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Integers without large prime factors in short intervals: Conditional results

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Abstract. Under the Riemann hypothesis and the conjecture that the order of growth of the argument of $\zeta(1/2+it)$ is bounded by $(\log t)^{\frac{1}{2}+o(1)}$, we show that for any given $\alpha > 0$ the interval $(X, X+\sqrt{X}(\log X)^{1/2+o(1)}]$ contains an integer having no prime factor exceeding X^{α} for all X sufficiently large.

Keywords. Smooth numbers; Riemann zeta function.

1. Introduction

Suppose $P(n)$ denotes the largest prime factor of an integer $n > 1$ and let us declare $P(1) = 1$. Given a positive real number y, an integer n is called y-smooth if $P(n) \leq y$. Smooth numbers are important in many branches of Number Theory as well as in Cryptography. We refer the reader to the articles by Granville [Gra00], Hildebrand and Tenenbaum [HT93] and Pomerance [Po94] for highly readable and informative discussions on these topics. This article is about distribution of smooth numbers in short intervals, namely intervals of type $(X, X+\sqrt{X}]$. See the next subsection for basic facts about distribution of smooth numbers. One expects that smooth numbers are uniformly distributed among intervals of moderate size. This means the following: Consider the function

$$\psi(x, y) = |\{1 \leq n \leq x\colon P(n) \leq y\}|$$

which counts the number of y-smooth numbers up to x. Then it is believed that the following asymptotic formula holds for wide ranges of the variables x and z,

$$\psi(x+z, y) - \psi(x, y) \sim \frac{z}{x}\psi(x, y).$$

However, this is known to be true only under the restriction that x/z is very small compared to x (see Hildebrand–Tenenbaum [HT93] and Friedlander–Granville [FG93]). The ranges of y and z in which such an asymptotic formula holds for almost all $n \leq x$ are also investigated in the two works cited above.

A challenging problem in this subject (see [Gra00] or [FG93]) is to prove that

$$\psi(x+x^{\beta}, x^{\alpha}) - \psi(x, x^{\alpha}) \gg x^{\beta} \tag{1}$$

holds for all $0 < \alpha, \beta < 1$. In a fundamental work, Balog [Ba87] proved this for all $\beta > 1/2$ and $\alpha > 0$ and his method was refined by Harman [Har91] who obtained the same result but with a much better smoothness condition, namely with $y = \exp((\log x)^{2/3+o(1)})$ in place of x^α. So far no one has been able to prove (1) even for $\beta = 1/2$ with $\alpha > 0$ arbitrary. However, breaking this '1/2'-barrier is crucial for application to Lenstra's elliptic curve factorization algorithm, though the smoothness required is even stronger. This algorithm finds a prime factor p of a large integer N in expected time if there are many $\exp(\sqrt{\log p \log\log p})$-smooth numbers in the interval $(p - \sqrt{p}, p + \sqrt{p})$. The expected time here is $O(\exp((\sqrt{2} + o(1))\sqrt{\log p \log\log p}))$. See [Gra00], §2f for Lenstra's algorithm.

Even the Riemann hypothesis does not solve the problem even though the zeros of the Riemann zeta function are intimately connected with distribution of smooth numbers as we shall see (see the end of §2.1 for a striking manifestation of this connection). Indeed, the strongest result in literature in this direction under the Riemann hypothesis (denoted RH henceforth), due to Xuan [Xu99], says

$$\psi(x + \sqrt{x}(\log x)^{1+o(1)}, x^\alpha) - \psi(x, x^\alpha) > 0$$

for all $\alpha > 0$. We have improved this to

$$\psi(x + \sqrt{x}(\log x)^{\frac{1}{2}+o(1)}, x^\alpha) - \psi(x, x^\alpha) > 0$$

under RH and a further hypothesis that $S(t) \ll (\log t)^{1/2+\varepsilon}$ which is widely believed to be true. See below for the definition of $S(t)$ and the reasons for believing this conjecture. Under RH alone, we have improved Xuan's result also, albeit by a very small amount (see Theorem 2).

Here and henceforth ε denotes fixed but arbitrarily small positive number whether or not it is explicitly mentioned.

