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The Atiyah bundle and connections on a principal bundle

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Abstract. Let M be a C^∞ manifold and G a Lie a group. Let E_G be a C^∞ principal G-bundle over M. There is a fiber bundle $\mathcal{C}(E_G)$ over M whose smooth sections correspond to the connections on E_G. The pull back of E_G to $\mathcal{C}(E_G)$ has a tautological connection. We investigate the curvature of this tautological connection.

Keywords. Principal bundle; connection; Atiyah bundle.

1. Introduction

Fix a Lie group G. Its Lie algebra will be denoted by $\mathfrak{g}$. Let M be a connected C^∞ manifold. Let $E_G \longrightarrow M$ be a C^∞ principal G-bundle. The adjoint vector bundle, which is the one associated to E_G for the adjoint action of G on $\mathfrak{g}$, is denoted by $\mathrm{ad}(E_G)$. The sheaf of G-invariant smooth vector fields on E_G defines a C^∞ vector bundle over M of rank $\dim M + \dim G$. This vector bundle is known as the Atiyah bundle, and is denoted by $\mathrm{At}(E_G)$. The Atiyah bundle fits in a short exact sequence of vector bundles over M,

$$0 \longrightarrow \mathrm{ad}(E_G) \longrightarrow \mathrm{At}(E_G) \longrightarrow TM \longrightarrow 0,$$

which is a known as the Atiyah exact sequence. A connection on the principal G-bundle E_G is a C^∞ splitting of the Atiyah exact sequence.

The sheaf of splittings of the Atiyah exact sequence defines a fiber bundle

$$\delta\colon \mathcal{C}(E_G) \longrightarrow M \tag{1.1}$$

(see §3). The pulled back principal G-bundle $\delta^* E_G \longrightarrow \mathcal{C}(E_G)$ has a tautological connection (see §3). This connection is denoted by D_0. The curvature of D_0 is a two-form on $\mathcal{C}(E_G)$ with values in $\delta^*\mathrm{ad}(E_G)$. We investigate the curvature of D_0.

Let $\mathcal{E}_G \longrightarrow \mathcal{B}_G$ be the universal principal G-bundle. Let

$$\mathcal{C}(\mathcal{E}_G) \longrightarrow \mathcal{B}_G$$

be the fiber bundle constructed as in (1.1) for the universal principal G-bundle. In a work in progress, we hope to show that the universal G-connection can be realized as a fiber bundle over $\mathcal{C}(\mathcal{E}_G)$. Turning this around, we hope to get an alternative construction of the universal G-connection. Also, this approach may yield a better understanding of the universal G-connection.

2. The Atiyah bundle

As before, G is a Lie group with Lie algebra $\mathfrak{g}$, and M is a connected C^∞ manifold. Let

$$p\colon E_G \longrightarrow M \tag{2.1}$$

be a C^∞ principal G-bundle over M. We recall that this means that E_G is a C^∞ manifold equipped with a C^∞ action

$$\psi\colon E_G \times G \longrightarrow E_G \tag{2.2}$$

of G, satisfying the following two conditions:

- $p \circ \psi$ coincides with $p \circ p_1$, where p_1 is the natural projection of $E_G \times G$ to E_G, and
- the map to the fiber product over M,

  $$(\psi, p_1)\colon E_G \times G \longrightarrow E_G \times_M E_G$$

  is a diffeomorphism. (We recall that $E_G \times_M E_G$ is the submanifold of $E_G \times E_G$ consisting of all points (x, y) with $p(x) = p(y)$.)

(See [4, 5].)

Let $\mathcal{F}C^\infty(M)$ denote the sheaf of all locally defined C^∞ functions on M. In other words, for any nonempty open subset $U \subset M$,

$$\mathcal{F}C^\infty(U) = C^\infty(U)$$

is the space of all C^∞ functions on U. Using E_G we will define a sheaf on M which is locally free over $\mathcal{F}C^\infty(M)$.

