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Embedding relations connected with strong approximation of Fourier series

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Abstract. We consider the embedding relation between the class $W^q H_\beta^\omega$, including only odd functions and a set of functions defined via the strong means of Fourier series of odd continuous functions. We establish an improvement of a recent theorem of Le and Zhou [*Math. Inequal. Appl.* **11(4)** (2008) 749–756] which is a generalization of Tikhonov's results [*Anal. Math.* **31** (2005) 183–194]. We also extend the Leindler theorem [*Anal. Math.* **31** (2005) 175–182] concerning sequences of Fourier coefficients.

Keywords. Strong approximation; Fourier series; embedding theorems.

1. Introduction

Let f be an odd continuous function of period 2π and let

$$\sum_{n=1}^{\infty} b_n \sin nx \tag{1}$$

be its Fourier series. The modulus of smoothness of order β (> 0) of the function $f \in C$ is given by

$$\omega_\beta(f;t) = \sup_{|h|\le t} \left\| \sum_{v=0}^{\infty} (-1)^v \binom{\beta}{v} f\left(x + (\beta - v)h\right) \right\|,$$

where

$$\binom{\beta}{v} = \begin{cases} \frac{\beta(\beta-1)\ldots(\beta-v+1)}{v!}, & \text{for } v \ge 1, \\ 1, & \text{for } v = 0 \end{cases} \quad \text{and} \quad \|f(\cdot)\| = \max_{x\in[0,2\pi]} |f(x)|.$$

Denote by $S_n(x) = S_n(f,x)$ the n-th partial sum of (1) and by $f^{(q)}$ the derivative of the function f of order $q \ge 0$ ($f^{(0)} = f$) in the Weyl sense (see [17]).

For any sequence $\lambda = (\lambda_n)$ of positive numbers, we set

$$\Lambda_n = \sum_{v=1}^{n} \lambda_v.$$

For $p > 0$, we define the following strong means:

$$h_n(f, \lambda, p; x) := \left\{ \frac{1}{\Lambda_n} \sum_{v=1}^{n} \lambda_v |f(x) - S_v(x)|^p \right\}^{\frac{1}{p}}$$

and

$$h_n(f, \lambda, p) := \left\| \left\{ \frac{1}{\Lambda_n} \sum_{v=1}^{n} \lambda_v |f(\cdot) - S_v(\cdot)|^p \right\}^{\frac{1}{p}} \right\|.$$

Let Ω be a set of all nondecreasing continuous functions defined on $[0, 2\pi]$ such that

$$\omega(0) = 0 \quad \text{and}$$
$$\omega(\delta_1 + \delta_2) \le \omega(\delta_1) + \omega(\delta_2) \ \text{ for any } \ 0 < \delta_1 \le \delta_2 \le \delta_1 + \delta_2 \le 2\pi.$$

Further, we define the following classes of functions:

$$H(\lambda, p, q, \omega) = \left\{ f \in C\colon h_n(f, \lambda, p) = O\left(n^{-q} \omega\left(\frac{1}{n}\right)\right)\right\},$$
$$W^q H_\beta^\omega = \{ f \in C\colon \omega_\beta(f^{(q)}, \delta) = O(\omega(\delta))\}.$$

Leindler, in [6] defined a new class of sequences in the following way:

DEFINITION 1

Let $\gamma := (\gamma_n)$ be a positive sequence. A null sequence $a := (a_n)$ of real numbers satisfying the inequality

$$\sum_{n=m}^{\infty} |a_n - a_{n+1}| \le K(a)\gamma_m, \quad m = 1, 2, \ldots$$

with a positive constant $K(a)$ is said to be a sequence of γ rest bounded variation, in symbol: $a \in \gamma$ RBVS.

If $\gamma \equiv a$ and $a_n > 0$, then we call the sequence a the rest bounded variation sequence; and briefly we write $a \in$ RBVS. Leindler [7] introduced the class of mean rest bounded variation sequences (MRBVS), where γ is defined by a certain arithmetical mean of the coefficients, e.g.,

$$\gamma_m := \frac{1}{m} \sum_{n \ge m/2}^{m} a_n. \tag{2}$$

It is easy to see that the class MRBVS includes the class RBVS, consequently the classes of almost monotone and monotone sequences, too. In [9] we proved that RBVS $\neq$ MRBVS.

A sequence $a := (a_n)$ of positive numbers is called quasimonotone ($a \in$ QMS) if there exists $\rho \ge 0$ such that $n^{-\rho} a_n \downarrow 0$.

We will also need the following notations:

$$C_1 = \left\{ f \in C\colon f(x) = \sum_{n=1}^{\infty} b_n \sin nx,\ (b_n) \in \text{QMS} \right\},$$

$$C_2 = \left\{ f \in C\colon f(x) = \sum_{n=1}^{\infty} b_n \sin nx,\ (b_n) \in \text{RBVS} \right\}$$

and

$$C_3 := \left\{ f \in C\colon f(x) = \sum_{n=1}^{\infty} b_n \sin nx,\ (b_n) \in \text{MRBVS} \right\}.$$

We shall write $I_1 \ll I_2$ if there exists a positive constant C such that $I_1 \le C I_2$.

A sequence $\gamma := (\gamma_n)$ of positive numbers will be called almost monotone (increasing) sequence ($(\gamma_n) \in$ AMS), if there exists a constant $K := K(\gamma) \ge 1$ such that

$$K\gamma_n \ge \gamma_m$$

holds for any $n \ge m$.

Tikhonov [12] proved the following theorem.

Theorem 1. *Let β, $p > 0$, $q \ge 0$, $\omega \in \Omega$ and $\lambda = (\lambda_n)$ be a sequence of positive numbers such that*

$$\Lambda_{2n} \ll \Lambda_n \ll n\lambda_n. \tag{3}$$

If ω is such that

$$\lambda_n n^{1-pq} \omega^p \left(\frac{1}{n}\right) \in \text{AMS}, \tag{4}$$

then

$$W^q H_\beta^\omega \cap C_j \subset H(\lambda, p, q, \omega),$$

where $j = 1$ or $j = 2$.

The above theorem is a generalization of the theorem of Leindler [3], who generalized the results of Mazhar [8]. In [10] we proved that under the assumption of the above theorem the following embedding relation $W^q H_\beta^\omega \cap C_3 \subset H(\lambda, p, q, \omega)$ holds.

The next generalization of Theorem 1 was proved by Leindler in [5]. He showed that this theorem is true with the assumptions (5) and (6) instead of (3) and (4), respectively.

Le and Zhou [1] defined the following new class of sequences ($\mathbb{R}_+ = [0, \infty)$):

$$\text{GBVS} = \left\{ (a_n) \in \mathbb{R}_+\colon \sum_{n=m}^{2m-1} |a_n - a_{n+1}| \le K(a) \max_{m \le n \le N+m} \{a_n\} \text{ for some integer } N \text{ and all } m \in \mathbb{N} \right\}.$$

Very recently, Yu, Zhou and Zhou [16] proposed a more general class:

$$\text{MVBVS} = \Bigg\{(a_n) \in \mathbb{R}_+ \colon \sum_{n=m}^{2m-1} |a_n - a_{n+1}| \le K(a) \sum_{n=[m/\lambda]}^{[\lambda m]} \frac{a_n}{n} \ \text{ for some } \ \lambda \ge 2 \ \text{ and all } \ m \in \mathbb{N}\Bigg\}.$$

Let

$$C_4 := \left\{ f \in C \colon f(x) = \sum_{n=1}^{\infty} b_n \sin nx,\, (b_n) \in \text{MVBVS} \right\}$$

Le and Zhou [2] proved the following generalization of Theorem 1.

