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Regularity of the interband light absorption coefficient

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Abstract. In this paper we consider the interband light absorption coefficient (ILAC), in a symmetric form, in the case of random operators on the d-dimensional lattice. We show that the symmetrized version of ILAC is either continuous or has a component which has the same modulus of continuity as the density of states.

Keywords. Light absorption coefficient; density of states; random operators.

1. Introduction

In the theory of disordered systems, one of the quantities that is widely studied is the integrated density of states, whose continuity properties and its behaviour near band edges (Lifshitz tails) were of great interest. Another quantity that is of interest is the interband light absorption coefficient (ILAC), which is an important quantitative characteristic of semiconductors.

When photons with sufficient energy are incident on a pure semiconductor crystal absorption of photons takes place with simultaneous creation of electron–hole pairs, which means excitation of electrons from valance band to the conduction band. This process is intrinsic interband absorption. The threshold electron energy required is related to the basic band gap. The absorption coefficient increases above the threshold.

To get the absorption coefficient one considers the transition of an electron between states in the same or a different band. The rate of absorption is then calculated using perturbation theory and the Fermi golden rule.

The theory of interband light absorption can be found in a book such as [3].

On the other hand, the presence of impurities cause electronic states to be produced in the forbidden band and this reduction of the band gap and the associated effect on the inter band light absorption coefficient is discussed in [10].

In experimental studies the absorption coefficient is a means to study the band gaps at different temperatures for a given material.

In mathematical terms this means that when there is a periodic potential (= pure crystal) there are bands and gaps and when one adds random potential to such a periodic background, spectrum extends into the original gaps.

The literature on density of states is vast, so we refer the reader to [6–8, 12, 18] and [21]. The continuity properties of the density of states and its Lifshitz tails behaviour in various models is widely understood. The physics literature is abound with works on the ILAC starting from [10] and for example [1]. On the other hand, rigorous work in this area seem to be minimal, see for example [14–17].

We consider a Borel probability space $(\Omega, \mathcal{B}, \mathbb{P})$ with $\mathbb{Z}^d$ acting on Ω such that $\mathbb{P}$ is invariant and ergodic with respect to this action. Let $V: \Omega \to \mathbb{R}^{\mathbb{Z}^d}$ (so that each $V(n)$ is measurable). We consider a self-adjoint operator $H_0 = \Delta$ (given in eq. (5)) on $\ell^2(\mathbb{Z}^d)$ and consider the family of operators,

$$H_\omega^\pm = H_0 \pm V^\omega, (V^\omega u)(n) = V^\omega(n)u(n), u \in \ell^2(\mathbb{Z}^d), \tag{1}$$

such that V^ω are covariant in the sense made precise in hypothesis (2.1) below.

We denote by δ_n the elements of the standard basis of $\ell^2(\mathbb{Z}^d)$ in the rest of the paper.

We define the density of states measures $n_\pm$ associated with $H_\omega^\pm$ by

$$n_\pm = \mathbb{E}(\langle \delta_0, E_{H_\omega^\pm}(\cdot)\delta_0\rangle). \tag{2}$$

Suppose $H_{\omega,\Lambda}^\pm$ are the restrictions of $H_\omega^\pm$ to $\ell^2(\Lambda)$, where $\Lambda \subset \mathbb{Z}^d$ is a finite set (usually taken to be a lattice cube centred at the origin) and $\lambda_\pm, u_{\lambda_\pm}$ are eigenvalues and eigenfunctions of $H_{\Lambda,\omega}^\pm$.

Then the interband light absorption coefficient A_Λ for such finite volume models can be defined by taking the measure $\rho_{\Lambda,\omega}$ as given below:

$$\rho_{\Lambda,\omega} = \frac{1}{|\Lambda|} \sum_{\lambda_\pm \in \sigma(H_{\omega,\Lambda}^\pm)} |\langle u_{\lambda_+}, v_{\lambda_-}\rangle|^2 \delta_{\lambda_+,\lambda_-}, \tag{3}$$

$$A_\Lambda(\lambda) = \rho_{\Lambda,\omega}(\{(\lambda_+, \lambda_-): \lambda_+ + \lambda_- \leq \lambda\}). \tag{4}$$

The operators $H_\omega^\pm$ may be unbounded. However, the finite volume operators $H_{\Lambda,\omega}$ are symmetric finite-dimensional matrices when Λ is a finite set, so their eigenvalues are finite in number and the eigenfunctions $\{u_{\lambda_\pm}: \lambda_\pm \in \sigma(H_{\omega,\Lambda}^\pm)\}$ are orthonormal (for each sign $\pm$). These properties show that the measure $\rho_{\Lambda,\omega}$ is a probability measure on $\mathbb{R}^2$, since

$$\sum_{\lambda_- \in \sigma(H_{\omega,\Lambda}^-)} |\langle u_{\lambda_+}, v_{\lambda_-}\rangle|^2 \delta_{\lambda_+,\lambda_-}(\mathbb{R}^2) = \|u_{\lambda_+}\|^2 = 1$$

and the second sum (over λ_+) is normalized by the size of the set $|\Lambda|$ which is precisely the number of eigenvalues of $H_{\Lambda,\omega}^+$.

