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On short wave stability and sufficient conditions for stability in the extended Rayleigh problem of hydrodynamic stability

M SUBBIAH and V GANESH*

Department of Mathematics, Pondicherry University, Kalapet, Pondicherry 605 014, India
*Department of Mathematics, Amrita School of Engineering, Amrita Vishwa Vidyapeetham University, Ettimadai, Coimbatore 641 105, India
E-mail: malaisubbiah@yahoo.com

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Abstract. We consider the extended Rayleigh problem of hydrodynamic stability dealing with the stability of inviscid homogeneous shear flows in sea straits of arbitrary cross section. We prove a short wave stability result, namely, if $k > 0$ is the wave number of a normal mode then $k > k_c$ (for some critical wave number k_c) implies the stability of the mode for a class of basic flows. Furthermore, if $K(z) = \frac{-(U_0'' - T_0 U_0')}{U_0 - U_{0s}}$, where U_0 is the basic velocity, T_0 (a constant) the topography and prime denotes differentiation with respect to vertical coordinate z then we prove that a sufficient condition for the stability of basic flow is $0 < K(z) \leq \left(\frac{\pi^2}{D^2} + \frac{T_0^2}{4}\right)$, where the flow domain is $0 \leq z \leq D$.

Keywords. Hydrodynamic stability; shear flows; variable bottom; sea straits.

1. Introduction

The stability analysis of homogeneous and stratified shear flows in sea straits of arbitrary cross section was initiated in [6] and a mathematical approach was adapted in [3]. In [3], the stability equation was derived and it was found to be an extended version of the well-known Taylor–Goldstein problem of hydrodynamic stability. A number of general analytical results have been obtained for this extended Taylor–Goldstein problem in [3], [9] and [4].

In the special case of homogeneous shear flows the problem reduces to the extended Rayleigh problem of hydrodynamic stability. For this problem [3] a necessary condition for instability is that $\left(\frac{U_0'}{b}\right)'$ changes sign atleast once in the flow domain where $U_0(z)$ is the basic velocity of the profile, $b(z)$ is the width function and a prime denotes differentiation with respect to z. If $\left(\frac{U_0'}{b}\right)'$ is equal to zero at $z = z_s$ and $U_{0s} = U_0(z_s)$ then it is proved in [3] that a necessary condition for instability is $\left(\frac{U_0'}{b}\right)'(U_0 - U_{0s}) < 0$ atleast once in the flow domain $0 \leq z \leq D$. Recently it was proved in [8] that the Howard's conjecture, namely, the growth rate $kc_i \to 0$ as the wave number $k \to \infty$.

In the present paper, we consider a basic flow with velocity $U_0(z)$ and constant topography $T = T_0$. For this class of flows, we prove mathematically that the waves shorter than some critical wavelength are stable, that is $c_i = 0$ when $k > k_c$ where k_c is some critical value of wave number k. This means that short waves are stable.

Furthermore, we prove that a sufficient condition for stability is that $0 < K(z) \leq \left(\frac{\pi^2}{D^2} + \frac{T_0^2}{4}\right)$. When $T_0 = 0$ we have the Rayleigh problem and in this case our result reduces to the sufficient condition of [2]. In this paper we have got two stability results both depending on $K(z)$ being non-negative. However, for the Rayleigh problem which corresponds to $T \equiv 0$, it has been found in [5] that the number of unstable modes is equal to the number of neutral modes when $K(z) = \frac{-U_0''}{(U_0 - U_{0s})} \geq 0$. It is possible that a similar result for the problem considered in this paper can be obtained by adopting the method of [5].

2. The extended Rayleigh problem

The extended Rayleigh problem is given by the second-order ordinary differential equation

$$W'' - \left[k^2 + \frac{U_0''}{U_0 - c}\right] W + \frac{1}{U_0 - c}[T(U_0 - c)W]' = 0, \tag{1}$$

with boundary conditions

$$W(0) = 0 = W(D). \tag{2}$$

Here, the real part of $W(z)\mathrm{e}^{ik(x-ct)}$ is the vertical velocity of a normal mode disturbance, $k > 0$ is the wave number, $c = c_r + ic_i$ is the complex phase velocity, $U_0(z)$ is the basic velocity and $T(z)$ is the topography.