Theorem 2. *Let* $\beta,\ p > 0,\ q \ge 0,\ \omega \in \Omega$ *and* $\lambda = (\lambda_n)$ *be a sequence satisfying* (3). *If* (4) *holds*, *then*

$$W^q H_\beta^\omega \cap C_4 \subset H(\lambda, p, q, \omega).$$

Tikhonov [13–15] defined the class of β-general monotone sequences as follows:

DEFINITION 2

Let $\beta := (\beta_n)$ be a nonnegative sequence. The sequence of nonnegative numbers $a := (a_n)$ is said to be β-general monotone, or $a \in \text{GM}\,(\beta)$, if the relation

$$\sum_{n=m}^{2m-1} |a_n - a_{n+1}| \le K(a)\beta_m$$

holds for all $m \in \mathbb{N}$.

Tikhonov in [15] considered the following examples of the sequences β_n:

(i) ${}_1\beta_n = a_n$,
(ii) ${}_2\beta_n = \sum_{k=n}^{n+N} a_k$ for some integer N,
(iii) ${}_3\beta_n = \sum_{\nu=0}^{N} a_{c^\nu n}$ for some integers N and $c > 1$,
(iv) ${}_4\beta_n = a_n + \sum_{k=n+1}^{[cn]} \frac{a_k}{k}$ for some $c > 1$,
(v) ${}_5\beta_n = \sum_{k=[n/c]}^{[cn]} \frac{a_k}{k}$ for some $c > 1$,

It is clear that GM $({}_2\beta) \equiv$ GBVS and GM $({}_5\beta) \equiv$ MVBVS. Moreover (see [15])

$$\text{GM}\,({}_1\beta +{}_2 \beta +{}_3 \beta +{}_4 \beta +{}_5 \beta) \equiv \text{GM}\,({}_5\beta).$$

In order to formulate our new results we define the next class of sequences.

DEFINITION 3 [11]

Let $\beta := (\beta_n)$ be a nonnegative sequence and r a natural number. The sequence of nonnegative numbers $a := (a_n)$ is said to be (β, r)-general monotone, or $a \in \mathrm{GM}\,(\beta, r)$, if the relation

$$\sum_{n=m}^{2m-1} |a_n - a_{n+r}| \leq K(a)\beta_m$$

holds for all $m \in \mathbb{N}$.

It is clear that $\mathrm{GM}\,(\beta, 1) \equiv \mathrm{GM}\,(\beta)$. Moreover, the embedding relation between $\mathrm{GM}\,(\beta, r)$ $(r > 1)$ and $\mathrm{GM}\,(\beta, 1)$ implies from the following remark (see [11]).

Remark 1. Let r be a natural number. If a nonnegative sequence $\beta := (\beta_n)$ is such that

$$\sum_{i=0}^{r-1} \beta_{n+i} \ll \beta_n$$

for all n, then

$$\mathrm{GM}\,(\beta, 1) \subseteq \mathrm{GM}\,(\beta, r).$$

Moreover, the classes $\mathrm{GM}\,({}_5\beta, r)$ have the following properties (see [11]):

Remark 2. Let $r_1, r_2 \in \mathbb{N}$ and $r_1 < r_2$. If $r_1 | r_2$, then $\mathrm{GM}\,({}_5\beta, r_1) \subsetneq \mathrm{GM}\,({}_5\beta, r_2)$.

Remark 3. If $r \in \mathbb{N}$ and $r > 1$, then $\mathrm{GM}\,({}_5\beta) \equiv \mathrm{GM}\,({}_5\beta, 1) \subsetneq \mathrm{GM}\,({}_5\beta, r)$.

Remark 4. Let $r_1, r_2 \in \mathbb{N}$. If $r_1 \nmid r_2$ and $r_2 \nmid r_1$, then the classes $\mathrm{GM}\,({}_5\beta, r_1)$ and $\mathrm{GM}\,({}_5\beta, r_2)$ are not comparable.

Remark 5. For any $r \geq 3$ there exists a sequence $a := (a_n) \in \mathrm{GM}\,({}_5\beta, r)$, with the properties (7) which does not belong to the class $\mathrm{GM}\,({}_5\beta, 2)$.

Finally, for $r \in \mathbb{N}$,

$$C_4(r) = \left\{ f \in C\colon f(x) = \sum_{n=1}^{\infty} b_n \sin nx,\ (b_n) \in \mathrm{GM}\,({}_5\beta, r) \right\}.$$

It is clear that $C_4 \equiv C_4(1)$.

In the present paper, we generalize the above theorem of Le and Zhou to the class $\mathrm{GM}\,({}_5\beta, 2)$. Moreover, we prove that Theorem 2 is also true if a sequence (b_n) belongs to the class $\mathrm{GM}\,({}_5\beta, r)$ $(r \geq 3)$ and satisfies (7).

2. Statement of the results

Theorem 3. *Let* β, $p > 0$, $q \geq 0$, $\omega \in \Omega$. *If* $\lambda = (\lambda_n)$ *be a sequence of positive numbers such that*

$$\Lambda_n n^{-pq} \omega^p \left(\frac{1}{n}\right) \in \mathrm{AMS} \tag{5}$$

and

$$\Lambda_{2n} \ll \Lambda_n, \tag{6}$$

then

$$W^q H_\beta^\omega \cap C_4(2) \subset H(\lambda, p, q, \omega.$$

Remark 6. If we confine our attention to the class GM $({}_5\beta)$ then by Remark 3 the Le and Zhou result follows from our Theorem 3. Moreover, from Theorem 3 we can derive the embedding relation from [5], [12] and [10].

Theorem 4. *Let β, $p > 0$, $q \geq 0$, $\omega \in \Omega$ and let $\lambda = (\lambda_n)$ be a sequence of positive numbers such that* (5) *and* (6) *hold. If*

$$\sum_{v=n}^{\infty} \sum_{k=1}^{[r/2]} |b_{r\cdot v+k} - b_{r\cdot v+r-k}| \ll n^{\varepsilon} \sum_{v=[n/c]}^{[cn]} \frac{b_v}{v} \quad (c > 1 \quad \text{and} \quad \varepsilon \in (0,1)) \tag{7}$$

holds for $r \geq 3$ and all n, then

$$W^q H_\beta^\omega \cap C_4(r) \subset H(\lambda, p, q, \omega).$$

3. Auxiliary results

To prove our theorem, the following lemmas are needed.