There are several earlier works, for example, Bellissard *et al* [4], Bouchlet *et al* [5] defining the density of states given in eq. (2) as average trace per unit volume, namely

$$n_\pm(\cdot) = \lim_{\Lambda \uparrow \mathbb{Z}^d} n_{\pm,\Lambda}(\cdot), n_{\pm,\Lambda}(\cdot) = \frac{1}{|\Lambda|} Tr(\chi_\Lambda E_{H_\omega^\pm}(\cdot)).$$

The above limit exists a.e. ω, in the weak sense for measures, by using Birkoff's ergodic theorem and the expression in eq. (7) is arrived at using covariance of the spectral measures.

However more classical definitions of density of states involves taking the operators $H_{\omega,\Lambda}^\pm = \chi_\lambda H_\omega^\pm \chi_\Lambda$ on $\ell^2(\Lambda)$, considering the average spectral measure, counting multiplicity,

$$n^\pm(\cdot) = \lim_{\Lambda \uparrow \mathbb{Z}^d} n_{\omega,\Lambda}^\pm(\cdot), n_{\omega,\Lambda}^\pm(\cdot) = \frac{1}{|\Lambda|} \sum_{\lambda \in \sigma(H_{\omega,\Lambda}^\pm)} \delta_\lambda$$

and taking their limits. In the limit both these definitions agree with that given in eq. (2).

It is well-known that for the models such as the one considered in eq. (1), $n_\pm = n^\pm$. The trace per unit volume definition also allows one to define the limits a.e.

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \mathrm{Tr}(\chi_\Lambda A_\omega B_\omega \chi_\Lambda)$$

for a pair of covariant operators, satisfying an integrability condition, which results in the limit being equal to $\mathbb{E}(\langle \delta_0, A_\omega B_\omega \delta_0 \rangle)$ and one also gets

$$\mathbb{E}(\langle \delta_0, A_\omega B_\omega \delta_0 \rangle) = \mathbb{E}(\langle \delta_0, B_\omega A_\omega \delta_0 \rangle).$$

In the case of continuous models (i.e. models on $L^2(\mathbb{R}^d)$) Kirsch–Pastur obtained in Theorem 2.1(i) [14], limits of the finite dimensional ILAC, using sub additivity properties of such finite dimensional quantities and they did not have to use the 'trace per unit volume' definition.

However in the present case the existence of such a limit is unclear for the quantities defined in eq. (3) when $\Lambda \uparrow \mathbb{Z}^d$. It would be nice to show such a result, which one might need to show for nice functions f,

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \mathrm{Tr}(f(\chi_\Lambda H_\omega \chi_\Lambda)) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \mathrm{Tr}(\chi_\Lambda f(H_\omega) \chi_\Lambda),$$

in other words, obtain Szego type asymptotics. We do not attempt this here since this is not the main aim of the paper.

Therefore we take directly the definition given in eq. (7) of the correlation measure in the infinite lattice case and define the ILAC as a marginal in direct analogy with the finite volume case; these definitions are consistent with the ones obtained in the continuous case.

The main theorems of this paper are Theorems 3.6 and 3.7. Theorem 3.6 obtains estimates on the correlation measure of balls of radius a given in terms of the uniform modulus of continuity of the density of states.

So taking the correlation measure ρ as in eq. (7) and denoting the density of states as n, if $\rho((a, b) \times \mathbb{R}) \leq |b - a|^\alpha$, uniformly for all (a, b),then our theorem is that $\rho(\{x \in \mathbb{R}^2 \colon |x - b| \leq r\}) \leq c r^{\alpha/2}$. So the regularity along a line through the origin says something for the whole measure, but not enough to conclude the regularity of marginals along other lines. (This is a general fact valid for any finite Borel measure on $\mathbb{R}^n$, regularity along one line through the origin implies some regularity for the whole measure.)

As an example even if the density of states are absolutely continuous, it will only imply that the measure ρ is $\frac{1}{2}$-Hölder continuous on $\mathbb{R}^2$ and this still leaves room for the measure to be supported on some lines, which means some marginals of the measure could have atomic components.

This is the main obstruction to obtaining regularity of the ILAC, which happens to be the distribution function of a marginal of the measure ρ along a diagonal direction. However, given that ρ has some modulus of continuity, it follows that, if along some direction it has a component that is not continuous at all, such a component should have continuity along an orthogonal direction.

This is the feature we exploit for Theorem 3.7.

In Theorem 3.7 we address the above question, observing that in the case when the operators $H_\omega^\pm$ have some further symmetry the marginals defined along the two orthogonal directions $\{\lambda \in \mathbb{R}^2 \colon \lambda_1 = \lambda_2\}$ and $\{x \colon x_1 = -x_2\}$ actually agree. Therefore we take a symmetric definition of the ILAC and work with it. In view of the comments made above, such a symmetric definition enables us to conclude some regularity of some components, though at first such a theorem seems surprising.

