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Geometric structures on loop and path spaces

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Abstract. The loop space associated to a Riemannian manifold admits a quasi-symplectic structure (that is, a closed 2-form which is non-degenerate up to a finite-dimensional kernel). We show how to construct a compatible almost-complex structure. Finally conditions to have contact structures on loop spaces are studied.

Keywords. Loop space; symplectic structures; contact structures.

1. Introduction

Let M be a smooth manifold, and consider the loop space $\mathcal{L}(M)$ consisting of $\mathcal{C}^\infty$ loops in M. This is naturally a Fréchet manifold. The tangent space to $\mathcal{L}(M)$ at a loop γ is $T_\gamma\mathcal{L}(M) \cong \Gamma(S^1, \gamma^*TM)$. The loop space $\mathcal{L}(M)$ is equipped with a natural section of its tangent bundle defined as

$$\begin{aligned} \xi\colon \mathcal{L}(M) &\to T\mathcal{L}(M) \\ \gamma &\mapsto \gamma'. \end{aligned}$$

Whenever we fix a Riemannian metric g on M, we can define an associated weak metric on the space of loops as

$$(g_{\mathcal{L}})_\gamma(X, Y) = \int_0^1 g(X(t), Y(t))\mathrm{d}t,$$

for $X, Y \in T_\gamma\mathcal{L}(M)$. Recall that a weak metric is a symmetric non-degenerate bilinear form which makes the space a pre-Hilbert space, and such that it extends to the completion giving rise to a (topological) isomorphism between the space and its dual. In our case, $g_{\mathcal{L}}$ gives the L^2-norm on $T_\gamma\mathcal{L}(M)$.

The section α and the metric $g_{\mathcal{L}}$ allow us to define a 1-form

$$\mu(X) = \frac{1}{2}\int_0^1 g(X(t), \gamma'(t))\mathrm{d}t, \tag{1}$$

whose exterior differential will be denoted as $\omega = \mathrm{d}\mu$. The forms μ and ω are known as the Atiyah forms of the loop space $\mathcal{L}(M)$ (see [At84] and [Se88]).

DEFINITION 1.1

A 2-form β on a pre-Hilbert space is weakly symplectic if it is non-degenerate and closed (but note that it may not produce an isomorphism between the Hilbert space obtained after completion and its dual).

A 2-form is quasi-symplectic if it has a finite-dimensional kernel and it is weakly symplectic on the orthogonal of the kernel.

In our case, we shall see that ω is quasi-symplectic with kernel

$$\ker(\omega_\gamma) = \{X \in \Gamma(S^1, \gamma^* TM); \nabla_{\gamma'} X = 0\}. \tag{2}$$

This quasi-symplectic structure can be enriched in many cases. This is well-known in the case of based loop groups (i.e., M is a Lie group). In this case it is possible to define an integrable complex structure making a finite codimensional closed manifold of a loop group into a Kähler manifold.

Now consider the path space $\mathcal{P}(M)$ consisting of $\mathcal{C}^\infty$-maps $\gamma: [0,1] \to M$. We again have the canonical section of the tangent bundle given by

$$\begin{aligned} \xi: \mathcal{P}(M) &\to T\mathcal{P}(M) \\ \gamma &\mapsto \gamma'. \end{aligned}$$

As in the case of the loop space, we will easily check that eq. (1) yields a 1-form whose differential is symplectic. Therefore we have the following:

PROPOSITION 1.2

The 2-form $\omega = \mathrm{d}\mu$ in $\mathcal{P}(M)$ induces a weakly symplectic structure. Moreover $\mathcal{L}(M)$ is a closed quasi-symplectic submanifold of $(\mathcal{P}(M), \omega)$.

Consider now

$$\mathcal{L}_p(M) = \{\gamma \in \mathcal{L}(M); \gamma(0) = p\}.$$

Then $(\mathcal{L}_p(M), \omega)$ is a symplectic submanifold of $\mathcal{P}(M)$. We shall show how to construct a weak almost complex structure J on $\mathcal{L}_p(M)$. This is a map from $T_\gamma \mathcal{L}_p(M)$ into its Hilbert completion, such that $J^2 = -\mathrm{Id}$.

PROPOSITION 1.3

$\mathcal{L}_p(M)$ has a weak almost complex structure J compatible with ω, that is $\omega(\cdot, J\cdot)$ is a weak metric on $\mathcal{L}_p(M)$.