Lemma 1 [11]. Let $r \in \mathbb{N}$, $l \in \mathbb{Z}$ and $a := (a_n) \in \mathbb{C}$. If $x = \frac{2l\pi}{r}$, then for all n,

$$\sum_{k=n}^{2n-1} a_k \sin kx = \frac{-1}{2\sin(rx/2)} \left\{ \sum_{k=n}^{2n-1} (a_k - a_{k+r}) \cos\left(k + \frac{r}{2}t\right) x + \sum_{k=2n}^{2n+r-1} a_k \cos\left(k - \frac{r}{2}\right) x - \sum_{k=n}^{n+r-1} a_k \cos\left(k - \frac{r}{2}\right) x \right\}. \tag{8}$$

Lemma 2. *Let $p > 0$, $q \geq 0$, $\omega \in \Omega$, $(b_n) \in$ GM $({}_5\beta, 2)$ and*

$$0 \leq b_n \ll n^{-q-1} \omega\left(\frac{1}{n}\right). \tag{9}$$

Then the Fourier series (1) *converges to f uniformly, i.e.*

$$f(x) := \sum_{n=1}^{\infty} b_n \sin nx.$$

Further, if (λ_n) be a sequence of positive numbers, which satisfies (5) *and* (6), *then $f \in H(\lambda, p, q, \omega)$.*

Proof. The first part of Lemma 2 is clear by generalization of Chaudy–Jollife's theorem for the class GM $({}_5\beta, 2)$ (see [11]) with the condition $b_n \ll n^{-q-1}\omega\left(\frac{1}{n}\right)$. In view of $S_k(f, 0) = S_k(f, \pi) = 0$, we may restrict $x \in (0, \pi)$.

First, assume that $x \in \left(0, \frac{\pi}{2}\right]$ and let $M = M(x) \geq 2$ be an integer such that

$$\frac{\pi}{M+1} < x \leq \frac{\pi}{M}. \tag{10}$$

Using (8) with $r = 2$, for $k < M$ we obtain

$$\begin{aligned}
&|f(x) - S_k(f; x)| \\
&\quad \leq \left|\sum_{v=k+1}^{M} b_v \sin vx\right| + \left|\sum_{v=M+1}^{\infty} b_v \sin vx\right| \\
&\quad = \left|\sum_{v=k+1}^{M} b_v \sin vx\right| + \left|\sum_{s=0}^{\infty} \sum_{v=2^s(M+1)}^{2^{s+1}(M+1)-1} b_v \sin vx\right| \\
&\quad \leq x \sum_{v=k+1}^{M} b_v v + \left|\frac{-1}{2\sin x} \sum_{s=0}^{\infty} \left\{\sum_{v=2^s(M+1)}^{2^{s+1}(M+1)-1} (b_v - b_{v+2}) \cos(v+1)x\right.\right. \\
&\qquad \left.\left. + \sum_{v=2^{s+1}(M+1)}^{2^{s+1}(M+1)+1} b_v \cos(v-1)x - \sum_{v=2^s(M+1)}^{2^s(M+1)+1} b_v \cos(v-1)x\right\}\right| \\
&\quad \leq x \sum_{v=k+1}^{M} b_v v + \frac{1}{2\sin x} \\
&\qquad \times \sum_{s=0}^{\infty} \left\{\sum_{v=2^s(M+1)}^{2^{s+1}(M+1)-1} |b_v - b_{v+2}| + \sum_{v=2^{s+1}(M+1)}^{2^{s+1}(M+1)+1} b_v + \sum_{v=2^s(M+1)}^{2^s(M+1)+1} b_v\right\}.
\end{aligned}$$

If $(b_n) \in$ GM $({}_5\beta, 2)$, then applying the inequality $\frac{2}{\pi}x \leq \sin x$ $\left(x \in \left[0, \frac{\pi}{2}\right]\right)$ and (10) we get

$$\begin{aligned}
&|f(x) - S_k(f; x)| \\
&\quad \ll \frac{1}{M} \sum_{v=k+1}^{M} b_v v + \frac{1}{\sin x} \sum_{s=0}^{\infty} \left\{\sum_{v=2^s(M+1)}^{2^{s+1}(M+1)-1} |b_v - b_{v+2}| + \sum_{v=2^s(M+1)}^{2^s(M+1)+1} b_v\right\} \\
&\quad \ll \frac{1}{M} \sum_{v=k+1}^{M} b_v v + \frac{1}{x} \sum_{s=0}^{\infty} \left\{\sum_{v=2^s(M+1)}^{2^{s+1}(M+1)-1} |b_v - b_{v+2}| + \sum_{v=2^s(M+1)}^{2^s(M+1)+1} b_v\right\} \\
&\quad \ll \frac{1}{M} \sum_{v=k+1}^{M} b_v v + M \sum_{s=0}^{\infty} \left\{\sum_{v=[2^s(M+1)/c]}^{[c2^s(M+1)]} \frac{b_v}{v} + \sum_{v=2^s(M+1)}^{2^s(M+1)+1} b_v\right\}.
\end{aligned} \tag{11}$$

Analogously, if $k \geq M$, then

$$|f(x) - S_k(f; xt)| \ll M \sum_{s=0}^{\infty} \left\{ \sum_{v=[2^s(k+1)/c]}^{[c2^s(k+1)]} \frac{b_v}{v} + \sum_{v=2^s(k+1)}^{2^s(k+1)+1} b_v \right\}. \tag{12}$$

Further, for $n > M$, we have

$$\sum_{k=1}^{n} \lambda_k |f(x) - S_k(f; x)|^p$$
$$= \sum_{k=1}^{M} \lambda_k |f(x) - S_k(f; x)|^p + \sum_{k=M+1}^{n} \lambda_k |f(x) - S_k(f; x)|^p := I_1 + I_2. \tag{13}$$

Using (11) we obtain

$$I_1 \ll \frac{1}{M^p} \sum_{k=1}^{M} \lambda_k \left(\sum_{v=k+1}^{M} b_v v \right)^p$$
$$+ \sum_{k=1}^{M} \lambda_k M^p \left(\sum_{s=0}^{\infty} \left\{ \sum_{v=[2^s(M+1)/c]}^{[c2^s(M+1)]} \frac{b_v}{v} + \sum_{v=2^s(M+1)}^{2^s(M+1)+1} b_v \right\} \right)^p := I_{11} + I_{12} \tag{14}$$

and by (12)

$$I_2 \ll \sum_{k=M+1}^{n} \lambda_k \left(M \sum_{s=0}^{\infty} \left\{ \sum_{v=[2^s(k+1)/c]}^{[c2^s(k+1)]} \frac{b_v}{v} + \sum_{v=2^s(k+1)}^{2^s(k+1)+1} b_v \right\} \right)^p. \tag{15}$$

Now, we shall estimate I_{11}. By (9) we have

$$I_{11} \ll \frac{1}{M^p} \sum_{k=1}^{M} \lambda_k \left(\sum_{v=k}^{M} v^{-q} \omega\left(\frac{1}{v}\right) \right)^p.$$

Let $p > 1$. Then using the Hölder inequality and (5) we obtain that

$$I_{11} \ll \frac{1}{M^p} \sum_{k=1}^{M} \lambda_k \left(\sum_{v=k}^{M} 1 \right)^{p-1} \left(\sum_{v=k}^{M} v^{-pq} \omega^p \left(\frac{1}{v}\right) \right)$$
$$\ll \frac{1}{M} \sum_{k=1}^{M} \lambda_k \sum_{v=k}^{M} v^{-pq} \omega^p \left(\frac{1}{v}\right) = \frac{1}{M} \sum_{v=1}^{M} v^{-pq} \omega^p \left(\frac{1}{v}\right) \sum_{k=1}^{v} \lambda_k$$
$$= \frac{1}{M} \sum_{v=1}^{M} v^{-pq} \omega^p \left(\frac{1}{v}\right) \Lambda_v \ll n^{-pq} \omega^p \left(\frac{1}{n}\right) \Lambda_n. \tag{16}$$