2. The symmetric and asymmetric ILAC

In this section we define the interband light absorption coefficient in analogy with the case of continuum models using a correlation measure. We argue that in some cases when the spectra of the operators in question have some symmetry properties, the ILAC can be taken to be the distribution function of the average of the marginals of the correlation measure along two diagonal lines.

We denote by $U_i, i \in \mathbb{Z}^d$ the unitary operators $(U_i u)(n) = u(n-i), u \in \ell^2(\mathbb{Z}^d)$.

Hypothesis 2.1.

(1) (Covariance) The potential V^ω satisfies $U_i^* V^\omega U_i = V^{T_i\omega}$, where $T_i\omega(n) = \omega(n+i)$.
(2) There is a bijection R of Ω to itself such that $V^{R\omega} = -V^\omega$ and $\mathbb{P}$ is invariant under R.
(3) The operators $H_\omega^\pm$ are self-adjoint with a common dense domain for a set of full measure in ω.
(4) The density of states measures $n_\pm$ are continuous.

Examples 2.2. Here are two extreme examples of operators satisfying the above conditions. Of course there are many more various varieties.

(1) *The Anderson model*:

$$V^\omega(n) = \omega(n), (\Delta u)(n) = \sum_{|i|=1} u(n+i), u \in \ell^2(\mathbb{Z}^d) \tag{5}$$

and $R\omega = -\omega$ and $\mathbb{P} = \times\mu$ with a probability measure μ on $\mathbb{R}$. If μ is continuous, then the density of states is continuous. We take μ to satisfy $\mu(B) = \mu(-B)$ for all Borel subsets of $\mathbb{R}$ and take $R\omega = -\omega$. Then $\mathbb{P}$ is invariant under R.

(2) *The almost Mathieu model*: Take $d = 1$ and $\Omega = \mathbb{T}$, $V^\omega(n) = \lambda\cos(\alpha n + \omega)$, $R\omega = \omega + \pi$ and $\mathbb{P}$ the rotation invariant measure on $\mathbb{T}$. The density of states of this model is absolutely continuous, when α is not rational and for $|\lambda| \neq 2$, see [2].

Remark 2.3. We note that, using the definition of $H_\omega^\pm$ and V^ω and the bijection R mentioned in the Hypothesis 2.1,

$$H_\omega^- = H_{R\omega}^+, H_\omega^+ = H_{R\omega}^-. \tag{6}$$

Therefore if $\mathbb{P}$ satisfies Hypothesis 2.1(2), then for any integrable function f of ω, we have

$$\mathbb{E}(f(\omega)) = \mathbb{E}(f(R\omega)).$$

The immediate consequence of our hypothesis is the equality of spectra of $H_\omega^\pm$.

Theorem 2.4. *Let $H_\omega^\pm$ be as in Hypothesis* 2.1. *Then we have*

$$\sigma(H_\omega^+) = \sigma(H_\omega^-), \quad a.e.\ \omega.$$

Proof. Under the assumptions of Hypothesis 2.1, it is well-known that the spectrum of the associated operators $\sigma(H_\omega^{\pm})$ are constant sets almost everywhere (Proposition V.2.4 of [6]). Hypothesis 2.1(2) implies that $H_\omega^+ = H_{R\omega}^-$ and also that the support of $\mathbb{P}$ is invariant under R. Therefore we have

$$\sigma(H_\omega^+) = \sigma(H_{R\omega}^+) = \sigma(H_\omega^-), \quad \text{a.e. } \omega,$$

proving the result. □

We consider $H_\omega^{\pm}$ as in eq. (1), their spectral measures $E_{H_\omega^{\pm}}$ and define the measure ρ as

$$\rho = \mathbb{E}(\langle \delta_0, E_{H_\omega^+}(\cdot) E_{H_\omega^-}(\cdot)\delta_0\rangle) \tag{7}$$

on $\mathbb{R}^2$. Let

$$\mathcal{I} = \mathbb{R} \cup \{(a,b]\colon a,b \in \mathbb{R}\} \cup \{(a,\infty)\colon a \in \mathbb{R}\} \cup \{(-\infty,a]\colon a \in \mathbb{R}\}.$$

This collection of sets forms a boolean semi-algebra on $\mathbb{R}$. We then consider the boolean semi-algebra $\mathcal{I} \times \mathcal{I}$ and there define the set function ρ by

$$\rho(\cup_{i=1}^k I_i \times J_i) = \sum_{i=1}^k \mathbb{E}(\langle \delta_0, E_{H_\omega^+}(I_i) E_{H_\omega^-}(J_i)\delta_0\rangle),\ I_i, J_i \in \mathcal{I},$$

where the $\{I_i \times J_i\ i = 1, \ldots, k\}$ are mutually disjoint rectangles. Then this ρ, takes values in [0, 1] and satisfies $\rho(\mathbb{R} \times \mathbb{R}) = 1$. The positivity of ρ follows from Proposition 2.5(2), and since intersection of rectangles of the form considered are again rectangles of the same form, ρ is also seen to be well defined. Hence it extends to a unique probability measure on the boolean algebra generated by $\mathcal{I} \times \mathcal{I}$ (see Exercises 1.4.4–1.4.6 and Proposition 1.4.7 of [19]). The unique extension of this to a probability measure on the Borel σ-algebra on $\mathbb{R}^2$ is again standard (see Proposition 2.5.1 of [19]).