If T is a constant, say $T = T_0$, then (1) becomes

$$W'' + T_0 W' - k^2 W - \frac{(U_0'' - T_0 U_0')}{U_0 - c} W = 0. \tag{3}$$

The transformation $W = \mathrm{e}^{\frac{-T_0 z}{2}} V$ reduces (3) to

$$V'' - \left[k^2 + \frac{(U_0'' - T_0 U_0')}{U_0 - c} + \frac{T_0^2}{4}\right] V = 0, \tag{4}$$

with boundary conditions

$$V(0) = 0 = V(D). \tag{5}$$

3. Stability results

Theorem 1. *A necessary condition for the existence of non-trivial solution with $c_i > 0$ is that the following integral relations are true*:

(i) $$\int [|V'|^2 + k^2|V|^2]\mathrm{d}z + \int \frac{(U_0'' - T_0 U_0')[U_0 - c_r]}{|U_0 - c|^2}|V|^2\mathrm{d}z + \frac{T_0^2}{4}\int |V|^2\mathrm{d}z = 0,$$

(ii) $$\int \frac{(U_0'' - T_0 U_0')}{|U_0 - c|^2}|V|^2\mathrm{d}z = 0.$$

Proof. Multiplying (4) by V^* (conjugate of V), integrating over [0, D] using integration by parts and applying (5), we get

$$\int |V'|^2 \mathrm{d}z + \int \left[k^2 + \frac{(U_0'' - T_0 U_0')}{U_0 - c} + \frac{T_0^2}{4} \right] |V|^2 \mathrm{d}z = 0. \tag{6}$$

The real part of (6) gives

$$\int [|V'|^2 + k^2 |V|^2] \mathrm{d}z + \int \frac{(U_0'' - T_0 U_0')[U_0 - c_r]}{|U_0 - c|^2} |V|^2 \mathrm{d}z + \frac{T_0^2}{4} \int |V|^2 \mathrm{d}z = 0. \tag{7}$$

The imaginary part of (6) gives

$$c_i \int \frac{(U_0'' - T_0 U_0')}{|U_0 - c|^2} |V|^2 \mathrm{d}z = 0.$$

Since $c_i > 0$, we have

$$\int \frac{(U_0'' - T_0 U_0')}{|U_0 - c|^2} |V|^2 \mathrm{d}z = 0. \tag{8}$$

This completes the proof of the theorem.

From (8) it follows that a necessary condition for instability is that there exists a $z_s \in$ [0, D] such that $U_0'' - T_0 U_0' = 0$ at $z = z_s$.

Let us denote $U_0(z_s)$ by U_{0s}.

Theorem 2. *A necessary condition for the existence of non-trivial solution with $c_i > 0$ is that the integral relation*

$$\int [|V'|^2 + k^2 |V|^2] \mathrm{d}z + \int \frac{(U_0'' - T_0 U_0')[U_0 - U_{0s}]}{|U_0 - c|^2} |V|^2 \mathrm{d}z + \frac{T_0^2}{4} \int |V|^2 \mathrm{d}z = 0.$$

Proof. Multiplying (8) by $(c_r - U_{0s})$ and adding the resultant equation to (7), we get

$$\int [|V'|^2 + k^2 |V|^2] \mathrm{d}z + \int \frac{(U_0'' - T_0 U_0')[U_0 - U_{0s}]}{|U_0 - c|^2} |V|^2 \mathrm{d}z + \frac{T_0^2}{4} \int |V|^2 \mathrm{d}z = 0. \tag{9}$$

This completes the proof of the theorem.

Theorem 3. *A necessary criterion of instability* $(c_i > 0)$ *is that* $(U_0'' - T_0 U_0')(U_0 - U_{0s}) < 0$ *at some point* $z = z_p \neq z_s$.