Let $T > 0$ and let $N(T)$ denote the number of zeros of the Riemann ζ function in the region $0 < \sigma < 1, 0 < t \le T$. If T is not the ordinate of a zero of ζ, let $S(T)$ denote the value of $\pi^{-1} \arg \zeta(1/2+it)$ obtained by continuous variation along the straight line joining $2, 2+iT, 1/2+iT$, starting with the value 0. If T is the ordinate of a zero of ζ, let $S(T) = S(T+0)$. Let

$$L(T) = \frac{1}{2\pi} T \log T - \frac{1 + \log 2\pi}{2\pi} T + \frac{7}{8}. \tag{2}$$

Then an application of the Stirling asymptotic formula for the gamma function yields (see Chapter 15 of [Da00]) the asymptotic formula

$$N(T) = L(T) + S(T) + O\left(\frac{1}{T}\right) \tag{3}$$

as $T \to \infty$.

Surprisingly little is known about the function $S(t)$. The best unconditional bound is $S(t) = O(\log t)$ (op. cit.) and it has not been improved upon for more than hundred years. Under RH, one can show that $S(t) O(\log t/\log\log t)$. Montgomery has shown that (see Theorem 2 of [Mo77]) under RH,

$$S(t) = \Omega_\pm((\log t/\log\log t)^{\frac{1}{2}}) \tag{4}$$

and further he conjectures that

$$S(t) \ll (\log t / \log\log t)^{\frac{1}{2}}. \tag{5}$$

Farmer, Gonek and Hughes [FGH07] have given arguments from random matrix theory that suggests

$$\limsup_{t\to\infty} \frac{S(t)}{\sqrt{\log t \log\log t}} = \frac{1}{\pi\sqrt{2}}.$$

Here we assume RH and a bound on $S(t)$ which is weaker than either of the two conjectures mentioned above, namely

$$S(t) \ll (\log t)^{1/2+\varepsilon}, \tag{6}$$

for ε arbitrarily small but positive.

Our main result is

Theorem 1. *Under RH and the conjectural bound* (6), *we have for any given* $\alpha > 0$ *and* $\varepsilon > 0$, *a positive number* $X_0 = X_0(\varepsilon, \alpha)$, *such that whenever* $X > X_0$, *and* $Y \geq \sqrt{X}(\log X)^{1/2+\varepsilon}$, *the interval* $(X, X+Y]$ *contains an integer having no prime factor exceeding* X^{α}.

We proceed along the lines of Xuan [Xu99]. However, the conjecture on $S(t)$ allows us to obtain a good bound on the growth of logarithmic derivative of $\zeta(s)$ on a vertical line sufficiently close to the critical line and this results in a better error term. We also choose M_1 and M_2 a little differently which gives us a little extra saving.

If we assume only RH, then in the proof of Lemma 5, we can use the bound $S(t) = O(\log t / \log\log t)$ and that will lead to the following result which gives a minute improvement over Xuan's result.

Theorem 2. *Under RH, we have for any given* $\alpha > 0$ *and* $\varepsilon > 0$, *a positive number* $X_0 = X_0(\varepsilon, \alpha)$, *such that whenever* $X > X_0$, *and* $Y \geq \sqrt{X}(\log X)(\log\log X)^{-1+\varepsilon}$, *the interval* $(X, X+Y]$ *contains an integer having no prime factor exceeding* X^{α}.

We give proof only of the first theorem since the proof of the second will be identical except that the bound for $S(t)$ will be different.

Remark 1. Recently Soundararajan [So10] has improved the result substantially on RH alone. He proves, on RH, that there are X^{α}-smooth numbers in intervals of length $c(\alpha)\sqrt{X}$.

Remark 2. Our proof shows that the number of X^{α}-smooth numbers in the intervals in question is actually $\gg X^{1/2-o(1)}$.

Notations and conventions. ε will denote positive real numbers which can be arbitrarily small and it need not be the same in different occurrences. s will denote a complex variable and its real and imaginary part will be denoted by σ and t respectively.

2. Preliminary steps

2.1 *Distribution of smooth numbers*

It is of interest to know how many y-smooth numbers are there between 1 and X. Let $\Psi(x, y)$ denote the number of y-smooth positive integers $\leq x$. Dickman [Di30] was the first to prove an asymptotic formula of the kind

$$\Psi(x, y) \sim \rho(u)x \quad \text{as } x \to \infty \text{ with } u = \frac{\log x}{\log y} \text{ fixed.} \tag{7}$$

The function ρ is monotonically decreasing, continuous and satisfies the following differential difference equation:

$$u\rho'(u) = -\rho(u-1) \quad (u > 1), \tag{8}$$

$$\text{with the initial condition} \quad \rho(u) = 1 \quad (0 \leq u \leq 1). \tag{9}$$