PROPOSITION 2.5

Consider the operators $H_\omega^{\pm}$, with $\omega \in \operatorname{supp}(\mathbb{P})$ and let ρ be as in eq. (7). *Then for any Borel subsets B, C of $\mathbb{R}$,*

(1) $\rho(B \times C) = \mathbb{E}(\langle \delta_0, E_{H_\omega^-}(C) E_{H_\omega^+}(B)\delta_0\rangle)$,
(2) $\rho(B \times C) = \mathbb{E}(\langle \delta_0, E_{H_\omega^-}(C) E_{H_\omega^+}(B) E_{H_\omega^-}(C)\delta_0\rangle)$,
(3) $\rho(B \times C) = \mathbb{E}(\langle \delta_0, E_{H_\omega^+}(B) E_{H_\omega^-}(C) E_{H_\omega^+}(B)\delta_0\rangle)$,
(4) *The following inequalities are valid*:

$$\rho(B \times C) \le n_+(B),\ \rho(B \times C) \le n_-(C).$$

Proof. When we consider operators $H_\omega^{\pm}$ satisfying Hypothesis 2.1, they form a covariant family of operators in the sense of Hypothesis 1 of [13] (taking $G = L = \mathbb{Z}^d$). Then the proof of this proposition is the same as that given in Proposition 1 of [13], so we omit it. □

To define the ILAC we need to look at the marginal of the measure ρ along the diagonal directions $\{(\lambda_1, \lambda_2)\colon \lambda_1 = \pm\lambda_2\}$. We rotate the co-ordinate axes of $\mathbb{R}^2$ so that these directions form the co-ordinate axes and to enable this we define the rotation T and look at the measure ρ from this new perspective.

The marginals of $\rho \circ T$ along the co-ordinate axes are precisely the marginals of ρ along the diagonal directions.

Let T be a transformation from $\mathbb{R}^2$ to itself given by the matrix

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Then T is an orthogonal matrix with $T = T$ and we have

$$T\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \frac{\lambda_1+\lambda_2}{\sqrt{2}} \\ \frac{\lambda_1-\lambda_2}{\sqrt{2}} \end{pmatrix}.$$

Using these we define the following.

DEFINITION 2.6

We consider the measure ρ defined in eq. (7) and set the asymmetric ILAC as

$$A_{\rm as}(\lambda) = \nu((-\infty, \lambda]), \quad \text{where } \nu(B) = \rho \circ T(B \times \mathbb{R}) \tag{8}$$

and the symmetric ILAC as

$$A_{\rm s}(\lambda) = \sigma((-\infty, \lambda]), \quad \text{where}$$
$$\sigma(B) = \frac{1}{2}(\rho \circ T(B \times \mathbb{R}) + \rho \circ T(\mathbb{R} \times B)). \tag{9}$$

In the above definitions and in our model we have dropped all the physical constants and also have dropped the band gap $E_{\rm g}$ that customarily appears in the definition since they play no role in the regularity properties as seen in the proofs of our theorems.

The reason we consider a symmetrized version of ILAC is that, in the case of disordered models where the spectrum is symmetric about 0, if λ is in the spectrum then $-\lambda$ is also in the spectrum. Therefore given a E we can have $\lambda^+ + \lambda^- = E$ and also $\lambda_+ - (\tilde{\lambda}_-) = E$ (of course, $\tilde{\lambda}_-$ would be $-\lambda_-$). Therefore in the definition of the finite distribution functions in eq. (4) we could also have taken the sum over $\lambda_+ - \lambda_- \leq E$. The distribution functions, however differ for these two different definitions. Therefore it might be more meaningful to take a symmetric definition.

3. Regularity properties

In this section we show a regularity of a symmetrized ILAC. The idea behind the proofs is the following. The ILAC is the distribution function of the marginal of a two-dimensional measure taken along the principal diagonal, while the marginals along the co-ordinate axes are the density of states. This measure ρ itself acquires a part of the regularity of the density of states. However even if ρ is smooth, it is possible for marginals along some directions to have atomic components.

This is possible only if the measure itself has its support (not the topological support) $\Sigma_1 \cup \Sigma_2$, with Σ_1 being a subset of a straight line which is disjoint from Σ_2. If this happens then restricted to the straight line containing Σ_1, the measure must be as regular as the density of states. This is precisely our conclusion.

Lemma 3.1. *Consider $H^{\pm}_{\omega}$ satisfying Hypothesis* 2.1. *Then*, $n_+ = n_-$ *and in this case* ρ *is symmetric*, *i.e.* $\rho(A \times B) = \rho(B \times A)$.