If $0 < p \leq 1$, then we choose two integers $\mu = \mu_M$ and $\nu = \nu_k$ such that

$$2^{\mu-1} \leq M < 2^{\mu} \quad \text{and} \quad 2^{\nu} \leq k < 2^{\nu+1}.$$

Then, by (5) the Hölder inequality gives

$$
\begin{aligned}
I_{11} &\ll \frac{1}{M^p}\sum_{k=1}^{M}\lambda_k\left(\sum_{l=\nu}^{\mu-1}\sum_{s=2^l}^{2^{l+1}-1}s^{-q}\omega\left(\frac{1}{s}\right)\right)^p \\
&\le \frac{1}{M^p}\sum_{k=1}^{M}\lambda_k\sum_{l=\nu}^{\mu-1}\left(\sum_{s=2^l}^{2^{l+1}-1}s^{-q}\omega\left(\frac{1}{s}\right)\right)^p \\
&\le \frac{1}{M^p}\sum_{k=1}^{M}\lambda_k\sum_{l=\nu}^{\mu-1}2^{lp(1-q)}\omega^p\left(\frac{1}{2^l}\right) \le \frac{1}{M^p}\sum_{l=1}^{\mu-1}2^{lp(1-q)}\omega^p\left(\frac{1}{2^l}\right)\sum_{k=1}^{2^l}\lambda_k \\
&= \frac{1}{M^p}\sum_{l=1}^{\mu-1}\Lambda_{2^l}(2^l)^{-pq}\omega^p\left(\frac{1}{2^l}\right)2^{lp} \ll n^{-pq}\omega^p\left(\frac{1}{n}\right)\Lambda_n\frac{1}{M^p}\sum_{l=1}^{\mu-1}2^{lp} \\
&\ll n^{-pq}\omega^p\left(\frac{1}{n}\right)\Lambda_n.
\end{aligned}\tag{17}
$$

The estimate I_{12} follows from (9), (5) and from the monotonicity of ω:

$$
\begin{aligned}
I_{12} &\ll M^p\Lambda_M\left(\sum_{s=0}^{\infty}\left\{\sum_{v=[2^s(M+1)/c]}^{[c2^s(M+1)]}\frac{b_v}{v}+\sum_{v=2^s(M+1)}^{2^s(M+1)+1}b_v\right\}\right)^p \\
&\ll M^p\Lambda_M\left(\sum_{s=0}^{\infty}\left\{\sum_{v=[2^s(M+1)/c]}^{[c2^s(M+1)]}v^{-q-2}\omega\left(\frac{1}{v}\right)+\sum_{v=2^s(M+1)}^{2^s(M+1)+1}v^{-q-1}\omega\left(\frac{1}{v}\right)\right\}\right)^p \\
&\ll M^p\Lambda_M\left(\sum_{s=0}^{\infty}\omega\left(\frac{1}{2^s(M+1)}\right)\frac{1}{(2^s(M+1))^{q+1}}\right)^p \\
&\ll \Lambda_M M^{-pq}\omega^p\left(\frac{1}{M}\right)\left(\sum_{s=0}^{\infty}\frac{1}{(2^s)^{q+1}}\right)^p \ll n^{-rp}\omega^p\left(\frac{1}{n}\right)\Lambda_n.
\end{aligned}\tag{18}
$$

Next, we estimate I_2. Using (9), (5) and the monotonicity of ω we obtain

$$
\begin{aligned}
I_2 &\ll M^p\sum_{k=M+1}^{n}\lambda_k \\
&\quad\times\left(\sum_{s=0}^{\infty}\left\{\sum_{v=[2^s(k+1)/c]}^{[c2^s(k+1)]}v^{-q-2}\omega\left(\frac{1}{v}\right)+\sum_{v=2^s(k+1)}^{2^s(k+1)+1}v^{-q-1}\omega\left(\frac{1}{v}\right)\right\}\right)^p \\
&\ll M^p\sum_{k=M+1}^{n}\lambda_k\left(\sum_{s=0}^{\infty}\omega\left(\frac{1}{2^s(k+1)}\right)\frac{1}{(2^s(k+1))^{q+1}}\right)^p
\end{aligned}
$$

$$\ll M^p \sum_{k=M+1}^{n} \Lambda_k k^{-pq} \omega^p \left(\frac{1}{k}\right) \frac{\lambda_k}{\Lambda_k} k^{-p} \left(\sum_{s=0}^{\infty} \frac{1}{(2^s)^{q+1}}\right)^p$$

$$\ll n^{-pq} \omega^p \left(\frac{1}{n}\right) \Lambda_n M^p \sum_{k=M}^{n} \frac{\lambda_k}{\Lambda_k} k^{-p}$$

and by (6) we have

$$I_2 \ll n^{-pq} \omega^p \left(\frac{1}{n}\right) \Lambda_n M^p \sum_{l=0}^{\infty} \sum_{i=2^l M}^{2^{l+1} M} \frac{\lambda_i}{\Lambda_i} (i)^{-p}$$

$$\leq n^{-pq} \omega^p \left(\frac{1}{n}\right) \Lambda_n M^p \sum_{l=0}^{\infty} (2^l M)^{-p} \frac{1}{\Lambda_{2^l M}} \sum_{i=2^l M}^{2^{l+1} M} \lambda_i$$

$$\leq n^{-pq} \omega^p \left(\frac{1}{n}\right) \Lambda_n \sum_{l=0}^{\infty} (2^l)^{-p} \frac{\Lambda_{2^{l+1} M}}{\Lambda_{2^l M}} \ll n^{-pq} \omega^p \left(\frac{1}{n}\right) \Lambda_n. \tag{19}$$

If $n \leq M$, then we can estimate

$$\sum_{k=1}^{n} \lambda_k |f(x) - S_k(f; x)|^p \tag{20}$$

in the same way as we have estimated I_1, with the only modification that instead of M we take n.

Thus, combining (13)–(19) we obtain that for $x \in \left(0, \frac{\pi}{2}\right]$ and every n,

$$\sum_{k=1}^{n} \lambda_k |f(x) - S_k(f; x)|^p \ll n^{-pq} \omega^p \left(\frac{1}{n}\right) \Lambda_n \tag{21}$$

holds.