Proof. In eq. (9), the first and third integrals are non-negative, so they can be dropped to give

$$\int \frac{(U_0'' - T_0 U_0')[U_0 - U_{0s}]}{|U_0 - c|^2} |V|^2 \mathrm{d}z < 0,$$

which implies that

$$(U_0'' - T_0 U_0')(U_0 - U_{0s}) < 0, \tag{10}$$

at some point $z = z_p \neq z_s$.

This completes the proof of the theorem.

Note that we can define $K(z)$ by $K(z) = \frac{-(U_0'' - T_0 U_0')}{U_0 - U_{0s}}$. Then the above theorem implies that a necessary condition for instability is $K(z) > 0$ atleast once, say, at $z = z_p \neq z_s$ in [0, D].

Theorem 4. *A necessary condition for the existence of non-trivial solutions with* $c_i > 0$ *is that the following integral relations are true*:

(i) $$\int |V''|^2 \mathrm{d}z + k^2 \int |V'|^2 \mathrm{d}z + \frac{T_0^2}{4} \int |V'|^2 \mathrm{d}z - \int \left[\frac{\left(k^2 + \frac{T_0^2}{4}\right)(U_0'' - T_0 U_0')[U_0 - U_{0s}] + (U_0'' - T_0 U_0')^2}{|U_0 - c|^2} \right] |V|^2 \mathrm{d}z = 0,$$

(ii) $$\int |V''|^2 \mathrm{d}z + 2k^2 \int |V'|^2 \mathrm{d}z + \frac{T_0^2}{4} \int |V'|^2 \mathrm{d}z + k^4 \int |V|^2 \mathrm{d}z + \frac{k^2 T_0^2}{4} \int |V|^2 \mathrm{d}z - \int \frac{(U_0'' - T_0 U_0')^2}{|U_0 - c|^2} |V|^2 \mathrm{d}z - \frac{T_0^2}{4} \int \frac{(U_0'' - T_0 U_0')[U_0 - U_{0s}]}{|U_0 - c|^2} |V|^2 \mathrm{d}z = 0.$$

Proof. Multiplying (4) by $(V^*)''$, integrating over $[0, D]$ using integration by parts and applying (5), we get

$$\int |V''|^2 \mathrm{d}z + k^2 \int |V'|^2 \mathrm{d}z + \frac{T_0^2}{4} \int |V'|^2 \mathrm{d}z - \int \frac{(U_0'' - T_0 U_0')}{U_0 - c} V (V^*)'' \mathrm{d}z = 0. \tag{11}$$

From (4) and by taking complex conjugate, we get

$$(V^*)'' = \left[k^2 + \frac{U_0'' - T_0 U_0'}{U_0 - c^*} + \frac{T_0^2}{4} \right] V^*. \tag{12}$$

Substituting (12) in (11), we get

$$\int |V''|^2 \mathrm{d}z + k^2 \int |V'|^2 \mathrm{d}z + \frac{T_0^2}{4} \int |V'|^2 \mathrm{d}z - k^2 \int \frac{(U_0'' - T_0 U_0')}{U_0 - c} |V|^2 \mathrm{d}z - \int \frac{(U_0'' - T_0 U_0')^2}{|U_0 - c|^2} |V|^2 \mathrm{d}z - \frac{T_0^2}{4} \int \frac{(U_0'' - T_0 U_0')}{U_0 - c} |V|^2 \mathrm{d}z = 0.$$

Taking real and imaginary parts, we get

$$\int |V''|^2 \mathrm{d}z + k^2 \int |V'|^2 \mathrm{d}z + \frac{T_0^2}{4} \int |V'|^2 \mathrm{d}z - k^2 \int \frac{(U_0'' - T_0 U_0')(U_0 - c_r)}{|U_0 - c|^2} |V|^2 \mathrm{d}z - \int \frac{(U_0'' - T_0 U_0')^2}{|U_0 - c|^2} |V|^2 \mathrm{d}z - \frac{T_0^2}{4} \int \frac{(U_0'' - T_0 U_0')(U_0 - c_r)}{|U_0 - c|^2} |V|^2 \mathrm{d}z = 0, \tag{13}$$

and

$$-k^2 c_i \int \frac{(U_0'' - T_0 U_0')}{|U_0 - c|^2}|V|^2 \mathrm{d}z - \frac{T_0^2}{4} c_i \int \frac{(U_0'' - T_0 U_0')}{|U_0 - c|^2}|V|^2 \mathrm{d}z = 0.$$

