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On the complex oscillation of differential polynomials generated by meromorphic solutions of differential equations in the unit disc

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Abstract. In this paper, we investigate the complex oscillation of differential polynomials generated by meromorphic solutions of differential equations

$$f^{(k)} + A(z)f = 0, \quad k \geq 2,$$

where the coefficient A is meromorphic in the unit disc $\mathbb{D} = \{z: |z| < 1\}$.

Keywords. Differential equation; meromorphic function; iterated order; the unit disc.

1. Introduction and main results

Let f be a meromorphic function in the disc $D(R) = \{z: |z| < R\}$, where $0 < R \leq \infty$. For every real number $x \geq 0$, we define $\log^+ x := \max\{0, \log x\}$. Assume that $n(r, f)$ counts the number of poles of f in $|z| \leq r$, each pole according to its multiplicity, and that $\bar{n}(r, f)$ counts the number of distinct poles of f in $|z| \leq r$, ignoring the multiplicity. The characteristic function of f is defined by

$$T(r, f) := m(r, f) + N(r, f),$$

where

$$N(r, f) := \int_0^r \frac{n(t, f) - n(0, f)}{t} \mathrm{d}t + n(0, f) \log r$$

and

$$m(r, f) := \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \mathrm{d}\theta.$$

The notation $\bar{N}(r, f)$ is similarly defined with $\bar{n}(r, f)$ instead of $n(r, f)$. For more notations and definitions of the Nevanlinna's value distribution theory of meromorphic functions, refer to [15, 28].

Let f be a function meromorphic in the unit disc $\mathbb{D} = \{z\colon |z| < 1\}$. We recall some definitions as follows, see [7, 8, 18, 20, 22, 23, 25]. Defining

$$D(f) = \limsup_{r\to 1^-} \frac{T(r, f)}{-\log(1-r)},$$

we say that f is non-admissible, if $D(f) < \infty$, while if $D(f) = \infty$, we say that f is admissible. For $n \in \mathbb{N}$, the iterated n-order of f is defined by

$$\sigma_n(f) = \limsup_{r\to 1^-} \frac{\log_n^+ T(r, f)}{-\log(1-r)},$$

where $\log_1^+ x = \log^+ x$, $\log_{n+1}^+ = \log^+ \log_n^+ x$. The growth index (or finiteness degree) of the order of f is defined by

$$i(f) = \begin{cases} 0, & \text{if } f \text{ is non-admissible,} \\ \min\{n \in \mathbb{N}\colon \sigma_n(f) < \infty\}, & \text{if } f \text{ is admissible and } \sigma_n(f) < \infty \\ & \quad \text{for some } n \in \mathbb{N}, \\ \infty, & \text{if } \sigma_n(f) = \infty \text{ for all } n \in \mathbb{N} \end{cases}.$$

The iterated n-convergence exponent of the sequence of distinct zeros in $\mathbb{D}$ of f is defined by

$$\bar{\lambda}_n(f) = \limsup_{r\to 1^-} \frac{\log_n^+ \bar{N}\left(r, \frac{1}{f}\right)}{-\log(1-r)}.$$

The growth index (or finiteness degree) of the convergence exponent of the sequence of distinct zeros in $\mathbb{D}$ of f is defined by

$$i_{\bar{\lambda}}(f) = \begin{cases} 0 & \text{if } \bar{N}\left(r, \frac{1}{f}\right) = O\left(\log \frac{1}{1-r}\right), \\ \min\{n \in \mathbb{N}\colon \lambda_n(f) < \infty\}, & \text{if some } n \in \mathbb{N} \text{ with } \lambda_n(f) < \infty \text{ exists,} \\ \infty, & \text{if } \lambda_n(f) = \infty \text{ for all } n \in \mathbb{N}. \end{cases}$$

For $a \in \bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the deficiency of f is defined by

$$\delta(a, f) = 1 - \limsup \frac{N\left(r, \frac{1}{f-a}\right)}{T(r, f)},$$

provided f has unbounded characteristic.

Let $\mathcal{L}(G)$ denote a differential subfield of the field $\mathcal{M}(G)$ of meromorphic functions in a domain $G \subset \mathbb{C}$. Throughout this paper, we simply denote $\mathcal{L}$ instead of $\mathcal{L}(\mathbb{D})$. Special case of such differential subfields used below are

$$\mathcal{L}_f := \{g \text{ meromorphic: } T(r, g) = S(r, f)\}$$

and

$$\mathcal{L}_{p+1,\sigma} := \{g \text{ meromorphic: } \sigma_{p+1}(g) < \sigma\},$$

where σ is a positive constant and $S(r, f) = O\left(\log^+\left(\frac{1}{1-r}T(r, f)\right)\right)$ possibly outside a set $E \subset [0, 1)$ with $\int_E \frac{\mathrm{d}r}{1-r} < \infty$.