This function is known as the Dickman function or the Dickman–de Bruijn function. Note that it is constant for all sufficiently large x if y is a constant power of x. de Bruijn [Br51] showed that

$$\Psi(x, y) = x\rho(u)\left\{1 + O\left(\frac{\log(u+1)}{\log y}\right)\right\} \tag{10}$$

holds uniformly in the range

$$y \geq 2, \quad 1 \leq u \leq (\log y)^{3/5-o(1)}; \quad \text{that is, for } y > \exp((\log x)^{5/8+o(1)}). \tag{11}$$

In 1986, Hildebrand [Hil86] improved the range (11) to

$$y \geq 2, \quad 1 \leq u \leq \exp\{(\log y)^{3/5-o(1)}\};$$

$$\text{that is, for } y > \exp((\log\log x)^{5/3+o(1)}). \tag{12}$$

It is natural to ask in what range we can expect this asymptotic to be valid. Hildebrand [Hil84] showed that the above asymptotic formula holds uniformly for

$$1 \leq u \leq y^{1/2-o(1)}; \quad \text{that is, for } y \geq (\log x)^{2+o(1)}, \tag{13}$$

if and only if the Riemann hypothesis is true.

2.2 *A mean value result for Dirichlet polynomials*

We record the following well-known bound (see, for example, Chap. 6 of [Mo71] or Theorem 9.1 of [IK04]) on the mean value of Dirichlet polynomials which will be required later.

Theorem 3. *For any sequence $\{b_n\}$ of complex numbers and any positive real number R, we have*

$$\int_0^R \left|\sum_{n\leq N} b_n n^{it}\right|^2 \mathrm{d}t \ll (R+N)\sum_{n\leq N}|b_n|^2.$$

3. The technique of counting smooth numbers

The basic technique of counting smooth numbers used here, which goes back to Balog [Ba87], is the following. Let α be any fixed positive real number. Define a sequence $\{a_m\}$ where

$$a_m = \begin{cases} 1, & \text{if } p|m \Rightarrow p \le X^{\alpha}, \\ 0, & \text{otherwise.} \end{cases}$$

Let $M_1 = \sqrt{2}X^{\frac{1}{2}-\frac{\alpha}{2}}$ and $M_2 = \frac{\sqrt{X}}{f(X)}$ where $f(X) < (\log X)^{\frac{1}{2}+\varepsilon}$. Define a Dirichlet polynomial

$$M(s) = \sum_{M_1 \le m \le M_2} \frac{a_m}{m^s},$$

and define, for any positive integer n,

$$A_n = \{(m_1, m_2)\colon M_1 < m_1, m_2 \le M_2, m_1 m_2 | n\}$$

and

$$d_n = \sum_{\substack{n=m_1 m_2 r,\\ (m_1,m_2)\in A_n}} a_{m_1} a_{m_2} \Lambda(r),$$

where Λ is the von Mangoldt function, defined by

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^t \text{ for some integer } t, \\ 0, & \text{otherwise,} \end{cases}$$

and the associated Dirichlet series can be written as

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = -\frac{\zeta'(s)}{\zeta(s)}.$$

If we can show that

$$\sum_{X < n \le X+Y} d_n > 0,$$

with $Y < X$, then there must be some integer $n = m_1 m_2 r$ between X and $X + Y$, with m_1 and m_2 smooth and therefore n itself is smooth, because, $r = n/m_1 m_2 \le (X+Y)/M_1^2 = X^{\alpha}$.

4. A bound on $\frac{\zeta'(s)}{\zeta(s)}$

We shall have an occasion to use a bound of $\frac{\zeta'(s)}{\zeta(s)}$ inside the critical strip. So in this section we obtain a conditional bound assuming the conjecture (6).

Theorem 4. *Under RH and the assumption* $S(t) \ll (\log t)^{1/2+\varepsilon}$,

$$\frac{\zeta'(s)}{\zeta(s)} \ll (\log(|t|+2))^{\frac{3}{2}+\varepsilon}$$

uniformly in $\frac{1}{2} + \frac{1}{\log t} \le \sigma = \operatorname{Re} s \le \sigma_1 < 1$.

To prove the above theorem we need the following lemma.

Lemma 5. *If* $S(t) \ll (\log t)^{\frac{1}{2}+\varepsilon}$ *then we have*

$$N\left(T + \frac{1}{\log T}\right) - N(T) = O((\log T)^{1/2+\varepsilon}), \tag{14}$$

for every $\varepsilon > 0$.