Since $c_i > 0$,

$$-k^2 \int \frac{(U_0'' - T_0 U_0')}{|U_0 - c|^2}|V|^2 \mathrm{d}z - \frac{T_0^2}{4} \int \frac{(U_0'' - T_0 U_0')}{|U_0 - c|^2}|V|^2 \mathrm{d}z = 0. \qquad (14)$$

Multiplying (14) by $(c_r - U_{0s})$ and adding the resultant to eq. (13), we get

$$\int |V''|^2 \mathrm{d}z + k^2 \int |V'|^2 \mathrm{d}z + \frac{T_0^2}{4} \int |V'|^2 \mathrm{d}z$$
$$- \int \left[\frac{\left(k^2 + \frac{T_0^2}{4}\right)(U_0'' - T_0 U_0')[U_0 - U_{0s}] + (U_0'' - T_0 U_0')^2}{|U_0 - c|^2} \right] |V|^2 \mathrm{d}z = 0. \qquad (15)$$

This proves the first relation in the statement of the theorem.

To (13), multiply (7) by k^2 and add, and multiply (8) by $\frac{T_0^2}{4}(c_r - U_{0s})$ and subtract to get

$$\int |V''|^2 \mathrm{d}z + 2k^2 \int |V'|^2 \mathrm{d}z + \frac{T_0^2}{4} \int |V'|^2 \mathrm{d}z + k^4 \int |V|^2 \mathrm{d}z$$
$$+ \frac{k^2 T_0^2}{4} \int |V|^2 \mathrm{d}z - \int \frac{(U_0'' - T_0 U_0')^2}{|U_0 - c|^2}|V|^2 \mathrm{d}z$$
$$- \frac{T_0^2}{4} \int \frac{(U_0'' - T_0 U_0')[U_0 - U_{0s}]}{|U_0 - c|^2}|V|^2 \mathrm{d}z = 0. \qquad (16)$$

This proves the second part of the theorem.

Theorem 5. *For an unstable normal mode with wave number* $k > 0$, *it is necessary that*

$$k^2 < \left[\frac{\frac{T_0^2}{4}(U_0'' - T_0 U_0')[U_0 - U_{0s}] + (U_0'' - T_0 U_0')^2}{-(U_0'' - T_0 U_0')[U_0 - U_{0s}]} \right]_{z=z_p}, \quad \textit{where} \quad z_p (\neq z_s) \in [0, D].$$

Proof. From eq. (15), it follows that there exists $z_p (\neq z_s) \in [0, D]$ at which

$$\left(k^2 + \frac{T_0^2}{4}\right)(U_0'' - T_0 U_0')[U_0 - U_{0s}] + (U_0'' - T_0 U_0')^2 > 0;$$

i.e., $[-k^2 (U_0'' - T_0 U_0')[U_0 - U_{0s}]]_{z=z_p}$

$$< \left[\frac{T_0^2}{4}(U_0'' - T_0 U_0')[U_0 - U_{0s}] + (U_0'' - T_0 U_0')^2 \right]_{z=z_o};$$

i.e., $$k^2 < \left[\frac{\frac{T_0^2}{4}(U_0'' - T_0 U_0')[U_0 - U_{0s}] + (U_0'' - T_0 U_0')^2}{-(U_0'' - T_0 U_0')[U_0 - U_{0s}]} \right]_{z=z_p}. \qquad (17)$$

This completes the proof of the theorem.