The complex oscillation theory of solutions of linear differential equations in the complex plane $\mathbb{C}$ was started by Bank and Laine [2, 3]. After their well-known work, many important results have been obtained on the complex oscillation theory of solutions of linear differential equations in $\mathbb{C}$, (see [21, 22]). In recent years, many authors [4–6, 10, 11, 23, 24, 26, 27] investigated the complex oscillation theory of differential polynomials generated by solutions of differential equations in $\mathbb{C}$.

An interesting subject on the problems of complex oscillation theory of differential polynomials generated by solutions of differential equations in the unit disc $\mathbb{D}$ arises naturally. Recently, there has been great interest in studying the growth of solutions of differential equations in $\mathbb{D}$ (see [7, 8, 14, 16–19, 22] and others). In [7], some results on the fixed points of analytic solutions of differential equations in $\mathbb{D}$ were obtained. The number of times that nontrivial solutions of the second order differential equation

$$f'' + A(z)f = 0, \tag{1}$$

in $\mathbb{D}$ can vanish, was investigated in [13]. In [9], the complex oscillation theory of analytic solutions in $\mathbb{D}$ of (1) was discussed. In this paper, we continue to consider this subject and investigate the complex oscillation theory of differential polynomials generated by meromorphic solutions of differential equations in $\mathbb{D}$. Considering (1), we have the following result.

Theorem 1.1. *Let A be an admissible meromorphic function of finite iterated order $\sigma_p(A) := \sigma > 0 (0 < p < \infty)$ in the unit disc $\mathbb{D}$ such that $\delta(\infty, A) > 0$, and let f be a non-zero meromorphic solution of the eq.* (1) *such that $\delta(\infty, f) > 0$. Moreover, let*

$$P[f] = P(f, f', \dots, f^{(\nu)}) = \sum_{j=0}^{\nu} p_j f^{(j)} \tag{2}$$

be a linear differential polynomial with coefficients $p_j \in \mathcal{L}_{p+1,\sigma}$, assuming that at least one of the coefficients p_j does not vanish identically. If $\varphi \in \mathcal{L}_{p+1,\sigma}$ is a non-zero meromorphic function in $\mathbb{D}$, and neither $P[f]$ nor $P[f] - \varphi$ vanishes identically. then we have $i(f) = i_{\bar{\lambda}}(P[f] - \varphi) = p + 1$, and

$$\bar{\lambda}_{p+1}(P[f] - \varphi) = \sigma_{p+1}(f) = \sigma_p(A) = \sigma$$

if $p > 1$, while

$$\sigma = \sigma_p(A) \le \bar{\lambda}_{p+1}(P[f] - \varphi) \le \sigma_{p+1}(f) \le \sigma_p(A) + 1 = \sigma + 1$$

if $p = 1$.

Considering an arbitrary k-order ($k \ge 2$) differential equation

$$f^{(k)} + A(z)f = 0, \tag{3}$$

we obtain another result as follows.

Theorem 1.2. *Let $k \geq 2$ and A be an admissible meromorphic function of finite iterated order $\sigma_p(A) = \sigma > 0$ $(0 < p < \infty)$ such that $\delta(\infty, A) > 0$ in the unit disc $\mathbb{D}$. Assume that $\varphi \in \mathcal{L}_{p+1,\sigma}$ is a non-zero meromorphic function in $\mathbb{D}$, and that*

$$\delta\left(\infty, \frac{\varphi^{(k-j)}}{A}\right) > 0 \quad or \quad \left(\frac{\varphi^{(k-j)}}{A}\right)^{(k)} + \varphi^{(k-j)} \not\equiv 0, \quad j = 0, 1, \ldots, k.$$

Then every non-zero moromorphic solution f, satisfying $\delta(\infty, f) > 0$, of the eq. (3) *satisfies that for $j = 0, 1, \ldots, k$,*

$$i_{\bar{\lambda}}(f^{(j)} - \varphi) = i_{\lambda}(f^{(j)} - \varphi) = i(f^{(j)} - \varphi) = i(f) = p + 1$$

and

$$\bar{\lambda}_{p+1}(f^{(j)} - \varphi) = \sigma_{p+1}(f^{(j)} - \varphi) = \sigma_{p+1}(f) = \sigma_p(A) = \sigma$$

if $p > 1$, while

$$\begin{aligned}\sigma_p(A) \leq \bar{\lambda}_{p+1}(f^{(j)} - \varphi) &= \sigma_{p+1}(f^{(j)} - \varphi)\\ &= \sigma_{p+1}(f) = \sigma_p(A) + 1 = \sigma + 1\end{aligned}$$

if $p = 1$.