Finally we discuss how to find contact hypersurfaces in loop spaces. We define

DEFINITION 1.4

A non-vanishing 1-form δ on a pre-Hilbert space is weakly contact if its exterior differential $\mathrm{d}\delta$ is non-degenerate when restricted to the kernel of δ (note that it may not produce an isomorphism between the Hilbert space obtained after completion of $\ker\delta$ and its dual).

A 1-form δ is quasi-contact if the restriction of its differential $\mathrm{d}\delta$ to the kernel $\ker\delta$ has finite dimensional kernel.

We recall that a vector field X in a Riemannian manifold (M, g) is locally gradient-like if the 1-form $X^\#$ is closed. We mean by the symbol # the operation of raising the index by means of the metric. This is equivalent to be the gradient for a local function in the neighborhood of any point.

The most natural construction is given by

Theorem 1.5. *Assume that the Riemannian manifold (M, g) admits a vector field X which satisfies $L_X g = g$ and is locally gradient-like, then the lift $\hat{X}$ of X to $\mathcal{L}(M)$ is a Liouville vector field for the weakly symplectic form ω. Moreover the lift $\hat{X}$ is transverse to the level sets of the length functional.*

A vector field is called Liouville for a symplectic form ω if it satisfies

$$L_X\omega = \omega.$$

As in the finite dimensional case, the existence of a Liouville vector field transverse to a hypersurface provides a contact form. Define

$$\mathcal{L}_1(M) = \{\gamma \in \mathcal{L}(M); \mathrm{length}(\gamma) = 1\}$$

and

$$\mathcal{L}_{p,1}(M) = \{\gamma \in \mathcal{L}_p(M); \mathrm{length}(\gamma) = 1\}.$$

They are smooth hypersurfaces of the manifolds $\mathcal{L}(M)$ and $\mathcal{L}_p(M)$ respectively. We have

COROLLARY 1.6

The form $\alpha = i_{\hat{X}}\omega$ is a quasi-contact form in $\mathcal{L}_1(M)$. The restriction of the form α to $\mathcal{L}_{p,1}$ is a weakly contact form in that space.

We will show in particular that stabilizing the manifold M, i.e. considering $M \times \mathbb{R}$, we obtain contact hypersurfaces in the loop space.

2. Symplectic structure

In this section we shall prove Theorem 1.2. The quasi-symplectic structure in the space of loops of a Riemannian manifold is defined by taking the exterior differential of the 1-form μ given by eq. (1). To do that we recall the formula

$$\mathrm{d}\alpha(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y]), \tag{3}$$

which is valid for any 1-form α and it does not depend on the vector fields X, Y chosen to extend $X(\gamma)$ and $Y(\gamma)$ for a given point (loop) $\gamma \in \mathcal{L}(M)$. In our case we start with two vectors $U, V \in \Gamma(S^1, \gamma^* TM) \simeq T_\gamma\mathcal{L}(M)$. First define

$$\theta\colon (-\varepsilon, \varepsilon)^2 \times S^1 \to M,$$

satisfying:

(i) $\theta(0,0,t)=\gamma(t)$,
(ii) $\frac{\partial\theta}{\partial u}(0,0,t)=U(t)$,
(iii) $\frac{\partial\theta}{\partial v}(0,0,t)=V(t)$.

Define $\gamma'=\frac{\partial\theta}{\partial t}$, $\hat{U}=\frac{\partial\theta}{\partial u}$ and $\hat{V}=\frac{\partial\theta}{\partial v}$. They clearly satisfy

$$[\hat{U},\hat{V}]=0, \tag{4}$$

since they are derivatives of the coordinates of a parametrization. This allows us to compute

$$\begin{aligned}\hat{U}(\mu(\hat{V})) &= \frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{1}{2}\int_0^1 g_{\theta(u,0,t)}\left(\frac{\partial\theta}{\partial v}(u,0,t),\frac{\partial\theta}{\partial t}(u,0,t)\right)\mathrm{d}t\right)\\ &= \frac{1}{2}\int_0^1 (g_{\theta(0,0,t)}(\nabla_{\hat{U}}\hat{V},\gamma'(t))+g_{\theta(0,0,t)}(V,\nabla_{\hat{U}}\gamma'))\mathrm{d}t. \end{aligned}\tag{5}$$