Proof. The hypothesis 2.1(2) says that for any integrable function f, $\mathbb{E}(f(\omega)) = \mathbb{E}(f(R\omega))$. Therefore taking $f(\omega) = \langle \delta_0, E_{H^+_{\omega}}(B)\delta_0\rangle$, for a fixed Borel set B, we see that it is integrable and satisfies $f(R\omega) = \langle \delta_0, E_{H^-_{\omega}}(B)\delta_0\rangle$. Therefore $n_+(B) = \mathbb{E}(f(\omega)) = \mathbb{E}(f(R\omega)) = n_-(B)$, this being valid for any Borel set B and the measures n_+ and n_- agree.

The symmetry of ρ follows from the following equalities, using the invariance of $\mathbb{P}$ under R.

$$\begin{aligned}\rho(B \times C) &= \mathbb{E}(\langle \delta_0, E_{H^+_{\omega}}(B)E_{H^-_{\omega}}(C)\delta_0\rangle)\\ &= \mathbb{E}(\langle \delta_0, E_{H^-_{R\omega}}(B)E_{H^+_{R\omega}}(C)\delta_0\rangle)\\ &= \mathbb{E}(\langle \delta_0, E_{H^-_{\omega}}(B)E_{H^+_{\omega}}(C)\delta_0\rangle) = \rho(C \times B). \end{aligned} \tag{10}$$

In the following we shall denote the marginals of $\rho \circ T$ by

$$\nu_1 = \frac{1}{2}\rho \circ T(B \times \mathbb{R}), \nu_2 = \frac{1}{2}\rho \circ T(\mathbb{R} \times B). \tag{11}$$

Then, clearly

$$A_{\mathrm{as}}(\lambda) = 2\nu_1((-\infty, \lambda]) \quad \text{and} \quad A_{\mathrm{s}}(\lambda) = (\nu_1 + \nu_2)((-\infty, \lambda]). \tag{12}$$

□

Remark 3.2. The measure ρ is quite nice and we can say more about it. We shall denote by $B_a(x)$ a ball of radius a with centre $x \in \mathbb{R}^2$. We denote by κ the marginal $\rho(\cdot \times \mathbb{R})$ and note that $\kappa = n_+$. In the case when $\rho(B \times \mathbb{R}) = \rho(\mathbb{R} \times B)$ for all Borel B, then we have $\kappa(B) = n_+(B) = n_-(B)$, from the definitions of $n_{\pm}$, ρ and κ.

DEFINITION 3.3

Given a probability measure μ and a bounded continuous function h on $[r, \infty)$, positive on (r, ∞) and vanishing at r, we say that μ has modulus of continuity h at a point x if

$$\limsup_{a>0} \frac{\mu(x-a, x+a)}{h(a+r)} < \infty.$$

We say that μ is uniformly h-continuous if the above condition is valid independent of x.

Examples 3.4.

(1) Let $r = 0$, $h(x) = x^{\alpha}, 0 \leq x \leq 1, h(x) = 1, x > 1$ for some $0 < \alpha \leq 1$. Then h-continuity of μ for this h is called α-Hölder continuity.
(2) If $r = 1$ and $h(x) = |(\ln(x))^{-\alpha}|, 0 \leq x \leq 1/2$ and some positive bounded continuous function on $(1/2, \infty)$, then h-continuity for this h is called α-log Hölder continuity.
(3) Let $r = 0$. Let $h(a) = \tau((y-a, y+a))$, $y \in \mathbb{R}$, for a probability τ, then h-continuity with this h means the modulus of continuity of μ at x is the same as that of τ at y.

Remark 3.5. In the theorems below we will only present the case when $r = 0$. The theorems easily follow even when we take $r \neq 0$ by taking $h(\cdot + r)$ to replace h.

Theorem 3.6. *Consider $H_\omega^\pm$ satisfying Hypothesis* 2.1. *Suppose the density of states $n = n_+ = n_-$ is uniformly h-continuous for some h as in Definition* 3.3 *with $r = 0$. Then, if $B_a(x)$ is a ball of radius a centred at x,*

$$\limsup_{a \to 0} \frac{\rho(B_a(x))}{h(a)} < C, \quad \textit{for all } x.$$

Proof. We consider the function $\psi(x) = \frac{1}{1+\|x\|^2}$, $x \in \mathbb{R}^2$, where $\|x\|^2 = x_1^2 + x_2^2$, $x = (x_1, x_2)$. Then ψ is integrable with respect to the probability measure ρ on $\mathbb{R}^2$. This ψ satisfies $\psi(x) \geq \frac{1}{2}$, whenever $\|x\| \leq 1$. So taking $\delta = 1$ in Theorem 4.1, it is enough to show that

$$\limsup_{a \to 0} \frac{1}{h(a)} \int \psi_a(y - x) \mathrm{d}\rho(y) < \infty.$$

To see this we note that

$$\psi_a(y - x) \leq \frac{1}{\left(1 + \frac{(y_1 - x_1)^2}{a^2}\right)},$$

so that

$$\frac{1}{h(a)} \int \psi_a(y - x) \mathrm{d}\rho(y) \leq \frac{1}{h(a)} \int \frac{1}{\left(1 + \frac{(y_1 - x_1)^2}{a^2}\right)} \mathrm{d}\kappa(y_1)$$

$$= \frac{1}{h(a)} \int \phi_a(y_1 - x_1) \mathrm{d}n(y_1), \tag{13}$$

where we have integrated over the variable y_2 on the right-hand side and used the definition of the measure κ, Remark 3.2 and have taken $\phi(y) = 1/(1+y^2)$, $\phi_a(y) = \phi(y/a)$, $y \in \mathbb{R}$.