The ideas and formulations of Theorems 1.1 and 1.2 are from [23] and [5], respectively, with modification from the complex place $\mathbb{C}$ to the unit disc $\mathbb{D}$. However, the case in $\mathbb{D}$ is more intricate than the case in $\mathbb{C}$. At the end of the section, we may raise a natural question below.

Question. Can we discuss $P[f]$ in Theorem 1.2 and obtain a result which generalizes Theorems 1.1 and 1.2.

2. Some lemmas

For the proofs of our main results, we need the following lemmas.

Lemma 2.1 [16, 25]. *Let f be a meromorphic function in the unit disc, and let $k \in \mathbb{N}$. Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r, f),$$

where $S(r, f) = O(\log^+ T(r, f)) + O\left(\log\left(\frac{1}{1-r}\right)\right)$, possibly outside a set $E \subset [0, 1)$ with $\int_E \frac{\mathrm{d}r}{1-r} < \infty$. If f is of finite order (namely, finite iterated 1-order) of growth, then

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\log\left(\frac{1}{1-r}\right)\right).$$

If f is non-admissible, then

$$m\left(r, \frac{f'}{f}\right) \leq \log\frac{1}{1-r} + (2 + o(1)) \log\log\frac{1}{1-r}.$$

Lemma 2.2 (*Lemma C of* [1]). *Let* $g\colon (0,1) \to \mathbb{R}$ *be monotone increasing functions such that* $g(r) \le h(r)$ *holds outside of an exceptional set* $E \subset [0,1)$ *of finite logarithmic measure. Then there exists* $d \in (0,1)$ *such that if* $s(r) = 1 - d(1-r)$ *then* $g(r) \le h(s(r))$ *for all* $r \in [0,1)$.

Lemma 2.3. *Let* $\Phi(r)$ *be a continuous and positive increasing function, defined for* r *on* $(0,1)$, *with* $\sigma_p(\Phi) = \limsup_{r\to 1^-} \frac{\log_p \Phi(r)}{-\log(1-r)}$. *Then for any subset* E *of* $[0,1)$ *that has finite linear measure, there exists a sequence* $\{r_n\}$ $(r_n \notin E)$ *such that*

$$\sigma_p(\Phi) = \lim_{r_n\to 1^-} \frac{\log_p \Phi(r_n)}{-\log(1-r_n)}.$$

Proof. Since $\sigma_p(\Phi) = \limsup_{r\to 1^-} \frac{\log_p \Phi(r)}{-\log(1-r)}$, there exists a sequence $\{r_n{}'\}(r_n{}' \to 1^-)$ such that

$$\lim_{r_n{}'\to 1^-} \frac{\log_p \Phi(r_n{}')}{-\log(1-r_n{}')} = \sigma_p(\Phi).$$

Set $\int_E \frac{\mathrm{d}r}{1-r} := \log\delta < \infty$. Since $\int_{r_n{}'}^{1-\frac{1-r_n{}'}{\delta+1}} \frac{\mathrm{d}r}{1-r} = \log(\delta+1)$, then there exists $r_n \in \left[r_n{}', 1-\frac{1-r_n{}'}{\delta+1}\right]\setminus E \subset [0,1)$ such that

$$\frac{\log_p^+ \Phi(r_n)}{-\log(1-r_n)} \ge \frac{\log_p^+ \Phi(r_n{}')}{\log(\frac{\delta+1}{1-r_n{}'})} = \frac{\log_p^+ \Phi(r_n{}')}{\log(\delta+1)-\log(1-r_n{}')}.$$

Hence

$$\liminf_{r_n\to 1^-} \frac{\log_p^+ \Phi(r_n)}{-\log(1-r_n)} \ge \lim_{r_n{}'\to 1^-} \frac{\log_p^+ \Phi(r_n{}')}{\log(\delta+1)-\log(1-r_n{}')} = \sigma_p(\Phi).$$

Therefore

$$\lim_{r_n\to 1^-} \frac{\log_p^+ \Phi(r_n)}{-\log(1-r_n)} = \sigma_p(\Phi).$$

■

Lemma 2.4. *Let* $A_0, A_1, \ldots, A_{k-1}$ *and* $F \not\equiv 0$ *be meromorphic functions in* $\mathbb{D}$ *and let* f *be a meromorphic solution of the equation*

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = F(z), \tag{4}$$

such that $i(f) = p+1$ $(0 < p < \infty)$. *If either*

$$\max\{i(F), i(A_j)\colon j = 0,1,\ldots,k-1\} < p+1$$

or

$$\max\{\sigma_{p+1}(F), \sigma_{p+1}(A_j)\colon j = 0,1,\ldots,k-1\} < \sigma_{p+1}(f),$$

then we have $i_{\bar\lambda}(f) = i_\lambda(f) = i(f) = p+1$ *and* $\bar\lambda_{p+1}(f) = \lambda_{p+1}(f) = \sigma_{p+1}(f)$.