Let $x \in \left[\frac{\pi}{2}, \pi\right)$ and $N = N(x) \geq 2$ be an integer such that

$$\pi - \frac{\pi}{N} \leq x < \pi - \frac{\pi}{N+1}. \tag{22}$$

Using (8) with $r = 2$ and the inequality $\sin x \leq \pi - x$ $(x \in (0, \pi))$, for $k < N$ we obtain

$$|f(x) - S_k(f; x)| \leq \left| \sum_{v=k+1}^{N} b_v \sin vx \right| + \left| \sum_{v=N+1}^{\infty} b_v \sin vx \right|$$

$$\leq (\pi - x) \sum_{v=k+1}^{N} b_v v + \frac{1}{2 \sin x} \sum_{s=0}^{\infty}$$

$$\times \left\{ \sum_{v=2^s(N+1)}^{2^{s+1}(N+1)-1} |b_v - b_{v+2}| + \sum_{v=2^{s+1}(N+1)}^{2^{s+1}(N+1)+1} b_v + \sum_{v=2^s(N+1)}^{2^s(N+1)+1} b_v \right\}.$$

If $(b_n) \in \text{GM}\,({}_5\beta, 2)$, then applying the inequality $2 - \frac{2}{\pi}x \le \sin x \ \left(x \in \left[\frac{\pi}{2}, \pi\right]\right)$ and (22) we get

$$\begin{aligned}
&|f(x) - S_k(f;x)| \\
&\quad \ll (\pi - x) \sum_{v=k+1}^{N} b_v v \\
&\qquad + \frac{1}{2\left(1 - \frac{1}{\pi}x\right)} \sum_{s=0}^{\infty} \left\{ \sum_{v=2^s(N+1)}^{2^{s+1}(N+1)-1} |b_v - b_{v+2}| + \sum_{v=2^s(N+1)}^{2^s(N+1)+1} b_v \right\} \\
&\quad \ll \frac{1}{N} \sum_{v=k+1}^{N} b_v v + N \sum_{s=0}^{\infty} \left\{ \sum_{v=[2^s(N+1)/c]}^{[c2^s(N+1)]} \frac{b_v}{v} + \sum_{v=2^s(N+1)}^{2^s(N+1)+1} b_v \right\}. \qquad (23)
\end{aligned}$$

Analogously, if $k \ge N$, then

$$|f(x) - S_k(f;x)| \ll N \sum_{s=0}^{\infty} \left\{ \sum_{v=[2^s(k+1)/c]}^{[c2^s(k+1)]} \frac{b_v}{v} + \sum_{v=2^s(k+1)}^{2^s(k+1)+1} b_v \right\}. \qquad (24)$$

If $n > N$, we have

$$\begin{aligned}
&\sum_{k=1}^{n} \lambda_k |f_0(x) - S_k(f_0;x)|^p \\
&\quad = \sum_{k=1}^{N} \lambda_k |f_0(x) - S_k(f_0;x)|^p + \sum_{k=N+1}^{n} \lambda_k |f_0(x) - S_k(f_0;x)|^p \\
&\quad := J_1 + J_2.
\end{aligned}$$

Using (23) we get

$$J_1 \ll \frac{1}{N^p} \sum_{k=1}^{N} \lambda_k \left(\sum_{v=k+1}^{N} b_v v \right)^p + \Lambda_N N^p \left(\sum_{s=0}^{\infty} \left\{ \sum_{v=[2^s(N+1)/c]}^{[c2^s(N+1)]} \frac{b_v}{v} + \sum_{v=2^s(N+1)}^{2^s(N+1)+1} b_v \right\} \right)^p$$

and by (24) we have

$$J_2 \ll \sum_{k=N}^{n} \lambda_k \left(N \sum_{s=0}^{\infty} \left\{ \sum_{v=[2^s(k+1)/c]}^{[c2^s(k+1)]} \frac{b_v}{v} + \sum_{v=2^s(k+1)}^{2^s(k+1)+1} b_v \right\} \right)^p.$$

Further we estimate the quantities J_1 and J_2 analogously as the quantities I_1 and I_2. The only difference is that instead of M we take N. If $n \le N$, then we can estimate the sum (20) in the same way as we have estimated J_1, with the only modification that instead of N we taken n. Thus, we obtain that for $x \in \left[\frac{\pi}{2}, \pi\right)$ and every n,

$$\sum_{k=1}^{n} \lambda_k |f(x) - S_k(f;x)|^p \ll n^{-pq} \omega^p \left(\frac{1}{n}\right) \Lambda_n. \qquad (25)$$

The inequalities (21) and (25) clearly show that $f \in H(\lambda, p, q, \omega)$. The proof is complete. ∎

Lemma 3. *Let* $p > 0$, $q \geq 0$, $\omega \in \Omega$, $(b_n) \in \mathrm{GM}\,({}_5\beta, r)$ $(r \geq 3)$. *If* (7) *and* (9) *hold, then the Fourier series* (1) *converges to* f *uniformly, i.e.*

$$f(x) := \sum_{n=1}^{\infty} b_n \sin nx.$$

Further, if (λ_n) *be a sequence of positive numbers, which satisfies* (5) *and* (6), *then* $f \in H(\lambda, p, q, \omega)$.

Proof. The first part of Lemma 3 is clear by generalization of Chaudy–Jollife's theorem for the class $\mathrm{GM}\,({}_5\beta, 2)$ (see [11]) with the conditions (7) and (9). Next, we shall show that (25) holds for any x. Since $S_k(f, 0) = S_k(f, \pi) = 0$, it suffices to prove (25) for $x \in (0, \pi)$.

First we shall show that (25) is valid for $x = \frac{2l\pi}{r}$, where l is an integer number such that $0 < 2l < r$. For any k there exist two numbers $p, u \in \mathbb{N} \cup \{0\}$ such that $k + 1 = r \cdot p + u$, where $0 \leq u < r$. Then

$$\begin{aligned}
&\left| f\left(\frac{2l\pi}{r}\right) - S_k\left(f; \frac{2l\pi}{r}\right) \right| \\
&= \left| \sum_{v=k+1}^{\infty} b_v \sin\left(v\frac{2l\pi}{r}\right) \right| \\
&\leq \left| \sum_{v=r\cdot p+u}^{r(p+1)-1} b_v \sin\left(v\frac{2l\pi}{r}\right) \right| + \left| \sum_{v=p+1}^{\infty} \sum_{i=0}^{r-1} b_{r\cdot v+i} \sin\left((rv+i)\frac{2l\pi}{r}\right) \right| \\
&= \left| \sum_{v=r\cdot p+u}^{r(p+1)-1} b_v \sin\left(v\frac{2l\pi}{r}\right) \right| + \left| \sum_{v=p+1}^{\infty} \sum_{i=1}^{r-1} b_{r\cdot v+i} \sin\left(i\frac{2l\pi}{r}\right) \right|.
\end{aligned}$$

If $r = 2s$ $(s = 2, 3, \ldots)$, then

$$\begin{aligned}
&\sum_{i=1}^{r-1} b_{r\cdot v+i} \sin\left(i\frac{2l\pi}{r}\right) \\
&= \sum_{k=1}^{2s-1} b_{2s\cdot v+i} \sin\left(i\frac{l\pi}{s}\right) = \sum_{i=1}^{s-1} (b_{2s\cdot v+i} - b_{2s\cdot v+2s-i}) \sin\left(i\frac{l\pi}{s}\right) \\
&= \sum_{i=1}^{s} (b_{2s\cdot v+i} - b_{2s\cdot v+2s-i}) \sin\left(i\frac{l\pi}{s}\right) \\
&= \sum_{i=1}^{r/2} (b_{r\cdot v+i} - b_{r\cdot v+r-i}) \sin\left(i\frac{2l\pi}{r}\right)
\end{aligned} \tag{26}$$

and if $r = 2s + 1$ $(s = 1, 2, \ldots)$ then

$$\sum_{i=1}^{r-1} b_{r\cdot v+i} \sin\left(i\frac{2l\pi}{r}\right) = \sum_{k=1}^{2s} b_{(2s+1)\cdot v+i} \sin\left(i\frac{2l\pi}{2s+1}\right)$$

$$= \sum_{k=1}^{s}(b_{(2s+1)\cdot v+i} - b_{(2s+1)\cdot v+2s+1-i}) \sin\left(i\frac{2l\pi}{2s+1}\right)$$

$$= \sum_{i=1}^{[r/2]}(b_{r\cdot v+i} - b_{r\cdot v+r-i}) \sin\left(i\frac{2l\pi}{r}\right). \tag{27}$$