Then using Theorem 4.3, we see that the lim sup of the right-hand side is finite for all x_1 once n is uniformly h-continuous. Therefore the lim sup of the left-hand side is finite for all x. □

This theorem shows that ρ has no atoms, that is for any point $x \in \mathbb{R}^2$, $\rho(\{x\}) = 0$. The marginals of ρ along the axes, namely $\rho(A \times \mathbb{R})$ and $\rho(\mathbb{R} \times A)$ both equal the density of states, as seen by using eq. (10) and the fact that $E_{H_\omega^\pm}(\mathbb{R}) = I$, and hence are continuous if the density of states has no atoms.

However it is possible that some marginal taken along other directions in $\mathbb{R}^2$ may have atoms. Consider, for example, a measure on $\mathbb{R}^2$ supported on the y-axis $\{(x_1, x_2) \in \mathbb{R}^2 \colon x_1 = 0\}$, then the marginal of this measure along the x-axis is atomic with an atom at the point 0.

Theorem 3.7. *Consider $H_\omega^\pm$ satisfying Hypothesis* 2.1 *and suppose the density of states n is uniformly h-continuous for some h as in Definition* 3.3.

If ν_1 or ν_2 defined in eq. (11) *has an atom, then the function $A_\mathrm{s}(\lambda)$ defined in eq.* (12) *has a uniformly h-continuous component.*

Proof. Suppose $\rho \circ T(A \times B) \neq 0$ for a pair of Borel subsets A, B of $\mathbb{R}$. Then $\rho \circ T(A \times \cdot)$ and $\rho \circ T(\cdot \times B)$ are both non-trivial finite positive measures on Borel subsets of $\mathbb{R}$. We also note that if $\rho \circ T(A \times \mathbb{R}) \neq 0$, for a given Borel set A, there must be a Borel set $B \subsetneq \mathbb{R}$, such that $\rho \circ T(A \times B) \neq 0$. (Otherwise if $\rho \circ T(A \times C) = 0$ for all Borel $C \subsetneq \mathbb{R}$, then taking any $C \neq \emptyset$, we see that $\rho \circ T(A \times \mathbb{R}) = \rho \circ T(A \times C) + \rho \circ T(A \times C^c) = 0$.) A similar statement is valid when $\rho(\mathbb{R} \times B) \neq 0$.)

Therefore if the marginal $\rho \circ T(A \times \mathbb{R})$ has an atom at a point x, then we can decompose the other marginal measure $\rho \circ T(\mathbb{R} \times B)$ as

$$\rho \circ T(\mathbb{R} \times B) = \rho \circ T((\mathbb{R} \backslash \{x\}) \times B) + \rho \circ T(\{x\} \times B).$$

Let S_ρ denote a finite subset of the set of atoms of $\rho \circ T(A \times \mathbb{R})$. Then we can write

$$\begin{aligned} \rho \circ T(\mathbb{R} \times B) &= \rho \circ T(\mathbb{R} \backslash S_\rho \times B) + \rho \circ T(S_\rho \times B) \\ &= \rho_1(B) + \rho_2(B). \end{aligned} \tag{14}$$

Similarly if S_σ is some finite subset of the set of atoms of $\rho \circ T(\mathbb{R} \times B)$, then we can write

$$\begin{aligned} \rho \circ T(A \times \mathbb{R}) &= \rho \circ T(A \times \mathbb{R} \backslash S_\sigma) + \rho \circ T(A \times S_\sigma) \\ &= \sigma_1(A) + \sigma_2(A). \end{aligned} \tag{15}$$

We have for each A, B, the following relations, which is easy to see from the above argument.

$$\begin{aligned} \rho_2(B) &= \sum_{x \in S_\rho} \rho \circ T(\{x\} \times B), \\ \sigma_2(A) &= \sum_{x \in S_\sigma} \rho \circ T(A \times \{x\}). \end{aligned} \tag{16}$$

Using the definition of A_{s} from eq. (12), decomposition in equations (15) and (16), we can write

$$\begin{aligned} A_{\mathrm{s}}(\lambda) &= \frac{1}{2}(\rho_1 + \sigma_1)((-\infty, \lambda]) + \frac{1}{2}(\rho_2 + \sigma_2)((-\infty, \lambda]) \\ &= A_{\mathrm{s},1}(\lambda) + A_{\mathrm{s},2}(\lambda), \end{aligned} \tag{17}$$

where $A_{\mathrm{s},1}$, $A_{\mathrm{s},2}$ are non-zero functions, as seen by the preceding arguments.