Proof. Let $\varepsilon > 0$ be fixed and let $T > 0$. From (3) we have

$$\begin{aligned}
&N\left(T + \frac{1}{\log T}\right) - N(T) \\
&= \frac{1}{2\pi}\left\{\left(T + \frac{1}{\log T}\right)\log\left(T + \frac{1}{\log T}\right) - T\log T\right\} \\
&\quad - \frac{1+\log\ 2\pi}{2\pi}\frac{1}{\log T} + S\left(T + \frac{1}{\log T}\right) - S(T) \\
&\ll \frac{1}{2\pi}T\log\left(1 + \frac{1}{T\log T}\right) + (\log T)^{1/2+\varepsilon} \\
&\ll (\log T)^{1/2+\varepsilon},
\end{aligned}$$

as $T \to \infty$. The lemma follows.

Now we shall prove the above theorem.

Proof. Let $R = \left[\frac{\log t}{\log\log t}\right]$ and $\frac{1}{2} + \frac{1}{\log t} \le \sigma \le \sigma_1 < 1$.
Then by the formula (see eq. (14.15.2) of [Ti86])

$$\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)} = \sum_{|t-\gamma|<1/\log\log t}\frac{1}{s-\rho} + O(\log t),$$

where $\rho = \frac{1}{2} + i\gamma$ varies over the zeros of ζ, we have

$$\begin{aligned}
\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)} &\ll \sum_{t<\gamma<t+\frac{1}{\log\log t}}\frac{1}{\sqrt{\frac{1}{(\log t)^2} + (\gamma-t)^2}} + O(\log t) \\
&\le \sum_{k=0}^{R}\ \sum_{t+\frac{k}{\log t}<\gamma\le t+\frac{k+1}{\log t}}\frac{1}{\sqrt{\frac{1}{(\log t)^2} + (\gamma-t)^2}} + O(\log t) \\
&\le \sum_{k=0}^{R}\ \sum_{t+\frac{k}{\log t}<\gamma\le t+\frac{k+1}{\log t}}\frac{\log t}{\sqrt{1+k^2}} + O(\log t) \\
&\ll \sum_{k=0}^{R}\frac{(\log t)^{\frac{3}{2}+\varepsilon}}{\sqrt{1+k^2}} + O(\log t), \quad \text{(by Lemma 5)}
\end{aligned}$$

$$\ll (\log t)^{\frac{3}{2}+\varepsilon} \int_{u=0}^{R} \frac{1}{\sqrt{1+u^2}} \mathrm{d}u + O(\log t)$$

$$\ll (\log t)^{\frac{3}{2}+\varepsilon} \log R \ll (\log t)^{\frac{3}{2}+\varepsilon} \log\log t.$$

Hence the theorem follows. □

5. The proof

Now, for any $x \in [X, X+Y]$, by the Perron formula,

$$\sum_{x<n\le x+Y} d_n = \frac{1}{2\pi i} \int_{2-iT}^{2+iT} -\frac{\zeta'(s)}{\zeta(s)} M^2(s) \frac{(x+Y)^s - x^s}{s} \mathrm{d}s + O\left(\frac{X^{2+\frac{1}{100}}}{T}\right) + O\left(X^{\frac{1}{100}}\right),$$

where T is some positive real number for the moment, but later we shall choose $T \gg X^4$.

We integrate this with respect to x, getting

$$\int_X^{X+Y} \left(\sum_{x<n\le x+Y} d_n\right) \mathrm{d}x = \frac{1}{2\pi i} \int_{2-iT}^{2+iT} -\frac{\zeta'(s)}{\zeta(s)} M^2(s) A(s) \mathrm{d}s + O\left(\frac{YX^{2+\frac{1}{100}}}{T}\right) + O(YX^{\frac{1}{100}}),$$

where

$$A(s) = \frac{(X+2Y)^{s+1} - 2(X+Y)^{s+1} + X^{s+1}}{s(s+1)}.$$

Now, to show that there is a smooth number between X and $X+2Y$, it is enough to show that the left-hand side is positive (for all X large enough), which is shown in the next section. This integration results in saving one $\log X$ factor.