Using (26) or (27) we obtain

$$\left|f\left(\frac{2l\pi}{r}\right) - S_k\left(f;\frac{2l\pi}{r}\right)\right| \le \sum_{v=r\cdot p+u}^{r(p+1)-1} b_v + \sum_{v=p+1}^{\infty}\sum_{i=1}^{[r/2]}|b_{r\cdot v+i} - b_{r\cdot v+r-i}|.$$

Thus, using (9), (7), (5) and (6), we get

$$\begin{aligned}
&\sum_{k=1}^{n}\lambda_k\left|f\left(\frac{2l\pi}{r}\right) - S_k\left(f;\frac{2l\pi}{r}\right)\right|^p \\
&\quad\ll \sum_{k=1}^{n}\lambda_k\left(\sum_{v=r\cdot p+u}^{r(p+1)-1} b_v\right)^p + \sum_{k=1}^{n}\lambda_k\left(\sum_{v=p+1}^{\infty}\sum_{i=1}^{[r/2]}|b_{r\cdot v+i} - b_{r\cdot v+r-i}|\right) \\
&\quad\ll \sum_{k=1}^{n}\lambda_k\left(\sum_{v=r\cdot p+u}^{r(p+1)-1} v^{-q-1}\omega\left(\frac{1}{v}\right)\right)^p + \sum_{k=1}^{n}\lambda_k\left(p^{\varepsilon}\sum_{v=[p/c]}^{[cp]}\frac{b_v}{v}\right)^p \\
&\quad\ll \sum_{k=1}^{n}\lambda_k k^{(-q-1)p}\omega^p\left(\frac{1}{k}\right) + \sum_{k=1}^{n}\lambda_k\left(p^{\varepsilon}\sum_{v=[p/c]}^{[cp]} v^{-q-2}\omega\left(\frac{1}{v}\right)\right)^p \\
&\quad\ll \sum_{k=1}^{n}\lambda_k k^{(-q-1)p}\omega^p\left(\frac{1}{k}\right) + \sum_{k=1}^{n}\lambda_k k^{(-q-1+\varepsilon)p}\omega^p\left(\frac{1}{k}\right) \\
&\quad\ll \sum_{k=1}^{n}\Lambda_k k^{-pq}\omega^p\left(\frac{1}{k}\right)k^{(\varepsilon-1)p}\frac{\lambda_k}{\Lambda_k} \\
&\quad\ll n^{-pq}\omega^p\left(\frac{1}{n}\right)\Lambda_n\sum_{k=1}^{n}\frac{\lambda_k}{\Lambda_k}k^{(\varepsilon-1)p} \\
&\quad\le n^{-pq}\omega^p\left(\frac{1}{n}\right)\Lambda_n M^p\sum_{l=0}^{\infty}\sum_{i=2^l}^{2^{l+1}}\frac{\lambda_i}{\Lambda_i}(i)^{(\varepsilon-1)p} \\
&\quad\le n^{-pq}\omega^p\left(\frac{1}{n}\right)\Lambda_n\sum_{l=0}^{\infty}(2^l)^{(\varepsilon-1)p}\frac{1}{\Lambda_{2^l}}\sum_{i=2^l}^{2^{l+1}}\lambda_i \\
&\quad\le n^{-pq}\omega^p\left(\frac{1}{n}\right)\Lambda_n\sum_{l=0}^{\infty}(2^l)^{(\varepsilon-1)p}\frac{\Lambda_{2^{l+1}}}{\Lambda_{2^l}} \ll n^{-pq}\omega^p\left(\frac{1}{n}\right)\Lambda_n.
\end{aligned}\tag{28}$$

Now, we prove that (25) holds for $\frac{2l\pi}{r} < x \le \frac{2l\pi}{r} + \frac{\pi}{r}$, where $0 \le 2l < r$.

Let $M := M(x) \geq r$ be the natural number such that

$$\frac{2l\pi}{r} + \frac{\pi}{M+1} < x \leq \frac{2l\pi}{r} + \frac{\pi}{M}. \tag{29}$$

Then, for $k < M$ we obtain

$$|f(x) - S_k(f;x)| \leq \left|\sum_{v=k+1}^{M} b_v \sin vx\right| + \left|\sum_{v=M+1}^{\infty} b_v \sin vx\right|.$$

Applying Lagrange's mean value theorem to the function $g(x) = \sin vx$ on the interval $\left[\frac{2l\pi}{r}, x\right]$ we obtain that there exists $y \in \left(\frac{2l\pi}{r}, x\right)$ such that

$$\sin vx - \sin\left(v\frac{2l\pi}{r}\right) = v\cos vy\left(x - \frac{2l\pi}{r}\right).$$

Using this, (29) and (8) we get

$$\begin{aligned}
&|f(x) - S_k(f;x)| \\
&\quad \leq \left|\sum_{v=k+1}^{M} vb_v \cos vy\left(x - \frac{2l\pi}{r}\right)\right| + \left|\sum_{v=k+1}^{M} b_v \sin\left(v\frac{2l\pi}{r}\right)\right| + \left|\sum_{v=M+1}^{\infty} b_v \sin vx\right| \\
&\quad \ll \frac{1}{M}\sum_{v=k+1}^{M} vb_v + \left|\sum_{v=k+1}^{M} b_v \sin\left(v\frac{2l\pi}{r}\right)\right| \\
&\quad + \left|\frac{-1}{2\sin(rx/2)}\sum_{s=0}^{\infty}\left\{\sum_{v=2^s(M+1)}^{2^{s+1}(M+1)-1}(b_v - b_{v+r})\cos\left(v+\frac{r}{2}\right)x\right.\right. \\
&\quad \left.\left. + \sum_{v=2^{s+1}(M+1)}^{2^{s+1}(M+1)+1} b_v \cos\left(v - \frac{r}{2}\right)x - \sum_{v=2^s(M+1)}^{2^s(M+1)+1} b_v \cos\left(v - \frac{r}{2}\right)x\right\}\right| \\
&\ll \frac{1}{M}\sum_{v=k+1}^{M} vb_v + \left|\sum_{v=k+1}^{M} b_v \sin\left(v\frac{2l\pi}{r}\right)\right| \\
&\quad + \frac{1}{2|\sin(rx/2)|}\sum_{s=0}^{\infty}\left\{\sum_{v=2^s(M+1)}^{2^{s+1}(M+1)-1}|b_v - b_{v+r}| + \sum_{v=2^{s+1}(M+1)}^{2^{s+1}(M+1)+1} b_v + \sum_{v=2^s(M+1)}^{2^s(M+1)+1} b_v\right\}.
\end{aligned}$$