In the same way, we obtain

$$\hat{V}(\mu(\hat{U}))=\frac{1}{2}\int_0^1 (g_{\theta(0,0,t)}(\nabla_{\hat{V}}\hat{U},\gamma'(t))+g_{\theta(0,0,t)}(U,\nabla_{\hat{V}}\gamma'))\mathrm{d}t.$$

We are using the torsion-free Levi-Civita connection for the computations, so $\nabla_{\gamma'}\hat{U}=\nabla_{\hat{U}}\gamma'$ and $\nabla_{\gamma'}\hat{V}=\nabla_{\hat{V}}\gamma'$. Also $\nabla_{\hat{U}}\hat{V}-\nabla_{\hat{V}}\hat{U}=[\hat{U},\hat{V}]=0$. We shall use the notation $\nabla_{\gamma'}U=\frac{\partial U}{\partial t}$. So we have, by applying the formula (3), that

$$\begin{aligned}\omega(U,V) &= \omega(\hat{U},\hat{V})=\mathrm{d}\mu(\hat{U},\hat{V})=\hat{U}(\mu(\hat{V}))-\hat{V}(\mu(\hat{U}))\\ &= \frac{1}{2}\int_0^1\left(g_{\gamma(t)}\left(V,\frac{\partial U}{\partial t}\right)-g_{\gamma(t)}\left(U,\frac{\partial V}{\partial t}\right)\right)\mathrm{d}t.\end{aligned}\tag{6}$$

Moreover we have

$$0=\int_0^1\left(\frac{\mathrm{d}}{\mathrm{d}t}g(U,V)\right)\mathrm{d}t=\int_0^1\left(g\left(\frac{\partial U}{\partial t},V\right)+g\left(U,\frac{\partial V}{\partial t}\right)\right)\mathrm{d}t, \tag{7}$$

which implies

$$\omega(U,V)=\int_0^1 g\left(\frac{\partial U}{\partial t},V\right)\mathrm{d}t. \tag{8}$$

Now the kernel of this 2-form at a point γ is given by the parallel vector fields along γ. Therefore $\dim\ker(\omega_\gamma)\leq n$.

There are several ways of removing the kernel of ω. The simplest one is to fix a point $p\in M$ and to define

$$\mathcal{L}_p(M)=\{\gamma\in\mathcal{L}(M);\gamma(0)=p\}.$$

This forces the tangent vectors to satisfy

$$X \in T_\gamma \mathcal{L}_p(M) \Rightarrow X \in \Gamma(S^1, \gamma^* TM), X(0) = 0.$$

Therefore any parallel vector field is null. So the manifold $\mathcal{L}_p(M)$ is (weakly) symplectic.

We shall take a second route. Extend our space to

$$\mathcal{P}(M) = \{\gamma : [0, 1] \to M\},$$

where it is still possible to repeat all the previous computations. We highlight the differences. Equation (5) is exactly the same as it is symmetric. Equation (6) remains without changes. We just need to rewrite eq. (7) which is not true anymore and so the final expression for the exterior differential of μ becomes

$$\begin{aligned} \mathrm{d}\mu(U, V) = \omega(U, V) &= \int_0^1 \left(g\left(\frac{\partial U}{\partial t}, V\right) - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} g(U, V)\right) \mathrm{d}t \\ &= \int_0^1 g\left(\frac{\partial U}{\partial t}, V\right) \mathrm{d}t - \frac{g(U(1), V(1)) - g(U(0), V(0))}{2}. \end{aligned}$$

It is obviously a closed (being exact) form. Let us compute its kernel. Assume that $X \in \ker(\omega_\gamma)$. Considering $\omega(X, V) = 0$ for all vectors $V \in T_\gamma \mathcal{P}(M)$ with $V(0) = V(1) = 0$, we obtain that

$$\frac{\partial X}{\partial t} = 0.$$

Now by choosing all $V \in T_\gamma \mathcal{P}(M)$ with $V(0) \neq 0$ and $V(1) = 0$, we conclude that $X(0) = 0$. By parallel transport, $X = 0$ and so the kernel of ω is trivial. Hence this form is (weakly) symplectic. This proves Proposition 1.2.