Proof. By eq. (4), we have

$$\frac{1}{f} = \frac{1}{F}\left(\frac{f^{(k)}}{f} + A_{k-1}\frac{f^{(k-1)}}{f} + \cdots + A_0\right). \tag{5}$$

If f has a zero at $z_0 \in \mathbb{D}$ of order $\gamma(> k)$ and if $A_0, A_1, \ldots, A_{k-1}$ are all analytic at z_0, then F has a zero at z_0 of order at least $\gamma - k$. Hence we have

$$n\left(r, \frac{1}{f}\right) \le k \cdot \bar{n}\left(r, \frac{1}{f}\right) + n\left(r, \frac{1}{F}\right) + \sum_{j=0}^{k-1} n(r, A_j),$$

$$N\left(r, \frac{1}{f}\right) \le k \cdot \bar{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{F}\right) + \sum_{j=0}^{k-1} N(r, A_j). \tag{6}$$

By Lemma 2.1 and (5), we get that

$$m\left(r, \frac{1}{f}\right) \le m\left(r, \frac{1}{F}\right) + \sum_{j=0}^{k-1} m(r, A_j) + O\left(\log\left(\frac{1}{1-r}T(r, f)\right)\right) \tag{7}$$

holds for all $|z| = r \notin E$, where the linear measure of E is finite. Therefore, by (6), (7) and the first main theorem, there holds

$$\begin{aligned} T(r, f) = T\left(r, \frac{1}{f}\right) + O(1) \le k\bar{N}\left(r, \frac{1}{f}\right) + T(r, F) \\ + \sum_{j=0}^{k-1} T(r, A_j) + O\left\{\log\left(rT(r, f)\right)\right\} \end{aligned} \tag{8}$$

for all $r \to 1^-$, $r \notin E$.

We assume first that $\max\{i(F), i(A_j)\colon j = 0, 1, \ldots, k-1\} < p + 1 = i(f)$, and hence, $\max\{\sigma_p(F), \sigma_p(A_j)\colon j = 0, 1, \ldots, k-1\} < \infty$. Thus there exists a finite constant $\alpha > 0$ such that

$$\max\{T(r, F), T(r, A_j)\colon j = 0, 1, \ldots, k-1\} \le \exp_{p-1}\left(\frac{1}{1-r}\right)^{\alpha}. \tag{9}$$

By Lemma 2.3, for the set E there exists a sequence $\{r_n\}$ ($r_n \notin E$) such that

$$\lim_{r_n \to \infty} \frac{\log_{p+1} T(r_n, f)}{-\log(1 - r_n)} = \sigma_{p+1}(f) := \sigma.$$

Hence, we get that for $r \to 1^-$, $r_n \notin E$, if $\sigma = 0$ then there holds

$$T(r_n, f) \ge \exp_{p-1}\left(\frac{1}{1-r_n}\right)^{\beta} \quad (\beta > \alpha), \tag{10}$$

and if $\sigma > 0$ then there holds

$$T(r_n, f) \ge \exp_p\left(\frac{1}{1-r_n}\right)^{\sigma-\varepsilon}, \tag{11}$$

for any given $\varepsilon (0 < 2\varepsilon < \sigma - b)$, where $b := \max\{\sigma_{p+1}(F), \sigma_{p+1}(A_j)(j = 0, 1, \ldots, k-1)\}$. By (9) and (10) or (11) we get that for $r \to 1^-$, $r_n \notin E_4$ there holds

$$\max\left\{\frac{T(r_n, F)}{T(r_n, f)}, \frac{T(r_n, A_j)}{T(r_n, f)}: j = 0, 1, \ldots, k-1\right\} \to 0, \quad (r_n \to 1^-). \quad (12)$$

Assume that $\max\{\sigma_{p+1}(F), \sigma_{p+1}(A_j)(j = 0, 1, \ldots, k-1)\} = b < \sigma_{p+1}(f)$. We have

$$\max\{T(r_n, F), T(r_n, A_j): j = 0, 1, \ldots, k-1\} \leq \exp_p\left(\frac{1}{1-r_n}\right)^{b+\varepsilon}. \quad (13)$$

Then by (13) and (11) we also have the conclusion (12).

