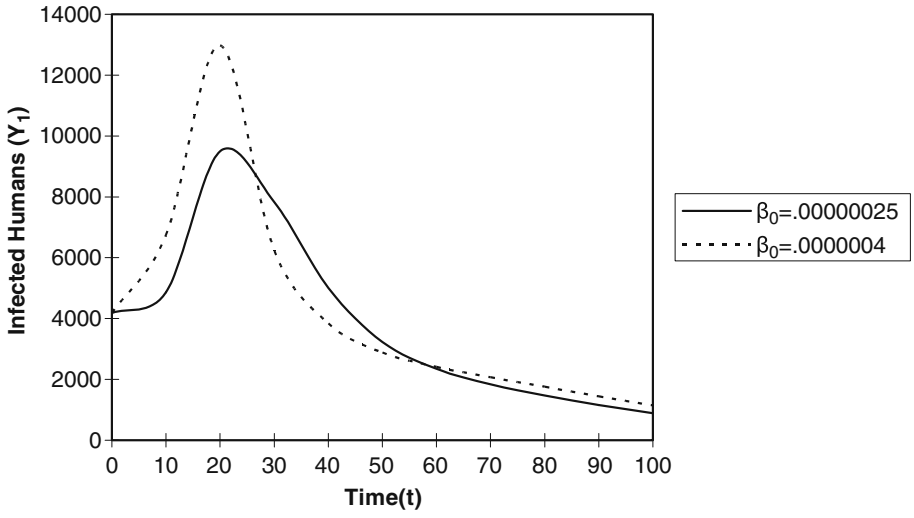

Figure 4. Variation in infective human population Y_1 with time for various values of the interaction coefficient β_0, between susceptible humans and infective mosquitoes, when the transmission coefficient for humans is a constant.

In figure 4, the infective human population is plotted against time for various values of β_0, the constant interaction coefficient between susceptible humans and infective mosquitoes. That an increase in β_0 increases the number of infected humans (Y_1) can be observed from the figure.

The time-dependent patterns of infection and immunity are illustrated in figure 5. With increase in time, infected humans increase and then slowly decrease. On the other hand, the proportion of recovered humans increase.

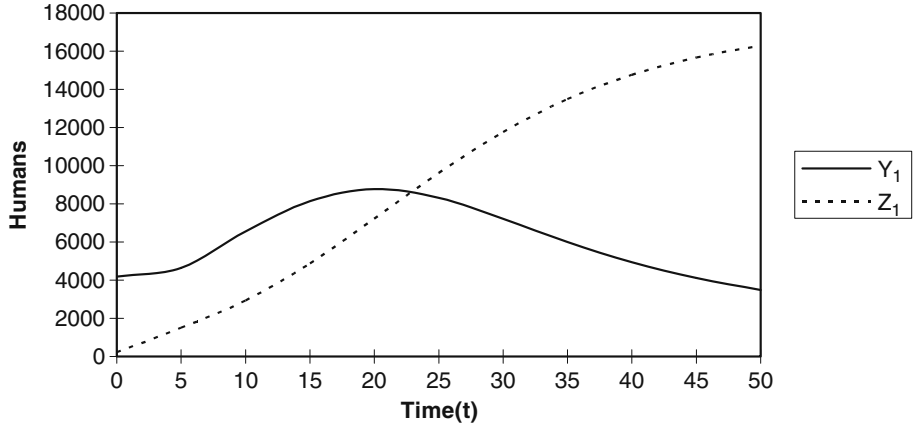

Figure 5. Variation in infective population and recovered population of humans with time when the transmission coefficient for humans is a constant.

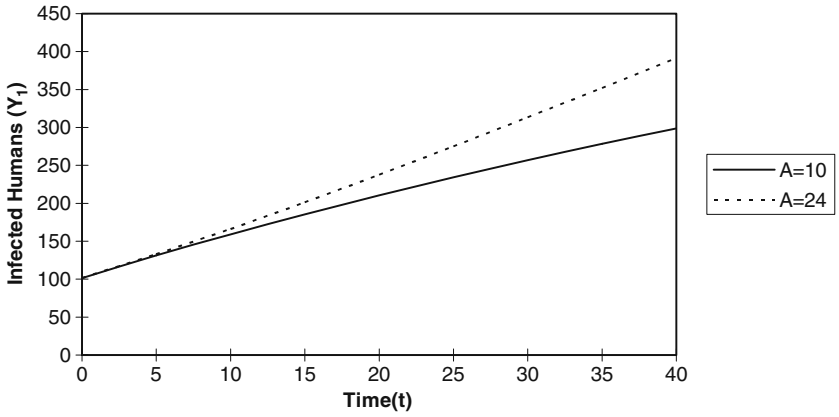

Figure 6. Variation in infective human population Y_1 with time for various values of the immigration constant A, when the transmission coefficient for humans is a variable.