If $(b_n) \in \text{GM}\,({}_5\beta, r)$ $(r \geq 3)$, then using the inequality $\frac{r}{\pi}x - 2l \leq \left|\sin\frac{rx}{2}\right|$ $\left(x \in \left[\frac{2l\pi}{r}, \frac{2l\pi}{r} + \frac{\pi}{r}\right]\right.$ and $\left.0 \leq 2l < r\right)$ and (29) we obtain

$$\begin{aligned}
&|f(x) - S_k(f;x)| \\
&\quad \ll \frac{1}{M}\sum_{v=k+1}^{M} vb_v + \left|\sum_{v=k+1}^{M} b_v \sin\left(v\frac{2l\pi}{r}\right)\right| \\
&\quad + \frac{1}{\frac{r}{\pi}x - 2l}\sum_{s=0}^{\infty}\left\{\sum_{v=2^s(M+1)}^{2^{s+1}(M+1)-1}|b_v - b_{v+r}| + \sum_{v=2^s(M+1)}^{2^s(M+1)+1} b_v\right\}
\end{aligned}$$

$$\ll \frac{1}{M}\sum_{v=k+1}^{M} v b_v + \left|\sum_{v=k+1}^{M} b_v \sin\left(v\frac{2l\pi}{r}\right)\right|$$
$$+ M\sum_{s=0}^{\infty}\left\{\sum_{v=[2^s(M+1)/c]}^{[c2^s(M+1)t]}\frac{b_v}{v} + \sum_{v=2^s(M+1)}^{2^s(M+1)+1} b_v\right\}. \tag{30}$$

Analogously, if $k \geq M$, then

$$|f(x) - S_k(f;x)| \ll M\sum_{s=0}^{\infty}\left\{\sum_{v=[2^s(k+1)/c]}^{[c2^s(k+1)]}\frac{b_v}{v} + \sum_{v=2^s(k+1)}^{2^s(k+1)+1} b_v\right\}. \tag{31}$$

Further, for $n > M$, using (30) and (31) we have

$$\sum_{k=1}^{n}\lambda_k|f(x) - S_k(f;x)|^p$$
$$= \sum_{k=1}^{M}\lambda_k|f(x) - S_k(f;x)|^p + \sum_{k=M+1}^{n}\lambda_k|f(x) - S_k(f;x)|^p$$
$$\ll \frac{1}{M^p}\sum_{k=1}^{M}\lambda_k\left(\sum_{v=k+1}^{M} b_v v\right)^p + \sum_{k=1}^{M}\lambda_k\left|\sum_{v=k+1}^{M} b_v\sin\left(v\frac{2l\pi}{r}\right)\right|^p$$
$$+ \sum_{k=1}^{M}\lambda_k M^p\left(\sum_{s=0}^{\infty}\left\{\sum_{v=[2^s(M+1)/c]}^{[c2^s(M+1)]}\frac{b_v}{v} + \sum_{v=2^s(M+1)}^{2^s(M+1)+1} b_v\right\}\right)^p$$
$$+ \sum_{k=M+1}^{n}\lambda_k\left(M\sum_{s=0}^{\infty}\left\{\sum_{v=[2^s(k+1)/c]}^{[c2^s(k+1)]}\frac{b_v}{v} + \sum_{v=2^s(k+1)}^{2^s(k+1)+1} b_v\right\}\right)^p$$
$$:= \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4.$$

The quantities Σ_1, Σ_3 and Σ_4 can be estimated in the same way as the quantities I_{11}, I_{12} and I_2 in the proof of Lemma 2, respectively.

Analogously as in (28) we can show that

$$\Sigma_2 \ll n^{-pq}\omega^p\left(\frac{1}{n}\right)\Lambda_n.$$

If $n \leq M$, then we can estimate (20) in the same way as we have estimated Σ_1, Σ_2 and Σ_3, with the only modification that instead of M we take n. Thus we obtain that (25) holds for $x \in \left[\frac{2l\pi}{r}, \frac{2l\pi}{r} + \frac{\pi}{r}\right]$, where $0 \leq 2l < r$ and every natural number n.

Finally, we prove that (25) is true for $\frac{2l\pi}{r} + \frac{\pi}{r} \leq x < \frac{2(l+1)\pi}{r}$, where $0 < 2(l+1) \leq r$. Let $N := N(x) \geq r$ be the integer such that

$$\frac{2(l+1)\pi}{r} - \frac{\pi}{N} \leq x < \frac{2(l+1)\pi}{r} - \frac{\pi}{N+1}. \tag{32}$$

Then, for $k < M$ we obtain

$$|f(x) - S_k(f;x)| \leq \left|\sum_{v=k+1}^{N} b_v \sin vx\right| + \left|\sum_{v=N+1}^{\infty} b_v \sin vx\right|.$$

Applying Lagrange's mean value theorem to the function $g(x) = \sin vx$ on the interval $\left[x, \frac{2(l+1)\pi}{r}\right]$ we obtain that there exists $z \in \left(x, \frac{2(l+1)\pi}{r}\right)$ such that

$$\sin\left(v\frac{2(l+1)\pi}{r}\right) - \sin vx = v\cos vz\left(\frac{2(l+1)\pi}{r} - x\right).$$

Using this, (32) and (8) we get

$$\begin{aligned}
&|f(x) - S_k(f;x)| \\
&\quad \leq \left|\sum_{v=k+1}^{N} vb_v \cos vz\left(x - \frac{2(l+1)\pi}{r}\right)\right| \\
&\qquad + \left|\sum_{v=k+1}^{N} b_v \sin\left(v\frac{2(l+1)\pi}{r}\right)\right| + \left|\sum_{v=N+1}^{\infty} b_v \sin vx\right| \\
&\quad \ll \frac{1}{N}\sum_{v=k+1}^{N} vb_v + \left|\sum_{v=k+1}^{N} b_v \sin\left(v\frac{2\,(l+1)\,\pi}{r}\right)\right| + \frac{1}{2|\sin(rx/2)|} \\
&\qquad \times \sum_{s=0}^{\infty}\left\{\sum_{v=2^s(N+1)}^{2^{s+1}(N+1)-1} |b_v - b_{v+r}| + \sum_{v=2^{s+1}(N+1)}^{2^{s+1}(N+1)+1} b_v + \sum_{v=2^s(N+1)}^{2^s(N+1)+1} b_v\right\}.
\end{aligned}$$

If $(b_n) \in \mathrm{GM}\,({}_5\beta, r)$ $(r \geq 3)$, then using the inequality $2(l+1) - \frac{r}{\pi}x \leq \left|\sin\frac{rx}{2}\right|$ $\left(x \in \left[\frac{2l\pi}{r} + \frac{\pi}{r}, \frac{2(l+1)\pi}{r}\right] \text{ and } 0 < 2(l+1) \leq r\right)$ and (32) we obtain

$$\begin{aligned}
&|f(x) - S_k(f;x)| \\
&\quad \ll \frac{1}{N}\sum_{v=k+1}^{N} vb_v + \left|\sum_{v=k+1}^{N} b_v \sin\left(v\frac{2(l+1)\pi}{r}\right)\right| \\
&\qquad + \frac{1}{2(l+1) - \frac{r}{\pi}x}\sum_{s=0}^{\infty}\left\{\sum_{v=2^s(N+1)}^{2^{s+1}(N+1)-1} |b_v - b_{v+r}| + \sum_{v=2^s(N+1)}^{2^s(N+1)+1} b_v\right\} \\
&\quad \ll \frac{1}{N}\sum_{v=k+1}^{N} vb_v + \left|\sum_{v=k+1}^{N} b_v \sin\left(v\frac{2(l+1)\pi}{r}\right)\right| \\
&\qquad + N\sum_{s=0}^{\infty}\left\{\sum_{v=[2^s(N+1)/c]}^{[c2^s(N+1)]} \frac{b_v}{v} + \sum_{v=2^s(N+1)}^{2^s(N+1)+1} b_v\right\} \qquad (33)
\end{aligned}$$