Now the result follows from Lemma 3.8 below. □

Lemma 3.8. *Assume the conditions of Theorem* 3.7. *Consider the function $A_{\mathrm{s},2}$ defined in eq.* (17). *If the density of states n is uniformly h-continuous for some h* (*as in Definition* 3.3), *then $A_{\mathrm{s},2}$ is uniformly h-continuous for the same h.*

Proof. We will prove that ρ_2 is uniformly h-continuous, the proof for σ_2 is similar. From these two statements the uniform h-continuity of $A_{\mathrm{s},2}(\lambda)$ is clear. Let the cardinality of S_ρ be N and let $E_1, \ldots, E_N$ be the elements of S_ρ. Then

$$\begin{aligned}
&\frac{1}{h(a)}\int \phi_a(y-x)\mathrm{d}\rho_2(y)\\
&\quad=\sum_{j=1}^{N}\frac{1}{h(a)}\int \phi_a(y-x)\mathrm{d}\rho\circ T(E_j,y)\\
&\quad=\sum_{j=1}^{N}\frac{1}{h(a)}\int_{T(\{E_j\}\times\mathbb{R})}\phi_a(w-(x+E_j))\mathrm{d}\rho\left(\frac{w}{\sqrt{2}},\frac{w-2E_j}{\sqrt{2}}\right). \qquad (18)
\end{aligned}$$

Since $T(\{E_j\}\times\mathbb{R})\subset\mathbb{R}\times\mathbb{R}$, the right-hand side is bounded by

$$\begin{aligned}
&\sum_{j=1}^{N}\frac{1}{h(a)}\int_{\mathbb{R}\times\mathbb{R}}\phi_a(w-(x+E_j))\mathrm{d}\rho\left(\frac{w}{\sqrt{2}},\frac{z}{\sqrt{2}}\right)\\
&\quad=\sum_{j=1}^{N}\frac{1}{h(a)}\int_{\mathbb{R}}\phi_a(\sqrt{2}w-(x+E_j))\mathrm{d}n(w),
\end{aligned}$$

where we used the fact that $\rho(\cdot\times\mathbb{R})=n(\cdot)$. The uniform h-continuity of the density of states n shows that the right-hand side is bounded uniformly in x, proving the lemma. □

4. Appendix

We present here some results that we use in the main part, whose proofs are essentially available elsewhere.

We have an abstract theorem that extends the theorem of Jensen–Krishna in [9]. In the following, let $(X,\|\cdot\|)$ be a normed vector space over complex numbers and ρ a probability measure on X with respect to the Borel σ-algebra. Denote by $B_a(x)$ the ball with centre x of radius a. Let ψ be a positive bounded continuous function on X taking value 1 at 0. Denote by $\psi_a(x)=\psi(x/a)$, $a>0$.

Theorem 4.1. *Let h be a function as in Definition* 3.3. *Suppose*

$$\limsup_{a>0}\frac{1}{h(a)}\int\psi_a(y-x)\mathrm{d}\rho(y)<\infty.$$

Then, there are constants C, $\delta>0$, depending upon ψ, such that

$$\rho(B_{a\delta}(x))\le Ch(a), a>0, x\in X.$$

Proof. Since ψ is continuous and is 1 at 0, there is a $\delta>0$ such that $\psi(y)\ge\frac{1}{2}$, whenever $\|x\|<\delta$. So we have

$$\frac{1}{h(a)}\int\psi_a(y-x)\mathrm{d}\rho(y)\ge\frac{1}{2h(a)}\rho(B_{a\delta}(x)).$$

We have used the fact that $\|x\|/a\le\delta\iff\|x\|\le a\delta$. Taking sup first on the left-hand side, which is finite since the lim sup of the left-hand side is finite by assumption, and then taking sup over a, for a fixed δ on the right-hand side shows that the right-hand side is finite for all x. □

Consider a function ψ satisfying:

Hypothesis 4.2. Let ψ be a continuous function on $\mathbb{R}$ with $\psi(0) = 1$ and $A_\psi = \int \psi(x)\mathrm{d}x \neq 0$. Further assume that

(1) ψ is bounded and positive.
(2) ψ is differentiable, even and satisfies

$$|\psi(x)| + |x\psi'(x)| \leq \langle x\rangle^{-\delta}, \quad \text{for some} \quad \delta > 1,$$

where $\langle x\rangle = (1 + x^2)^{1/2}$.
(3) Let h be as in Definition 3.3. Let $K(y) = \sup_{0<a<1} \left|\frac{h(ay)}{h(a)}\right|$, $y > 0$. Then $\int K(y)$ $|\psi'(y)|\mathrm{d}y < \infty$.

In most cases the assumption (3) on ψ above follows from (2), but we include it for generality.