The results regarding the second case when β_1 is a function of N_2 are discussed below:

For various values of the immigration constant (A), it is observed that the infective human population (Y_1) increases as A increases. This is shown in figure 6.

We explore various values of δ, which measures the rate at which recovered humans lose immunity and move to the susceptible class. The values of δ are 0.00008, 0.00908 and 0.01808. In figure 7, it is observed that the infective human population (Y_1) increases as δ increases.

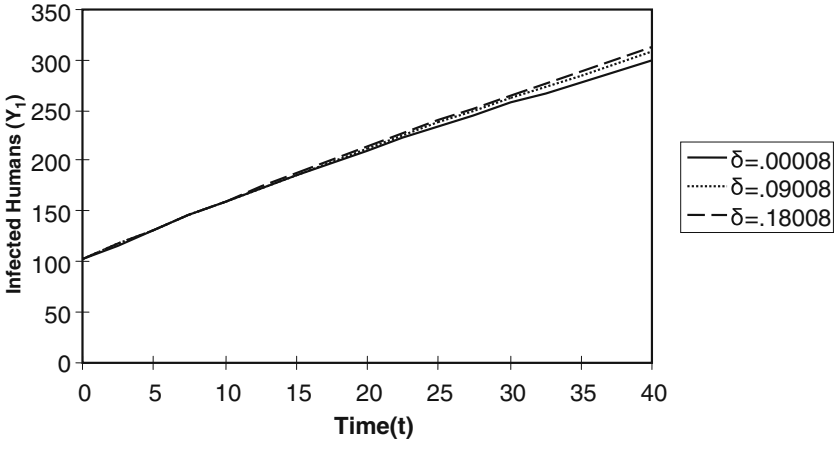

Figure 7. Variation in infective human population Y_1 with time for various values of the rate of immunity loss of recovered humans δ, when the transmission coefficient for humans is a variable.

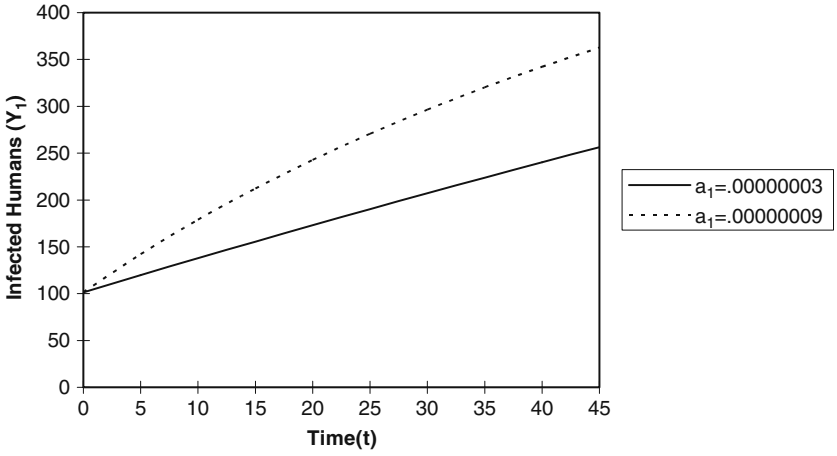

Figure 8. Variation in infective human population Y_1 with time for various values of a_1, when the transmission coefficient for humans is a variable.

In figure 8, the infective human population (Y_1) is plotted against time for different values of a_1(since $\beta_1 = a_0 + a_1 N_2$). It can be seen that a rise in a_1 which means a rise in β_1 (the interaction coefficient between susceptible humans and infective mosquitoes) leads to a rise in the number of infective humans Y_1.

A rise in a_0 also leads to an increase in β_1, which in turn increases the number of infective humans Y_1. This is shown in figure 9.

In figure 10, the infective human population (Y_1) is plotted against time for two values of N_2. It is observed that an increase in N_2 leads to an increase in infected humans Y_1.