Analogously, if $k \geq M$, then

$$|f(x) - S_k(f;x)| \ll N\sum_{s=0}^{\infty}\left\{\sum_{v=[2^s(k+1)/c]}^{[c2^s(k+1)]} \frac{b_v}{v} + \sum_{v=2^s(k+1)}^{2^s(k+1)+1} b_v\right\}. \qquad (34)$$

Using (33) and (34) we get that for $n > N$,

$$\begin{aligned}
&\sum_{k=1}^{n} \lambda_k |f(x) - S_k(f;x)|^p \\
&\quad = \sum_{k=1}^{N} \lambda_k |f(x) - S_k(f;x)|^p + \sum_{k=N+1}^{n} \lambda_k |f(x) - S_k(f;x)|^p \\
&\quad \ll \frac{1}{N^p} \sum_{k=1}^{N} \lambda_k \left(\sum_{v=k+1}^{N} b_v v \right)^p + \sum_{k=1}^{N} \lambda_k \left| \sum_{v=k+1}^{N} b_v \sin\left(v \frac{2(l+1)\pi}{r} \right) \right|^p \\
&\qquad + \sum_{k=1}^{N} \lambda_k N^p \left(\sum_{s=0}^{\infty} \left\{ \sum_{v=[2^s(N+1)/c]}^{[c2^s(N+1)]} \frac{b_v}{v} + \sum_{v=2^s(N+1)}^{2^s(N+1)+1} b_v \right\} \right)^p \\
&\qquad + \sum_{k=N+1}^{n} \lambda_k \left(N \sum_{s=0}^{\infty} \left\{ \sum_{v=[2^s(k+1)/c]}^{[c2^s(k+1)]} \frac{b_v}{v} + \sum_{v=2^s(k+1)}^{2^s(k+1)+1} b_v \right\} \right)^p \\
&\quad := \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4.
\end{aligned}$$

Further, we estimate the quantities Δ_1, Δ_2, Δ_3 and Δ_4 analogously as the quantities Σ_1, Σ_2, Σ_3 and Σ_4, respectively. The only difference is that instead of M we take N. If $n \le N$, then we can estimate the sum (20) in the same way as we have estimated Δ_1, Δ_2 and Δ_3, with the only modification that instead of N we taken n. Thus, we obtain that (25) is true for $x \in \left[\frac{2l\pi}{r} + \frac{\pi}{r}, \frac{2(l+1)\pi}{r}\right]$, where $0 < 2(l+1) \le r$ and every natural number n.

Collecting our partial estimations we obtain that (25) holds for any x. Thus $f \in H(\lambda, p, q, \omega)$ and this ends our proof. ■

Lemma 4. *Let* $(c_n) \in \mathrm{GM}\,(\beta, r)$, *where* $r \ge 1$ *and a nonnegative sequence* $\beta = (\beta_n)$ *satisfies*

$$\sum_{k=[n/2]}^{n} \beta_k \ll \sum_{k=[n/\gamma]}^{[\gamma n]} c_k \tag{35}$$

for some $\gamma > 1$. *Then*

$$n c_n \ll \sum_{k=[n/\gamma]}^{[\gamma n]} c_k, \qquad \gamma > 1.$$

Proof. If $n \le r$, then the inequality obviously holds. Now, let $n > r$. For $j = n+1, n+2, \ldots, 2n$, we get

$$\sum_{k=n}^{j-1} |c_k - c_{k+r}| \ge \left| \sum_{k=n}^{j-1} (c_k - c_{k+r}) \right| \ge \sum_{k=n}^{j-1} c_k - \sum_{k=n+r}^{j+r-1} c_k$$

and for $j \geq n + r + 1$ we obtain

$$c_n \leq \sum_{k=n}^{n+r-1} c_k \leq \sum_{k=n}^{j-1} |c_k - c_{k+r}| + \sum_{k=j}^{j+r-1} c_k$$

$$\leq \sum_{k=[j/2]}^{2[j/2]-1} |c_k - c_{k+r}| + \sum_{k=j}^{j+r-1} c_k \ll \beta_{[j/2]} + \sum_{k=j}^{j+r-1} c_k.$$

Summing up on j and using (35) we get

$$nc_n = \sum_{j=n+1}^{2n} c_n = \sum_{j=n+1}^{n+r} c_n + \sum_{j=n+r+1}^{2n} c_n$$

$$\ll rc_n + \sum_{j=n+r+1}^{2n} \left(\beta_{[j/2]} + \sum_{k=j}^{j+r-1} c_k \right)$$

$$\leq rc_n + \sum_{j=n+r+1}^{2n} \beta_{[j/2]} + \sum_{j=0}^{r-1} \sum_{k=n+1+j}^{2n-1+j} c_k$$

$$\leq \sum_{j=[n/2]}^{n} \beta_j + r \sum_{k=n}^{2n+r-2} c_k \ll \sum_{k=[n/\gamma]}^{[\gamma n]} c_k.$$

The proof is complete. ■

4. Proofs of the results

4.1 *Proof of Theorem 3*

To prove the assertion of Theorem 3 we show that if $f \in W^r H_\beta^\omega \cap C_4(2)$, then $f \in H(\lambda, p, q, \omega)$ also holds. Let $f \in W^q H_\beta^\omega \cap C_4(2)$. Then using the technique of Tikhonov, we get that

$$\omega\left(\frac{1}{n}\right) \gg n^{-(\beta+1)} \sum_{k=1}^{n} k^{q+\beta+1} b_k.$$

Let $m = [\gamma n] + 1$ and $(b_k) \in \mathrm{GM}\ ({}_5\beta, 2)$. Then

$$\omega\left(\frac{1}{n}\right) \gg \omega\left(\frac{1}{m}\right) \gg m^{-(\beta+1)} \sum_{k=1}^{m} k^{q+\beta+1} b_k$$

$$\gg m^{-(\beta+1)} \sum_{k=1}^{m} b_k \sum_{j=1}^{k} j^{q+\beta} = m^{-(\beta+1)} \sum_{k=1}^{m} k^{q+\beta} \sum_{j=k}^{m} b_j$$

$$\gg (\gamma n)^{-(\beta+1)} \sum_{k=1}^{[\gamma n]} k^{q+\beta} \sum_{j=\left[\frac{n}{\gamma}\right]}^{[\gamma n]} b_j$$

and by Lemma 4 with $r = 2$,

$$\omega\left(\frac{1}{n}\right) \gg n^{-(\beta+1)} \sum_{k=1}^{[\gamma n]} k^{q+\beta} n b_n \gg n^{q+1} b_n$$

i.e.

$$b_n \ll n^{-q-1} \omega\left(\frac{1}{n}\right). \tag{36}$$

Furthermore, by the hypothesis of Theorem 3, all the assumptions of Lemma 2 hold, and consequently we obtain that $f \in H(\lambda, p, q, \omega)$. This completes our proof.

4.2 *Proof of Theorem 4*

Analogously, as in the above proof, using Lemma 4 we get that (36) is true. Moreover, by the hypothesis of Theorem 4, all the assumptions of Lemma 3 hold. Hence $f \in H(\lambda, p, q, \omega)$ and this ends the proof.

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