Hence, by (8) and (12) we get that for $r \to 1^-$, $r_n \notin E$ there holds

$$(1 + o(1))T(r_n, f) \leq k\bar{N}\left(r_n, \frac{1}{f}\right).$$

This implies that $i_{\bar{\lambda}}(f) = i_{\lambda}(f) = i(f) = p + 1$ and $\bar{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \sigma_{p+1}(f)$. ■

Lemma 2.5 [12]. *Suppose that* $0 < \rho < r < t < R < 1$ *and that the path* $\Gamma = \Gamma(\theta_0, \rho, t)$ *is given by the segment*

$$\Gamma_1: z = \tau e^{i\theta_0}, \quad \rho \leq \tau \leq t < \frac{1}{4}(3r + R),$$

followed by the circle

$$\Gamma_2: z = te^{i\theta}, \quad \theta_0 \leq \theta \leq \theta_0 + 2\pi.$$

We suppose that f is a meromorphic solution of the equation

$$f^{(k)} + a_{k-1}(z)f^{(k-1)} + \cdots + a_1(z)f' + a_0(z)f = 0, \quad (14)$$

where the coefficients $a_0, a_1, \ldots, a_{k-1}$ *are meromorphic in the unit disc* $\mathbb{D}$. *We also define*

$$\begin{aligned} C &= C(a_n, \rho, r, R) \\ &= (k+2)\exp\left\{\frac{20R}{R-r}\sum_{n=0}^{k-1} T(R, a_n) + \left(\sum_{n=0}^{k-1} p_n\right)\log\left(\frac{R}{\rho}\right)\right\}, \end{aligned}$$

where p_n *is the multiplicity of the pole of* a_n *at the origin if* $a_n(0) = \infty$, *and* $p_n = 0$ *otherwise. If* $\delta = \delta(\infty, f) > 0$ *and* ε *is fixed,* $0 \leq \varepsilon < \delta$, *we have for* $r_1(\varepsilon) < r < R$,

$$T(r, f) \leq \left(\frac{1}{\delta - \varepsilon}\right)(2\pi + 1)RC.$$

Lemma 2.6. *Let* $k \geq 2$ *and* A *be an admissible meromorphic function in* $\mathbb{D}$ *satisfying* $i(A) = p$ $(0 < p < \infty)$ *and* $\delta(\infty, A) > 0$, *and let* f *be a nonzero meromorphic solution of* (3). *If* $\delta(\infty, f) > 0$, *then* $i(f) = p + 1$, *and* $\sigma_{p+1}(f) = \sigma_p(A)$ *if* $p > 1$, *while* $\sigma_p(A) + 1 \geq \sigma_{p+1}(f) \geq \sigma_p(A)$ *if* $p = 1$.

Proof. Assume that $i(f) < p+1$, namely, $\sigma_p(f) := \beta < \infty$. Rewrite (3) as

$$A = -\frac{f^{(k)}}{f}. \tag{15}$$

By Lemma 2.1, there exist a set $E \subset [0, 1)$ with $\int_E \frac{dr}{1-r} < \infty$ such that

$$\begin{aligned} m\left(r, \frac{f^{(k)}}{f}\right) &= O(\log^+ T(r, f)) + O\left(\log\left(\frac{1}{1-r}\right)\right) \\ &= O\left(\exp_{p-2}\left(\frac{1}{1-r}\right)^{\beta+\varepsilon}\right) \end{aligned} \tag{16}$$

for $r \notin E$. Consider the assumption $\delta(\infty, A) := \delta > 0$. Then for $r \to 1^-$ we have

$$m(r, A) \geq \frac{\delta}{2} T(r, A). \tag{17}$$

By (15)–(17), there holds

$$T(r, A) \leq \frac{2}{\delta} m(r, A) = \frac{2}{\delta} m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\exp_{p-2}\left(\frac{1}{1-r}\right)^{\beta+\varepsilon}\right) \tag{18}$$

for $r \to 1^-$, $r \notin E$. Thus by Lemma 2.2 and (18) we have $i(A) < p$. This contradicts the assumption that $i(A) = p$. Hence $i(f) \geq p+1$, namely $\sigma_p(f) = \infty$.

Using a similar discussion as in the above, it is easy to get that $\sigma_p(A) \leq \sigma_{p+1}(f)$.