Remark 2.1. The Riemannian metric determines the symplectic form ω. But conversely, the form ω determines the Riemannian metric of M as follows: consider M embedded in $\mathcal{L}(M)$ as the space of constant loops. Given $p \in M$, consider the constant loop $\gamma(t) = p$, for all $t \in S^1$. Given $v \in T_p M$ and $f(t)$, a smooth real-valued function on S^1 with $f(0) = 0$, write $X_{v,f} \in T_\gamma \mathcal{L}(M)$ for the vector field defined as $X_{v,f}(t) = f(t)v$. Then for $v_1, v_2 \in T_p M$, one has

$$\omega_\gamma(X_{v_1,f_1}, X_{v_2,f_2}) = \langle v_1, v_2 \rangle \int_0^1 f_1'(t) f_2(t) \mathrm{d}t.$$

Therefore we can recover the metric of M out of ω.

3. Almost complex structures

There is a canonical almost-complex structure compatible with ω in $\mathcal{L}_p(M)$. Let us construct it. Given a curve $\gamma : [0, 1] \to M$, denote P_s^t the parallel transport isometry along γ. There is an isometric isomorphism between $\gamma^* TM$ and the trivial $T_{\gamma(0)} M$ bundle over I with constant metric $g_{\gamma(0)}$. This allows to translate any section $U(t) \in \gamma^* TM$ to a section $P_t^0(U(t)) = \hat{U}(t) \in T_{\gamma(0)} M$. This gives rise to a 'développement' map

$$T_\gamma \mathcal{L}_p(M) \to \bar{\mathcal{L}}_0(T_{\gamma(0)} M),$$

where $\bar{\mathcal{L}}_0(T_{\gamma(0)}M)$ is the space of loops based at 0 in the tangent space $T_\gamma(0)M$ such that they are C^∞ in (0, 1) and continuous at the origin. Realize that the derivatives, though bounded, are not continuous in general at the point 0. This is a continuous injective linear map. Moreover, realize that the map extends to an isomorphism between the completions of the spaces $T_\gamma\mathcal{L}_p(M)$ and $\bar{\mathcal{L}}_0(T_{\gamma(0)}M)$.

Note that if we apply this map to $\gamma'(t)$ itself, we get a curve $x(t) \in T_{\gamma(0)}M$. Now we define

$$a(t) = \int_0^t x(s)\mathrm{d}s,$$

which is known as the 'développement de Cartan' of the curve γ in the tangent space $T_{\gamma(0)}M$. As the covariant derivative along γ becomes the ordinary derivative in $T_{\gamma(0)}M$, we have that γ is a geodesic just when its développement de Cartan is a line.

Define a weak almost complex structure $\hat{J}$ in $T_\gamma\mathcal{L}_p(M)$ as follows: take any vector field $U \in \Gamma_0(S^1, \gamma^*TM)$ and compute its 'développement' $\hat{U}$, which obviously satisfies $\hat{U}(0) = \hat{U}(1) = 0$, since $U(t) \in T_\gamma\mathcal{L}_p(M)$. Fixing an isomorphism $T_{\gamma(0)}M \cong \mathbb{R}^n$, we have

$$\hat{U}(t)\colon [0, 1] \to \mathbb{R}^n,$$

which is $\mathcal{C}^\infty$ in (0, 1), continuous in [0, 1] and $\hat{U}(0) = \hat{U}(1) = 0$. Take its Fourier series expansion,

$$\hat{U}(t) = \sum_{k=-\infty}^{\infty} a_k \mathrm{e}^{2\pi ikt},$$

where $a_k \in \mathbb{C}^n$ and $a_{-k} = \bar{a}_k$. Then define

$$\tilde{J}(\hat{U})(t) = \sum_{k<0}(-ia_k)\mathrm{e}^{2\pi ikt} + a_0 + \sum_{k>0} ia_k\mathrm{e}^{2\pi ikt}. \tag{9}$$

We subtract the constant vector $\tilde{J}(\hat{U})(0)$ to get the almost-complex structure. So we have

$$\hat{J}(\hat{U})(t) = \tilde{J}(\hat{U})(t) - \tilde{J}(\hat{U})(0) \in \bar{\mathcal{L}}_0(T_{\gamma(0)}M).$$

To check that it is an almost complex structure we compute

$$\begin{aligned}\hat{J}\hat{J}(\hat{U}) &= \hat{J}(\tilde{J}(\hat{U}) - \tilde{J}(\hat{U})(0))\\ &= \tilde{J}\tilde{J}(\hat{U}) - \tilde{J}(\hat{U})(0) - \tilde{J}\tilde{J}(\hat{U})(0) + \tilde{J}(\hat{U})(0)\\ &= \tilde{J}\tilde{J}(\hat{U}) - \tilde{J}\tilde{J}(\hat{U})(0)\\ &= (-\hat{U} + 2a_0) - (-\hat{U}(0) + 2a_0) = -\hat{U}.\end{aligned}$$