Remark. As $\lambda = \frac{2\pi}{k}$ is the length of the wave it follows that the wavelength of an unstable mode must be sufficiently large as k small corresponds to λ large. Let

$$k_c^2 = \max_{z\in[0,D]\setminus\{z_s\}} \left[\frac{\frac{T_0^2}{4}(U_0'' - T_0U_0')[U_0 - U_{0s}] + (U_0'' - T_0U_0')^2}{-(U_0'' - T_0U_0')[U_0 - U_{0s}]} \right].$$

Then it follows that $k > k_c$ implies that the normal mode with wave number k is stable provided k_c is finite. Here $U_0'' - T_0U_0'$ should be zero atleast once for instability. It does not matter if it becomes zero more than once. However it cannot be identically equal to zero for unstable flows. The points where $U_0'' - T_0U_0' = 0$ may not be inflexion points of the velocity profile. i.e., U_0'' need not be zero where $U_0'' - T_0U_0'$ is zero. Now we see that k_c^2 may become infinity if the numerator is not zero where the denominator is zero. The denominator is zero at $z = z_s$.

Now expansion around $z = z_s$ gives $U_0'' - T_0U_0' = (U_0'' - T_0U_0')(z_s) + (U_0'' - T_0U_0')'(z_s)(z - z_s) + \cdots$, $U_0 - U_{0s} = U_0'(z_s)(z - z_s) + \cdots$,

$$\lim_{z\to z_s} \left[\frac{\frac{T_0^2}{4}(U_0'' - T_0U_0')[U_0 - U_{0s}] + (U_0'' - T_0U_0')^2}{-(U_0'' - T_0U_0')[U_0 - U_{0s}]} \right]$$

$$= \lim_{z\to z_s} \frac{U_0'' - T_0U_0'}{-(U_0 - U_{0s})} = \frac{(U_0'' - T_0U_0')'(z_s)}{-U_0'(z_s)} = \frac{U_0'''(z_s) - T_0U_0''(z_s)}{-U_0'(z_s)}$$

and this will be finite if $U_0'(z_s) \neq 0$. This condition will be satisfied for monotonic profiles. Also $(U_0 - U_{0s})$ will not be zero at any other point for monotonic profiles. Hence k_c^2 will be finite for monotonic profiles. Since the exchange flows $U_0 = \left(z - \frac{D}{2}\right)$ and $U_0 = \tanh\left(z - \frac{D}{2}\right)$ are monotonic profiles the above result is applicable to these flows.

For non-monotonic profiles with $U_0'(z_s) \neq 0$ it is possible that $(U_0 - U_{0s})$ is zero at other points also. In this case it is necessary for the finiteness of k_c^2 that the numerator is also zero at those points and that the order of zero of the numerator is greater or equal to the order of zero of the denominator.

In the absence of topography $T_0 = 0$, this result reduces to that of [1]. It may be noted here that a short wave stability result has been proved recently for the Rayleigh problem in [7]. However in this paper no explicit value for k_c has been found as in [1]. Furthermore the basic flow considered in [7] are also different from those considered here.

Theorem 6. *An estimate for the growth rate of unstable mode is given by*

$$k^2c_i^2 \leq \left[\frac{(U_0'' - T_0U_0')^2}{K(z)} \left(\frac{K(z) - \left[\frac{\pi^2}{D^2} + \frac{T_0^2}{4}\right]}{k^2} \right) \right]_{\max}.$$

Proof. Multiplying (9) by $\left(\frac{-\pi^2}{D^2}\right)$ and adding the resultant to (16), we get

$$\left[\int |V''|^2\mathrm{d}z - \frac{\pi^2}{D^2}\int |V'|^2\mathrm{d}z\right] + k^2\left[\int |V'|^2\mathrm{d}z - \frac{\pi^2}{D^2}\int |V|^2\mathrm{d}z\right]$$

$$+ \frac{T_0^2}{4}\left[\int |V'|^2\mathrm{d}z - \frac{\pi^2}{D^2}\int |V|^2\mathrm{d}z\right] + k^2\int |V'|^2\mathrm{d}z + \frac{k^2T_0^2}{4}\int |V|^2\mathrm{d}z$$

$$+k^4\int |V|^2\mathrm{d}z - \int \frac{(U_0''-T_0U_0')^2}{|U_0-c|^2}|V|^2\mathrm{d}z$$
$$-\frac{T_0^2}{4}\int \frac{(U_0''-T_0U_0')(U_0-U_{0s})}{|U_0-c|^2}|V|^2\mathrm{d}z$$
$$-\frac{\pi^2}{D^2}\int \frac{(U_0''-T_0U_0')(U_0-U_{0s})}{|U_0-c|^2}|V|^2\mathrm{d}z = 0. \tag{18}$$