On the other hand, if $\delta(\infty, f) > 0$, then by Lemma 2.5, for any fixed ε, $0 \leq \varepsilon < \delta_1 := \delta(\infty, f)$ and $r_1(\varepsilon) < r < t < R := \frac{1+r}{2} < 1$,

$$T(r, f) \leq \left(\frac{1}{\delta_1 - \varepsilon}\right)(2\pi + 1)RC \tag{19}$$

holds on the path $\Gamma = \Gamma(\theta_0, \rho, t)$ chosen in accordance with Lemma 2.5, where

$$C = (k+2)\exp\left\{\frac{20R}{R-r} T(R, A) + p_0 \log\left(\frac{R}{\rho}\right)\right\},$$

p_0 is the multiplicity of the pole of A at the origin if $A(0) = \infty$, and $p_0 = 0$ otherwise. By (19) and Lemma 2.2, we immediately get $\sigma_{p+1}(f) \leq \sigma_p(A)$ if $p > 1$, and $\sigma_{p+1}(f) \leq \sigma_p(A) + 1$ if $p = 1$. Hence we obtain that $i(f) = p+1$ and $\sigma_{p+1}(f) = \sigma_p(A)$ if $p > 1$, while $\sigma_p(A) \leq \sigma_{p+1}(f) \leq \sigma_p(A) + 1$ if $p = 1$. ■

3. Proof of Theorem 1.1

By Lemma 2.6, we have $i(f) = p+1$, and $\sigma_{p+1}(f) = \sigma_p(A) = \sigma$ if $p > 1$, while $\sigma = \sigma_p(A) \leq \sigma_{p+1}(f) \leq \sigma_p(A)+1 = \sigma+1$ if $p = 1$. Since $\bar{\lambda}_{p+1}(P[f]-\varphi) \leq \sigma_{p+1}(f)$, we may assume that

$$\bar{\lambda}_{p+1}(P[f] - \varphi) < \sigma_p(A) = \sigma. \tag{20}$$

Obviously, $A \in \mathcal{L}_{p+1,\sigma}$. We may assume that $\nu \leq 1$. Indeed, if $\nu \geq 2$, then by repeated differentiation of (1) we obtain that $f^{(k)} = q_{k,0}f + q_{k,1}f'$, $q_{k,0}, q_{k,1} \in \mathcal{L}_{p+1,\sigma}$ for $k = 2, 3, \ldots, \nu$. Substituting into the form of $P[f]$ yields the required reduction. Hence, we may assume, from now on, that $P[f] = p_0 f + p_1 f'$, where at least one of the coefficients $p_0, p_1 \in \mathcal{L}_{p+1,\sigma}$ does not vanish identically.

Since

$$T\left(r, \frac{(P[f]-\varphi)'}{P[f]-\varphi}\right) = m\left(r, \frac{(P[f]-\varphi)'}{P[f]-\varphi}\right) + \bar{N}(r, P[f]-\varphi) + \bar{N}\left(\frac{1}{P[f]-\varphi}\right) \tag{21}$$

and $P[f] - \varphi$ may have poles at the poles of A, p_0, p_1 and φ only, (20) and (21) imply that for some $\beta < \sigma$ and $r \to 1^-$, there holds

$$T\left(r, \frac{(P[f]-\varphi)'}{P[f]-\varphi}\right) = O\left(\exp_p\left(\frac{1}{1-r}\right)^{\beta}\right).$$

Hence, there exists $h \in \mathcal{L}_{p+1,\sigma}$ such that

$$(P[f]-\varphi)' = h(P[f]-\varphi). \tag{22}$$

Using the fact that $f'' = -Af$, we may rewrite (1) as

$$b_1 f' + b_0 f + h\varphi - \varphi' = 0, \tag{23}$$

where $b_1 = p_0 + p_1{}' - hp_1$ and $b_0 = p_0{}' - p_1 A - hp_0$.

We fist assume that $b_1(z) \equiv 0$ and $b_0(z) \not\equiv 0$. Then $f = \frac{\varphi' - h\varphi}{b_0}$. Hence, $f \in \mathcal{L}_{p+1,\sigma}$ and so $\sigma_{p+1}(f) < \sigma$, a contradiction.

Assume that $b_0(z) \equiv 0$ and $b_1(z) \not\equiv 0$. Then $f' = \frac{\varphi' - h\varphi}{b_1}$. Hence, $f' \in \mathcal{L}_{p+1,\sigma}$ and so $\sigma_{p+1}(f) = \sigma_{p+1}(f') < \sigma$, also a contradiction.