We set, given a ψ,

$$C^h_{\mu,\psi}(x) = \limsup_{a>0} \frac{\psi_a * \mu}{h(\delta a)}(x),\ D^h_{\mu,\psi}(x) = \limsup_{a>0} \frac{\mu((x-a, x+a))}{h(a)}(x).$$

Theorem 4.3. *Let μ be a probability measure and let ψ satisfy Hypothesis* 4.2. *Then $C^h_{\mu,\psi}$ is finite for any x, iff $D^h_\mu(x)$ is finite for the same x.*

Proof. We note that as in eq. (1.3.4) [9], we have by integration by parts,

$$\frac{1}{h(a)}\psi_a * \mu(x) = -\frac{1}{h(a)} \int_0^\infty \psi'(y)\frac{h(ay)}{h(a)}\frac{\Phi_\mu(x+ay) - \Phi_\mu(x-ay)}{h(ay)}\mathrm{d}y,$$

where Φ_μ is the distribution function of μ. Then taking lim sup as $a \to 0$, using the condition that $K(y)\psi'(y)$ is integrable by assumption we see that the finiteness of $D^h_{\mu,\psi}(x)$ implies that of $C^h_{\mu,\psi}(x)$.

To see the other direction, note that since ψ is a positive continuous function on $\mathbb{R}$, it attains a positive minimum on $[-1, 1]$, say β. Then we have the estimate

$$\frac{1}{h(a)}\psi_a * \mu(x) \geq \frac{1}{h(a)} \int_{-a}^{a} \psi_a(y)\mathrm{d}\mu(x+y) \geq \beta\frac{\mu((x-a, x+a))}{h(a)},$$

from which we conclude that finiteness of $C^h_{\mu,\psi}(x)$ implies that of $D^h_{\mu,\psi}(x)$. □

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References

[1] Atoyan M S, Kazaryan E M and Sarkisyan H A, Interband light absorption in parabolic quantum dot in the presence of electrical and magnetic fields, *Physica* **E31** (2006) 83–85

[2] Avila A and Damanik D, Absolute continuity of the integrated density of states for the almost Mathieu operator with non-critical coupling, *Invent. Math.* **172** (2008) 439–453
[3] Balkanski M and Wallis R F, Semiconductor Physics and Applications (Oxford University Press) (2000)
[4] Bellissard J, van Elst A and Schulz-Baldes H, The non commutative geometry of the quantum Hall effect, Topology and physics, *J. Math. Phys.* **35(10)** (1994) 5373–5451
[5] Bouclet Jean-Marc, Germinet Francois, Klein Abel and Schenker Jeffrey H, Linear response theory for magnetic Schrödinger operators in disordered media, *J. Funct. Anal.* **226(2)** (2005) 301–372
[6] Carmona R and Lacroix J, Spectral Theory of Random Schrödinger Operators (Boston: Birkhäuser Verlag) (1990)
[7] Cycone H, Froese R, Kirsch W and Simon B, Schrödinger Operators, Texts and Monographs in Physics (Springer Verlag) (1985)
[8] Combes J M, Hislop P and Klopp F, An optimal Wegner estimate and its application to the global continuity of the integrated density of states for random Schrödinger operators, *Duke Math. J.* **140** (2007) 469–498
[9] Demuth M and Krishna M, Determining Spectra in Quantum Theory, Progress in Mathematical Physics, vol. 44 (Boston: Birkhauser) (2005)
[10] Efros A L, Density of states and interband absorption of light in strongly doped semiconductors, *Soviet Phys. Usp.* **16(6)** (1974) 789–805
[11] Figotin A and Pastur L, Spectra of random and almost-periodic operators (Berlin: Springer Verlag) (1992)
[12] Kirsch W and Metzger B, The integrated density of states for random Schrodinger operators, Spectral theory and mathematical physics: a Festschrift in honor of Barry Simon's 60th birthday, *Proc. Sympos. Pure Math.* **76** (2007) 649–696
[13] Kirsch W and Krishna M, Lifshitz Tails for the Interband Light Absorption Coefficient, Preprint (2009)
[14] Kirsch W and Pastur L A, The large-time asymptotics of some Wiener integrals and the interband light absorption coefficient in the deep fluctuation spectrum, *Comm. Math. Phys.* **132** (1990) 365–382
[15] Kirsch W and Pastur L A, The interband light absorption coefficient in the weak disorder regime: an asymptotically exactly solvable model, *J. Phys.* **A27** (1994) 2527–2543
[16] Kirsch W, Pastur L A and Stork H, Asymptotics of the interband light absorption coefficient near the band edge for an alloy-type model, *J. Statist. Phys.* **92** (1998) 1173–1191
[17] Khoruzenko B, Kirsch W and Pastur L A, The interband light absorption coefficient in the weak disorder regime: An asymptotically exactly solvable model, *J. Phys.* **A27** (1994) 2527–2543
[18] Krishna M, Continuity of integrated density of states – independent randomness, *Proc. Ind. Acad. Sci. (Math. Sci.)* **173** (2007) 401–410
[19] Parthasarathy K R, Introduction to Probability and Measure, Texts and Readings in Mathematics, Vol. 33 (New Delhi: Hindustan Book Agency) (2005)
[20] Reed M and Simon B, Methods of Modern Mathematical Physics, Functional Analysis (Academic Press) (1972)
[21] Veselic Ivan, Existence and Regularity Properties of the Integrated Density of States of Random Schrödinger Operators, Lecture Notes in Mathematics No. 1917 (Springer Verlag) (2008)