The first term is non-negative by Lemma 1 of [2], the second and third terms are non-negative by use of the well-known Rayleigh–Ritz inequality. The fourth and fifth terms are obviously non-negative, therefore dropping these terms, we get

$$k^4\int |V|^2\mathrm{d}z - \int \Bigg[(U_0''-T_0U_0')^2 + \frac{T_0^2}{4}(U_0''-T_0U_0')(U_0-U_{0s})$$
$$+\frac{\pi^2}{D^2}(U_0''-T_0U_0')(U_0-U_{0s})\Bigg]\frac{|V|^2}{|U_0-c|^2}\mathrm{d}z \le 0.$$

Hence it follows that

$$\int\left[k^4 - \frac{(U_0''-T_0U_0')^2}{|U_0-c|^2} - \frac{\left(\frac{\pi^2}{D^2}+\frac{T_0^2}{4}\right)(U_0''-T_0U_0')(U_0-U_{0s})}{|U_0-c|^2}\right]|V|^2\mathrm{d}z \le 0.$$

Consequently there exists a point $z_q \in [0, D]$ such that

$$k^4c_i^2 \le \left[(U_0''-T_0U_0')^2 + \left(\frac{\pi^2}{D^2}+\frac{T_0^2}{4}\right)(U_0''-T_0U_0')(U_0-U_{0_s})\right]_{z=z_q}.$$

Because for all U_{0s}, there exists a z_q satisfying preceding inequality, it follows that

$$k^2c_i^2 \le \left[\frac{(U_0''-T_0U_0')^2 + \left[\frac{\pi^2}{D^2}+\frac{T_0^2}{4}\right](U_0''-T_0U_0')(U_0-U_{0s})}{k^2}\right]_{\max}.$$

Let $K(z) = \frac{-(U_0''-T_0U_0')}{U_0-U_{0s}}$, then

$$k^2c_i^2 \le \left[\frac{(U_0''-T_0U_0')^2}{K(z)}\left(\frac{K(z)-\left[\frac{\pi^2}{D^2}+\frac{T_0^2}{4}\right]}{k^2}\right)\right]_{\max}. \tag{19}$$

This completes the proof of the theorem.

Hence, if $K(z) \le \left(\frac{\pi^2}{D^2}+\frac{T_0^2}{4}\right)$ it follows that $c_i = 0$ which means stability of the flow. We state this result in the following theorem.

Theorem 7. *A sufficient criterion of stability is that* $0 < K(z) \le \left(\frac{\pi^2}{D^2}+\frac{T_0^2}{4}\right)$ *in the flow domain* $[0, D]$.

4. Concluding remarks

In this paper we have obtained some analytical results on the extended Rayleigh problem of hydrodynamic stability. We consider basic flows for which the basic velocity $U_0(z)$ is any twice continuously differentiable function and the topography T_0 is a constant. For this class of flows we prove a short wave stability result, namely, a normal mode disturbance with wave number $k > 0$ is stable when $k > k_c$, where the critical wave number

$$k_c^2 = \max_{z \in [0,D] \setminus \{z_s\}} \left[\frac{\frac{T_0^2}{4}(U_0'' - T_0 U_0')[U_0 - U_{0s}] + (U_0'' - T_0 U_0')^2}{-(U_0'' - T_0 U_0')[U_0 - U_{0s}]} \right].$$

In the absence of topography this result reduces to the result of [1] for the Rayleigh problem. It may be remarked that a short wave stability result has recently been proved for the Rayleigh problem in [7]. However, no explicit value for the critical wave number k_c is given and moreover the basic flow velocity $U_0(z)$ is considered to be an analytic function of z. Furthermore, we prove that a sufficient condition for stability of basic flows is that $0 < K(z) \leq \left(\frac{\pi^2}{D^2} + \frac{T_0^2}{4}\right)$ in $[0, D]$.

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