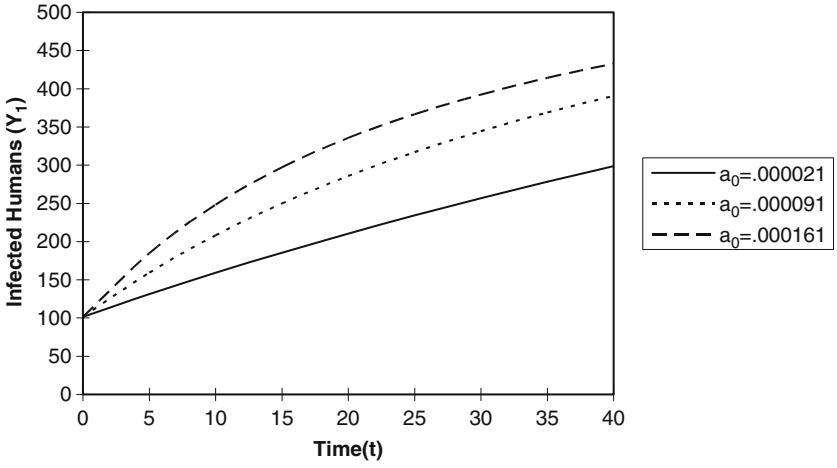

Figure 9. Variation in infective human population Y_1 with time for various values of a_0, when the transmission coefficient for humans is a variable.

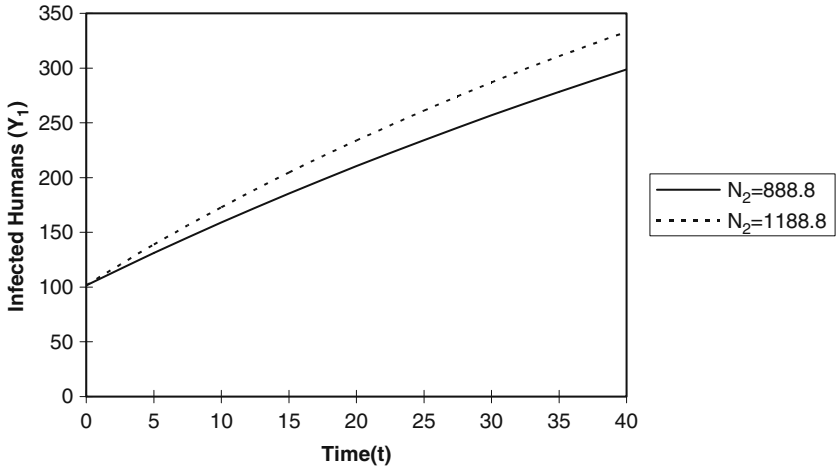

Figure 10. Variation in infective human population Y_1 with time for various values of N_2, when the transmission coefficient for humans is a variable.

4. Conclusion

In this paper, a model for malaria is formulated taking into account both the human and mosquito populations. An SIRS model with immigration is formulated for humans. We further consider the human population to act as a reservoir. An SI model is formulated for mosquito. Mosquito dynamics is studied along with human dynamics because mosquito population determines to a large extent whether a malaria outbreak will occur or not.

The transmission coefficient between susceptible humans and infected mosquitoes is an important factor in the spread of malaria. We first considered it a constant and then to be dependent on the mosquito population.

The threshold condition for the spread of malaria is derived in the above two cases. Disease persists if the threshold exceeds one. The stability condition of the equilibrium points are derived in both cases. It is seen that in each case if the threshold is greater than one, the non trivial equilibrium is always locally asymptotically stable under some conditions.

Simulations are performed by varying some parameters of the model. If the immigration constant A or the rate of loss of immunity of recovered δ or the mosquito population N_2 increases, then it will lead to an increase in the number of infected humans, giving an endemic shape to the disease. Also, a rise in a_0 or a_1 as depicted in figures 8 and 9 leads to an increase in the transmission coefficient between susceptible humans and infected mosquitoes and results in the increased spread of malaria.

In this model, the transmission coefficient β_1 between susceptible humans and infective mosquitoes was once taken as a constant i.e. $\beta_1 = \beta_0$ and once as a function of N_2, i.e. $\beta_1 = a_0 + a_1 N_2$. In the first case, it is seen that a rise in β_0 leads to an increase in infected humans. In the second case, increase in a_0, a_1 and N_2 resulted in an increase in infected humans. So one way to control the outbreak of malaria is to have a check on the transmission coefficient β_1, i.e. to curb the mosquito population.

In conclusion, there is need for effective control program against malaria vectors as they contribute to the outbreak of malaria. Spraying insecticides and proper drainage system should be a major target to eliminate mosquitoes. Early diagnosis and proper treatment with anti-malarial drugs can help control the spread of malaria.

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