Our goal now is to show that $\int_X^{X+Y} \left(\sum_{x<n\le x+Y} d_n\right) \mathrm{d}x > 0$ for all X sufficiently large, and $Y = f(X)\sqrt{X}/2$, f satisfying the conditions of the theorem. We move the contour to $\operatorname{Re} s = \eta = \frac{1}{2} + \frac{1}{\log X}$, and apply the residue theorem of Cauchy, getting

$$\int_X^{X+Y} \left(\sum_{x<n\le x+Y} d_n\right) \mathrm{d}x = Y^2 M^2(1) + \frac{1}{2\pi i} \int_{2-iT}^{\eta-iT} + \frac{1}{2\pi i} \int_{\eta-iT}^{\eta+iT} + \frac{1}{2\pi i} \int_{\eta+iT}^{2+iT} + O\left(\frac{X^{2+\frac{1}{100}}Y}{T}\right) + O\left(YX^{\frac{1}{100}}\right) \quad (15)$$

since $\operatorname{Res}_{s=1}\left(-\frac{\zeta'(s)}{\zeta(s)}\right) = 1$, and $A(1) = Y^2$. Now, by (10),

$$M(1) = \sum_{M_1\le m\le M_2} \frac{a_m}{m} = \int_{M_1}^{M_2} \frac{1}{t} d\left(\sum_{m\le t} a_m\right) \gg \int_{M_1}^{M_2} \frac{1}{t} \rho(1/\alpha) \mathrm{d}t \gg \log X. \quad (16)$$

So the first term, $Y^2M^2(1) \gg Y^2(\log X)^2$, and we shall show that this term dominates all other terms. We have the bound

$$\frac{(X+Y)^s - X^s}{s} \ll \min\left\{YX^{\sigma-1}, \frac{X^\sigma}{|t|}\right\},$$

where $s = \sigma + it$ is as usual. This implies,

$$A(s) \ll \min\left\{Y^2X^{\sigma-1}, \frac{X^{\sigma+1}}{|t|^2}\right\}. \tag{17}$$

The horizontal integrals have T in the denominator and will be shown to be very small by trivial estimation. Namely, using the second bound for $A(s)$, and the bound $\left|\frac{\zeta'(\sigma+iT)}{\zeta(\sigma+iT)}\right| \ll (\log T)^2$, which we can ensure by choosing T suitably, avoiding the zeros of $\zeta(s)$ (see Chap. 17 of [Da00]),

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{2-iT}^{\eta-iT} -\frac{\zeta'(s)}{\zeta(s)}M^2(s)A(s)\mathrm{d}s \\
&\ll \int_2^\eta -\frac{\zeta'(\sigma+iT)}{\zeta(\sigma+iT)}M^2(\sigma+iT)A(\sigma+iT)\mathrm{d}\sigma \\
&\ll X^{3+\frac{1}{2}}(\log T)^2T^{-2} \ll X^{-4},
\end{aligned}$$

by choosing $T \gg X^4$. And similarly we get the same bound for the other integral $\int_{\eta+iT}^{2+iT}$.

Now for estimating the vertical integral from $\eta - iT$ to $\eta + iT$, we break up the interval $[0, T]$ into $[0, X/Y]$ and $[X/Y, T]$.

$$\begin{aligned}
I_1 &= \frac{1}{2\pi}\int_0^{X/Y} -\frac{\zeta'(\eta+it)}{\zeta(\eta+it)}M^2(\eta+it)A(\eta+it)\mathrm{d}t \\
&\ll Y^2X^{\eta-1}(X/Y + M_2)M_1^{1-2\eta}(1-2\eta)^{-1}(\log X)^{\frac{3}{2}+\frac{\varepsilon}{2}}
\end{aligned}$$

by Theorem 3, Theorem 4 and the bound (17). Hence,

$$\begin{aligned}
I_1 &\ll Y^2X^{-\frac{1}{2}+\frac{1}{\log X}}(X/Y)X^{\left(\frac{1}{2}-\frac{\alpha}{2}\right)(1-2\eta)}(\log X)(\log X)^{\frac{3}{2}+\frac{\varepsilon}{2}} \\
&\ll Y^2\frac{(\log X)^{\frac{5}{2}+\frac{\varepsilon}{2}}}{f(X)}.
\end{aligned}$$