Assume that $b_0(z) \equiv 0$ and $b_1(z) \equiv 0$. Then we have $h = \frac{\varphi'}{\varphi}$ because of $\varphi(z) \not\equiv 0$. Hence,

$$b_0 = p_0{}' - p_1 A - \frac{\varphi' p_0}{\varphi} = 0 \tag{24}$$

and

$$b_1 = p_0 + p_1{}' - \frac{\varphi' p_1}{\varphi} = 0. \tag{25}$$

hold. By (24) and (25) we get

$$A = -\frac{p_1{}''}{p_1} + \frac{\varphi''}{\varphi} + 2\frac{\varphi'}{\varphi}\frac{p_1{}'}{p_1} - 2\left(\frac{\varphi'}{\varphi}\right)^2. \tag{26}$$

By the assumption that $\delta := \delta(\infty, A) > 0$, we deduce from (26) that

$$T(r, A) \leq \frac{2}{\delta} m(r, A) = O\left(\log\left(\frac{1}{1-r}T(r, p_1)\right) + \log\left(\frac{1}{1-r}T(r, \varphi)\right)\right) = O\left(\exp_{p-1}\left(\frac{1}{1-r}\right)^{\beta}\right) \tag{27}$$

for some $\beta < \sigma$ outside a possible exceptional set of finite linear measures. Hence, we have $\sigma_p(A) \le \beta < \sigma$, a contradiction.

Therefore, we may now assume that neither b_0 nor b_1 vanishes identically. Rewrite eq. (23) as

$$b_0 f + b_1 f' = \varphi' - h\varphi. \tag{28}$$

Differentiating eq. (28) and making use of $f'' = -Af$, we have

$$(b_0{}' - b_1 A)f + (b_0 + b_1{}')f' = (\varphi' - h\varphi)'. \tag{29}$$

If the pair of eqs (28) and (29) determine that f and f' have a nonidentically vanishing determinant, and we must have

$$(b_0^2 + b_0 b_1{}' - b_1 b_0{}' + b_1^2 A)f = -(\varphi' - h\varphi)(b_0 + b_1{}') + (\varphi' - h\varphi)' b_1. \tag{30}$$

Hence, we have $f \in \mathcal{L}_{p+1,\sigma}$, and thus $\sigma_{p+1}(f) < \sigma$, a contradiction. Hence, the determinant vanishes, and thus we have

$$b_0^2 + b_0 b_1{}' - b_1 b_0{}' + b_1^2 A = 0 \tag{31}$$

and

$$-(\varphi' - h\varphi)(b_0 + b_1{}') + (\varphi' - h\varphi)' b_1 = 0. \tag{32}$$

If now $\varphi'(z) - h(z)\varphi(z) \not\equiv 0$, then by a simple computation we deduce from (32) and (31) that

$$\frac{b_0}{b_1} = \frac{((\varphi' - h\varphi)/b_1)'}{(\varphi' - h\varphi)/b_1}$$

and

$$A = \left(\frac{b_0}{b_1}\right)' - \left(\frac{b_0}{b_1}\right)^2$$

hold. Again by making use of the assumption $\delta := \delta(\infty, A) > 0$, we obtain

$$\begin{aligned} T(r, A) \le \frac{2}{\delta} m(r, A) &= O\left(\log\left(\frac{1}{1-r} T\left(r, \frac{\varphi' - h\varphi}{b_1}\right)\right)\right) \\ &= O\left(\exp_{p-1}\left(\frac{1}{1-r}\right)^{\beta}\right) \end{aligned}$$

for some $\beta < \sigma$, and thus $\sigma_p(A) \le \beta < \sigma$, a contradiction. Therefore, we must have $\varphi'(z) - h(z)\varphi(z) \equiv 0$, and thus $h = \frac{\varphi'}{\varphi}$. Integrating (22) we have

$$P[f] = p_0 f + p_1 f' = C\varphi, \tag{33}$$

where $C \ne 0, 1$ by assumption, while eq. (28) reduces to

$$b_0 f + b_1 f' = 0. \tag{34}$$

As the determinant of the pair (33) and (34) obviously has to be nonzero, we obtain $f = \frac{C\varphi}{p_0 b_1 - b_0 p_1}$. We also obtain $f \in \mathcal{L}_{p+1,\sigma}$, and thus $\sigma_{p+1}(f) < \sigma$, a contradiction.