It is important to realize that the map naturally has its image in the completion of $\bar{\mathcal{L}}_0(T_{\gamma(0)}M)$, and in fact it does not preserve the space without compactifying it. So we obtain a *weak* almost complex structure on $\bar{\mathcal{L}}_0(T_{\gamma(0)}M)$. This means a map from a pre-Hilbert space into its completion, which is a linear and bounded map satisfying that the square is minus the identity (note that J does not extend to the Hilbert completion of $\bar{\mathcal{L}}_0(T_{\gamma(0)}M)$). Finally, using the 'développement' we have a weak almost complex structure in $T_\gamma\mathcal{L}_p(M)$, that is, in $\mathcal{L}_p(M)$.

To check that $\hat{J}$ is compatible with the symplectic form ω we just compute the value of ω when trivialized in the 'développement' to obtain

$$\omega(\hat{U}, \hat{V}) = \omega\left(\sum_p a_p e^{2\pi i p t}, \sum_q b_q e^{2\pi i q t}\right) = \sum_k 2\pi i k \langle a_k, b_k\rangle, \tag{10}$$

where $a_k, b_k \in \mathbb{C}^n$ and $\langle\,,\,\rangle$ is the standard Hermitian product in $\mathbb{C}^n$. The associated metric

$$g(\hat{U}, \hat{V}) = \omega(\hat{U}, J\hat{V}) = \sum_{k>0} 2\pi k \operatorname{Re}(\langle a_k, b_k\rangle + \langle a_{-k}, b_{-k}\rangle)$$

is clearly Riemannian. Moreover, $\hat{J}$ is smooth, meaning that at each tangent space $T_\gamma\mathcal{L}_p(M)$, the map $\hat{J}$ between the respective completions is a linear bounded map.

Moreover, $\hat{J}$ also depends smoothly on γ. Fix $\gamma \in \mathcal{L}_p(M)$ and consider an open neighbourhood as follows: Let $B \subset \Gamma_0([0,1], \gamma^*TM)$ a small ball (in the Fréchet topology), and parametrize a neighbourhood U of γ by means of the exponential map:

$$X(t) \mapsto \gamma_X(t) = \exp_{\gamma(t)}(X(t)).$$

Now we trivialize the tangent bundle over U by means of the 'développement': $T_{\gamma_X}\mathcal{L}_p(M) \cong \bar{\mathcal{L}}_0(T_pM)$, which produces a chart $TU \cong U \times \bar{\mathcal{L}}_0(T_pM)$. With respect to this chart, $\hat{J}$ is a constant operator, hence smooth. This corrects the folklore statement saying that this almost complex structure is not smooth in general (see page 355 of [Wu95]).

The case of Lie groups

In the case in which G is a Lie group, there is an alternative way of defining an almost complex structure for the space of loops based at the neutral element $e \in G$. To do it we just use the left multiplication to take the tangent space $T_\gamma\mathcal{L}_e(G)$ to $\Gamma_0(S^1, T_eG)$ [PS86], so we obtain an isomorphism

$$T_\gamma\mathcal{L}_e(G) \cong \Gamma_0(S^1, \mathbb{R}^n) = \{f \in \mathcal{C}^\infty(S^1, \mathbb{R}^n);\ f(0) = 0\},$$

preserving the metric by construction. So every particular vector field $X \in T_\gamma\mathcal{L}_e(G)$ is transformed via the isomorphism to a loop in $\mathbb{R}^n$. Recall that the isomorphism does not coincide with the one induced by the 'développement' unless the group is flat (an abelian group). Once we have set up the previous identification, the formula (9) provides again an almost complex structure. We remark that it does not coincide with the previous one in the cases when both are defined.

This almost complex structure is compatible with the metric. Moreover, it is smooth. To check it, recall that the map

$$\begin{aligned}\mathcal{F}\colon \Gamma_0(S^1, \mathbb{R}^n) &\to \mathcal{S} \subset (\mathbb{C}^n)^\infty\\ f &\mapsto (a_1, a_2, a_3, \ldots),\end{aligned}$$

where $\mathcal{S}$ is the Schwartz space of sequences of vectors in $\mathbb{C}^n$ with decay faster than polynomial, and $\{a_k\}$ are the Fourier coefficients of f, is a topological isomorphism (we take in $\mathcal{S}$ the Fréchet structure given by the norms $\|(a_k)\|_t = \sum k^t |a_k|$). (Note that the Fourier coefficients satisfy $a_{-k} = \bar{a}_k$ and $a_0 = -\sum_{k\neq 0} a_k$.) The map $\hat{J}$ is conjugated under $\mathcal{F}$ to the map

$$\begin{aligned} \mathcal{J} \colon \mathcal{S} &\to \mathcal{S}, \\ (a_1, a_2, \dots) &\mapsto (ia_1, ia_2, \dots), \end{aligned}$$

which is smooth. So $\hat{J}$ is smooth.