Recall that we have taken $M_2 = \sqrt{X}/f(X)$ and $Y = \sqrt{X}f(X)/2$. The second integral is estimated by integration by parts, and we get

$$\begin{aligned}
I_2 &= \frac{1}{2\pi}\int_{X/Y}^{T}\left(-\frac{\zeta'(\eta+it)}{\zeta(\eta+it)}\right)M^2(\eta+it)A(\eta+it)\mathrm{d}t \\
&\ll X^{1+\eta}(\log X)^{\frac{3}{2}+\frac{\varepsilon}{2}}\int_{X/Y}^{T}\frac{|M^2(\eta+it)|}{|t|^2}\mathrm{d}t \\
&\ll X^{1+\eta}(\log X)^{\frac{3}{2}+\frac{\varepsilon}{2}}\frac{1}{(X/Y)^2}(X/Y+M_2)M_1^{1-2\eta}(1-2\eta)^{-1} \\
&\ll Y^2\frac{(\log X)^{\frac{5}{2}+\frac{\varepsilon}{2}}}{f(X)}.
\end{aligned}$$

by again using the same bounds. Finally,

$$\int_X^{X+Y} \left(\sum_{x<n\le x+Y} d_n \right) \mathrm{d}x = Y^2 M^2(1) + O\left(Y^2 \frac{(\log X)^{\frac{5}{2}+\frac{\varepsilon}{2}}}{f(X)} \right) + O(X^{-4}) + O\left(\frac{X^{2+\frac{1}{100}} Y}{T} \right) + O(YX^{\frac{1}{100}}).$$

Since $M^2(1) \gg (\log X)^2$ by (16) and $T \gg X^4$, if we take $f(X) > (\log X)^{\frac{1}{2}+\varepsilon}$ then we can conclude that

$$\int_X^{X+Y} \left(\sum_{x<n\le x+Y} d_n \right) \mathrm{d}x > 0$$

for all X sufficiently large, which is what we wanted to prove.

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References

[Ba87] Balog A, On the distribution of integers having no large prime factors, *Astérisque* **147–148** (1987) 27–31

[Br51] de Bruijn N G, On the number of positive integers $\le x$ and free of prime factors $> y$, *Nederl. Akad. Wetensch. Proc.* **A54** (1951) 50–60

[Da00] Davenport H, Multiplicative number theory, Third edition, Revised and with a preface by Hugh L Montgomery, Graduate Texts in Mathematics 74 (New York: Springer-Verlag) (2000)

[Di30] Dickman K, On frequency of numbers containing prime factors of a certain relative magnitude, *Ark. Mat. Astr. Fys.* **22** (1930) 1–14

[FG93] Friedlander J B and Granville A, Smoothing "smooth" numbers, *Philos. Trans. Soc. London* **A345** (1993) 339–347

[FGH07] Farmer D W, Gonek S M and Hughes C P, The maximum size of L-functions, *J. Reine Angew. Math.* **609** (2007) 215–236

[Gra00] Granville A, Smooth Numbers: Computational Number Theory and Beyond, Proc. MSRI Conf. Algorithmic Number Theory: Lattices, Number Fields, Curves and Cryptography (eds) J Buhler and P Stevenhagen (Berkeley: Cambridge University Press) (2000) pp. 1–56

[Har91] Harman G, Short intervals containing numbers without large prime factors, *Math. Proc. Cambridge Philos. Soc.* **109(1)** (1991) 1–5

[Hil84] Hildebrand A, Integers free of large prime factors and the Riemann hypothesis, *Mathematika* **31** (1984) 258–271

[Hil86] Hildebrand A, On the number of positive integers $\leq x$ and free of prime factors $> y$, *J. Number Theory* **22** (1986) 289–307

[HT93] Hildebrand A and Tenenbaum G, Integers without large prime factors, *J. Thor. Nombres Bordeaux* **5(2)** (1993) 411–484

[IK04] Iwaniec H and Kowalski E, Analytic Number Theory American Mathematical Society Colloquium Publications, 53 (Providence, RI: American Mathematical Society) (2004)

[Mo77] Montgomery H L, Extreme values of the Riemann zeta function, *Comm. Math. Helv.* **52** (1977) 511–518

[Mo71] Montgomery H L, Topics in multiplicative number theory, Lecture Notes in Mathematics, vol. 227 (Berlin-New York: Springer-Verlag) (1971)

[Po94] Pomerance C, The role of smooth numbers in number-theoretic algorithms, Proceedings of the International Congress of Mathematicians, vols 1, 2 (Zrich, 1994) pp. 411–422 (Basel: Birkhauser) (1995)

[So10] Soundararajan K, Smooth numbers in short intervals, http://arxiv.org/abs/1009.1591v1

[Ti86] Titchmarsh E C, The theory of the Riemann zeta function, Second edition, edited and with a preface by D R Heath-Brown (New York: The Clarendon Press, Oxford University Press) (1986)

[Xu99] Xuan T Z, On smooth integers in short intervals under the Riemann Hypothesis, *Acta Arith.* **88** (1999) 327–332