Therefore, we have $i(f) = i_{\bar{\lambda}}(P[f] - \varphi) = p + 1$ and

$$\bar{\lambda}_{p+1}(P[f] - \varphi) \leq \sigma_{p+1}(f) = \sigma_p(A) = \sigma$$

if $p > 1$, while

$$\sigma_p(A) \leq \bar{\lambda}_{p+1}(P[f] - \varphi) \leq \sigma_{p+1}(f) \leq \sigma_p(A) + 1 = \sigma + 1$$

if $p = 1$.

4. Proof of Theorem 1.2

Suppose that $f(z) \not\equiv 0$ is a meromorphic solution of eq. (3). Set $w_j = f^{(j)} - \varphi$ ($j = 0, 1, \ldots, k$), where $\varphi \in \mathcal{L}_{p+1,\sigma}$. Then for $j = 0, 1, \ldots, k$, we deduce by Lemma 2.6 that $i(w_j) = i(f) = p + 1$ and $\sigma_{p+1}(w_j) = \sigma_{p+1}(f) = \sigma_p(A) = \sigma$ if $p > 1$, while $\sigma = \sigma_p(A) \leq \sigma_{p+1}(w_j) = \sigma_{p+1}(f) \leq \sigma_p(A) + 1 = \sigma + 1$ if $p = 1$. Differentiating both sides of $w_j = f^{(j)} - \varphi$ and making use of $f^{(k)} = -Af$, we obtain that

$$w_j^{(k-j)} = -Af - \varphi^{(k-j)}, \quad j = 0, 1, \ldots, k.$$

Thus we have

$$f = -\frac{w_j^{(k-j)} + \varphi^{(k-j)}}{A}. \tag{35}$$

Combining (3) and (35) we obtain

$$\left(\frac{w_j^{(k-j)}}{A}\right)^{(k)} + w_j^{(k-j)} = -\left(\left(\frac{\varphi^{(k-j)}}{A}\right)^{(k)} + \varphi^{(k-j)}\right),$$

and thus

$$\begin{aligned} &w_j^{(2k-j)} + g_{2k-j-1} w_j^{(2k-j-1)} + \cdots + g_{k-j} w_j^{(k-j)} \\ &\quad = -A\left(\left(\frac{\varphi^{(k-j)}}{A}\right)^{(k)} + A\left(\frac{\varphi^{(k-j)}}{A}\right)\right), \end{aligned} \tag{36}$$

where $g_{k-j}, \ldots, g_{2k-j-1} \in \mathcal{L}_{p+1,\sigma}$ ($j = 0, 1, \ldots, k$) are meromorphic functions in $\mathbb{D}$.

By our assumption we get that

$$F := A\left(\frac{\varphi^{(k-j)}}{A}\right)^{(k)} + A\left(\frac{\varphi^{(k-j)}}{A}\right) \not\equiv 0.$$

Obviously, there holds either

$$\max\{i(g_{k-j}), \ldots, i(g_{2k-j-1}), i(F)\} < p + 1$$

or

$$\max\{\sigma_{p+1}(g_{k-j}), \ldots, \sigma_{p+1}(g_{2k-j-1}), \sigma_{p+1}(F)\} < \sigma \leq \sigma_{p+1}(w_j)$$

for $j = 0, 1, \ldots, k$. By Lemma 2.4 we have

$$i_{\bar{\lambda}}(w_j) = i_{\lambda}(w_j) = i(w_j) = p + 1 \quad \text{and} \quad \bar{\lambda}_{p+1}(w_j) = \sigma_{p+1}(w_j),$$

where $j = 0, 1, \ldots, k$. Hence, for $j = 0, 1, \ldots, k$, we obtain our assertion that

$$i_{\bar{\lambda}}(f^{(j)} - \varphi) = i_{\lambda}(f^{(j)} - \varphi) = i(f^{(j)} - \varphi) = i(f) = p + 1$$

and

$$\bar{\lambda}_{p+1}(f^{(j)} - \varphi) = \sigma_{p+1}(f^{(j)} - \varphi) = \sigma_{p+1}(f) = \sigma_p(A) = \sigma$$

if $p > 1$, while

$$\sigma = \sigma_p(A) \leq \bar{\lambda}_{p+1}(f^{(j)} - \varphi) = \sigma_{p+1}(f^{(j)} - \varphi)$$
$$= \sigma_{p+1}(f) = \sigma_p(A) + 1 = \sigma + 1$$

if $p = 1$.

Note that there holds $\frac{\varphi^{(k-j)}}{A} \in \mathcal{L}_{p+1,\sigma}$. If $\delta\big(\infty, \frac{\varphi^{(k-j)}}{A}\big) > 0$, $j = 0, 1, \ldots, k$, then by Lemma 2.6 we get $F \not\equiv 0$. Using a similar discussion as the above, our assertion is also true.

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