4. Contact structures

We want to check whether the symplectic manifold $\mathcal{L}_p(M)$ has hypersurfaces of contact type on it. We now prove Theorem 1.5.

Proof of Theorem 1.5. Let X be a vector field on M satisfying $L_X g = g$. Then $\nabla X \in \Gamma(M, \mathrm{End}(TM))$, and its symmetrization is $\frac{1}{2}$ Id. This follows since, for Y, Z vector fields on M, we have

$$\begin{aligned} &g(\nabla_Z X, Y) + g(\nabla_Y X, Z) \\ &\quad = g(\nabla_X Z, Y) + g(\nabla_X Y, Z) - g(L_X Z, Y) - g(L_X Y, Z) \\ &\quad = X(g(Y, Z)) + (L_X g)(Y, Z) - L_X(g(Y, Z)) \\ &\quad = g(Y, Z), \end{aligned}$$

where we have used that $L_X Z = \nabla_X Z - \nabla_Z X$ on the second line. The anti-symmetrization of ∇X is $\mathcal{A}(\nabla X) = \mathcal{A}(\nabla X^\#)_\# = (\mathrm{d}X^\#)_\#$, where $X^\#$ is the 1-form associated to X ('raising the index'), and the $(\cdot)_\#$ means 'lowering the index' with the metric. Recall that a vector field is 'locally gradient-like' in a neighborhood U for a metric g if it is g-dual of some exact 1-form $\mathrm{d}f$, where f is a function $f \colon U \to \mathbb{R}$. Thus, if X is locally gradient-like, then $X^\#$ is a locally exact, i.e. closed, 1-form and so $\mathcal{A}(\nabla X) = 0$. Then $\nabla X = \frac{1}{2}$ Id.

Associated to X there is an induced vector field $\hat{X}$ on $\mathcal{L}(M)$. It is defined as follows: for $\gamma \in \mathcal{L}(M)$, $\hat{X}_\gamma \in T_\gamma \mathcal{L}(M)$ is given by $\hat{X}_\gamma(t) = X(\gamma(t))$. We want to check that $L_{\hat{X}} \mu = \mu$. For $Y \in T_\gamma \mathcal{L}(M)$, we have

$$\begin{aligned} \alpha(Y) &= i_{\hat{X}} \omega(Y) = \omega(\hat{X}, Y) \\ &= \int_0^1 g\left(\frac{\partial \hat{X}_\gamma}{\partial t}, Y\right) \mathrm{d}t = \int_0^1 g(\nabla_{\gamma'} X, Y) \mathrm{d}t \\ &= \frac{1}{2} \int_0^1 g(\gamma', Y) \mathrm{d}t = \mu(Y). \end{aligned}$$

So $\alpha = \mu$. Then

$$L_{\hat{X}} \mu = \mathrm{d} i_{\hat{X}} \mu + i_{\hat{X}} \mathrm{d}\mu = \mathrm{d} i_{\hat{X}} i_{\hat{X}} \omega + i_{\hat{X}} \omega = 0 + \alpha = \mu.$$

From this, it follows that $L_{\hat{X}} \omega = L_{\hat{X}} \mathrm{d}\mu = \mathrm{d} L_{\hat{X}} \mu = \mathrm{d}\mu = \omega$, as required. □

Remark 4.1. The manifolds to which the previous result applies, that is, those satisfying $L_X g = g$ with X locally gradient-like, are locally of the form $(N \times \mathbb{R}, \mathrm{e}^t(g + \mathrm{d}t^2))$ with expanding vector field $X = \frac{\partial}{\partial t}$. This follows by writing $X = \operatorname{grad} f$, with $f > 0$ and putting $t = \log(f)$.

Two examples are relevant:

- $M = N \times \mathbb{R}$, with (N, g) a compact Riemannian manifold. Give to M the metric $\mathrm{e}^t(g + \mathrm{d}t^2)$.
- Let (N, g) be an open Riemannian manifold with a diffeomorphism $\varphi\colon N \to N$ such that $\varphi^*(g) = \mathrm{e}^{\lambda} g$, $\lambda > 0$. Then take $M = (N \times [0, \lambda])/\sim$, where $(x, 0) \sim (\varphi(x), \lambda)$ and M has the metric induced by $\mathrm{e}^t(g + \mathrm{d}t^2)$.

To finish, let us check that the familiar finite dimensional picture translates to this case.

PROPOSITION 4.2

Let (M, g) be a Riemannian manifold which has a locally gradient-like vector field X satisfying $L_X g = g$. Then the hypersurface

$$\mathcal{L}_1(M) = \{\gamma \in \mathcal{L}(M); \mathit{length}(\gamma) = 1\}$$

is a quasi-contact hypersurface of $\mathcal{L}(M)$.

Proof. We define a family of loops as follows:

$$\gamma_s(t) = \psi_s(\gamma(t)),$$

where $\psi_s\colon M \to M$ is the flow associated to the vector field X. Let us compute the following derivative:

$$\begin{aligned}
\left.\frac{\mathrm{d}\,\mathrm{length}(\gamma_s)}{\mathrm{d}s}\right|_{s=0} &= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \int_0^1 g(\gamma_s'(t), \gamma_s'(t))^{1/2}\mathrm{d}t \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \int_0^1 g(\mathrm{d}\psi_s(\gamma'(t)), \mathrm{d}\psi_s(\gamma'(t)))^{1/2}\mathrm{d}t \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \int_0^1 ((\psi_s^* g)(\gamma'(t), \gamma'(t)))^{1/2}\mathrm{d}t \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \int_0^1 ((\mathrm{e}^s g)(\gamma'(t), \gamma'(t)))^{1/2}\mathrm{d}t \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \left(\mathrm{e}^s \int_0^1 (g(\gamma'(t), \gamma'(t)))^{1/2}\mathrm{d}t\right) \\
&= \mathrm{length}(\gamma).
\end{aligned}$$

Therefore the vector field $\hat{X}$ is tranverse to the level sets of the functional length in $\mathcal{L}(M)$.

Now, we need to check that $\alpha = i_{\hat{X}}\omega$ is a weak contact form on $\mathcal{L}_1(M)$. First we claim that the form α is nowhere zero on that submanifold. If this were not the case, then we would have that

$$(i_{\hat{X}}\omega)_{\gamma_0}|_{T_\gamma \mathcal{L}_1(M)} = 0, \qquad \text{for certain } \gamma_0 \in \mathcal{L}_1(M), \tag{11}$$

and we know that $i_{\hat{X}}\omega(\hat{X}) = 0$. Therefore we have that $i_{\hat{X}}\omega = 0$ and then the vector field $\hat{X}$ belongs to $\ker \omega$. Thus it satisfies eq. (2). Since the Levi-Civita connection is torsion-free we have that

$$\nabla_{\gamma_s'} X = \nabla_X \gamma_s'.$$

To set up the previous formula, we are extending γ_0 to the family of loops γ_s in order to correctly define the commutators. We finish by computing

$$\nabla_X g(\gamma_0', \gamma_0') = g(\nabla_X \gamma_0', \gamma_0') + g(\gamma_0', \nabla_X \gamma_0') = 0,$$

because of the compatibility condition of the connection. So, we get that $\frac{\mathrm{d}\,\mathrm{length}(\gamma_s)}{\mathrm{d}s}\big|_{s=0} = 0$, that is a contradiction. So α is nowhere zero.

Now we have the distribution $(\ker \alpha, \omega = \mathrm{d}\alpha)$ on $\mathcal{L}_1(M)$. To finish we check that it is quasi-symplectic. Assume that $Y \in \ker \alpha \subset T_\gamma \mathcal{L}_1(M)$ satisfies that

$$\omega(Y, Z) = 0,$$

for all $Z \in \ker \alpha \subset T_\gamma \mathcal{L}_1(M)$. Moreover we have that $\omega(Y, \hat{X}) = -\alpha(Y) = 0$. Hence $i_Y \omega = 0$, and we get that $Y \in \ker \omega$, that is, finite dimensional. □

COROLLARY 4.3

The submanifold of $\mathcal{L}_p(M)$ defined as

$$\mathcal{L}_{p,1}(M) = \{\gamma \in \mathcal{L}_p(M); \mathrm{length}(\gamma) = 1\}$$

is a weak contact submanifold.

Proof. We have checked that $\mathcal{L}_1(M)$ is quasi-contact. By choosing the submanifold $\mathcal{L}_{p,1}(M)$ we are removing the kernel as in the symplectic case. □

COROLLARY 4.4

Given a Riemannian manifold (M, g), the manifold $(M \times \mathbb{R}, e^{\lambda}(g + d\lambda^2))$ has an associated space of loops of length one with a canonical contact form. For a loop γ and Y vector field along γ, we denote $\gamma = (\gamma_1, \gamma_2)$ and $Y = (Y_1, Y_2)$ according to the decomposition $M \times \mathbb{R}$. Then the contact form is given by

$$\alpha(Y) = \mu(Y) = \frac{1}{2} \int_0^1 e^{\gamma_2(t)} (g(\gamma_1'(t), Y_1(t)) + \gamma_2'(t) Y_2(t)) \mathrm{d}t.$$

Reeb vector fields

Since the manifold

$$\mathcal{L}_1(M) = \{\gamma \in \mathcal{L}(M);\ \mathrm{length}(\gamma) = 1\}$$

is quasi-contact, there is not a unique Reeb vector field, i.e. a vector field R satisfying

$$i_R \alpha = 1,$$
$$i_R \mathrm{d}\alpha = 0.$$

However, this pair of equations has solution always as we now show.

Lemma 4.5. *A Reeb vector field associated to* $(\mathcal{L}_1(M), \alpha)$ *is the vector*

$$R = 2\frac{\gamma'}{\|\gamma'\|}. \tag{12}$$

Proof. The condition for a vector $V \in T_\gamma \mathcal{L}(M)$ to belong to $T_\gamma \mathcal{L}_1(M)$ is that there is a curve variation γ_s with $\gamma_0 = \gamma$, $\frac{\mathrm{d}\gamma_s}{\mathrm{d}s}|_{s=0} = V$ and length$(\gamma_s) = 1$. At first order, this is equivalent to:

$$\lim_{s\to 0} \frac{1}{s} \int_0^1 \|\gamma' + s\nabla_{\gamma'} V\|\mathrm{d}t = 0.$$

So we are asking the vector to satisfy

$$\int_0^1 g\left(\frac{\partial V}{\partial t}, \gamma'(t)\right) \frac{1}{\|\gamma'\|}\mathrm{d}t = 0.$$

Thus the previous equation can be rewritten as

$$\omega\left(V, \frac{\gamma'}{\|\gamma'\|}\right) = \mathrm{d}\alpha\left(V, \frac{\gamma'}{\|\gamma'\|}\right) = 0.$$

Imposing this condition for every non-zero vector $V \in T_\gamma \mathcal{L}_1(M)$, we get that the Reeb vector field is a positive multiple of $\frac{\gamma'}{\|\gamma'\|}$. Now we compute

$$\begin{aligned}\alpha\left(\frac{\gamma'}{\|\gamma'\|}\right) &= i_{\hat{X}}\omega\left(\frac{\gamma'}{\|\gamma'\|}\right) = \omega\left(\frac{\gamma'}{\|\gamma'\|}, \hat{X}_\gamma\right)\\ &= \frac{1}{2}\int_0^1 g\left(\frac{\gamma'}{\|\gamma'\|}, \gamma'\right) = \frac{1}{2}\int_0^1 \|\gamma'\| = \frac{1}{2}.\end{aligned}$$

Therefore

$$R = 2\frac{\gamma'}{\|\gamma'\|}$$

is a Reeb vector field. □

There are more solutions to the Reeb vector field equation since for any parallel vector field w along γ, we can add it to R, so that $R + w$ defines another solution to the equation. However those solutions are not continuous (as vector fields on $\mathcal{L}_1(M)$) in general).

We have the following:

Lemma 4.6. *All the Reeb orbits for* (12) *in* $\mathcal{L}_1(M)$ *are closed of period* 1/2.

Proof. Take $\gamma\colon S^1 \to M$. The arc-length parametrization of $\gamma(S^1)$ is denoted as γ_p, where p is the point of $\gamma(S^1)$ in which the arc-length parameter starts. So we define $\theta(s,t) = \gamma_{\gamma(t)}(2s)$ which is clearly the Reeb orbit starting at γ. It is periodic with period $1/2$. □

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