For any nonempty open subset $U \subset M$, let

$$\mathcal{A}(E_G)(U) := \Gamma(p^{-1}(U), Tp^{-1}(U))^G$$

be the space of all G-invariant vector fields on $p^{-1}(U)$ (the action of G on $p^{-1}(U)$ is given by ψ in (2.2)). Note that $C^\infty(U)$ acts on $\mathcal{A}(E_G)(U)$. The action of any smooth function f on U sends a vector field θ to $(f \circ p) \cdot \theta$, where p is the projection in (2.1).

Take any point $x \in M$. Let $U \subset M$ be an open subset containing x such that

- the restriction $E_G|_U$ of E_G to U is trivializable, and
- the tangent bundle TU is trivializable.

Fix a trivialization of $E_G|_U$ by choosing a C^∞ isomorphism of principal G-bundles

$$\beta\colon U \times G \longrightarrow E_G|_U. \tag{2.3}$$

Also, fix a trivialization of the tangent bundle of U

$$\gamma\colon TU \longrightarrow U \times \mathbb{R}^d, \tag{2.4}$$

where d is the dimension of M. Let

$$\theta\colon p^{-1}(U) \longrightarrow Tp^{-1}(U)$$

be any G invariant C^∞ vector field on $p^{-1}(U)$. Using the trivialization β in (2.3), the vector field θ gives a vector field on $U \times G$. Let

$$\tilde{\theta}\colon U \times G \longrightarrow T(U \times G) \tag{2.5}$$

be the vector field defined by θ. Note that using the left invariant vector fields, we have

$$TG = G \times \mathfrak{g},$$

where $\mathfrak{g}$ is the Lie algebra of G. Therefore, using the trivialization γ in (2.4), we have

$$T(U \times G) = (U \times G) \times (\mathfrak{g} \oplus \mathbb{R}^d).$$

Consequently, the restriction of $\tilde{\theta}$ (see (2.5)) to $U \times \{e\} \subset U \times G$, where $e \in G$ is the identity element, defines a C^∞ function

$$\tilde{\theta}'\colon U \longrightarrow \mathfrak{g} \oplus \mathbb{R}^d.$$

Conversely, given any C^∞ function

$$\eta_0\colon U \longrightarrow \mathfrak{g} \oplus \mathbb{R}^d,$$

using γ (see (2.4)), the function η_0 defines a smooth G-invariant vector field on $U \times G$. Now, using the trivialization β in (2.3), this G-invariant vector field on $U \times G$ produces a G-invariant vector field on $E_G|_U$. In other words, we get a bijective linear map between $\mathcal{A}(E_G)(U)$ (the space of smooth G-invariant vector fields on $E_G|_U$) and $\Gamma(U, \mathfrak{g} \oplus \mathbb{R}^d)$ (the space of smooth maps from U to $\mathfrak{g} \oplus \mathbb{R}^d$). This map

$$\mathcal{A}(E_G)(U) \longrightarrow \Gamma(U, \mathfrak{g} \oplus \mathbb{R}^d)$$

clearly commutes with the actions of $C^\infty(U)$ on $\mathcal{A}(E_G)(U)$ and $\Gamma(U, \mathfrak{g} \oplus \mathbb{R}^d)$ defined by multiplication.

Therefore, the sheaf $\mathcal{A}(E_G)$ is locally free over $\mathcal{F}C^\infty(M)$ of rank $\dim(\mathfrak{g} \oplus \mathbb{R}^d)$. Hence $\mathcal{A}(E_G)$ defines a C^∞ vector bundle over M of rank $\dim(\mathfrak{g} \oplus \mathbb{R}^d)$. Let

$$\mathrm{At}(E_G) \longrightarrow M \tag{2.6}$$

be the vector bundle of rank $\dim(\mathfrak{g} \oplus \mathbb{R}^d)$ defined by the sheaf $\mathcal{A}(E_G)$.

The vector bundle $\mathrm{At}(E_G)$ in (2.6) is known as the *Atiyah bundle* for E_G (see [1]).

There is a natural homomorphism of vector bundles

$$\mu\colon p^*\mathrm{At}(E_G) \longrightarrow TE_G \tag{2.7}$$

which we will describe. Take any point $x \in M$. Fix a point

$$y \in p^{-1}(x) \subset E_G.$$

The homomorphism $\mu(y)$ sends a G-invariant vector field ξ, defined around $p^{-1}(x) \subset E_G$, to the evaluation $\xi(y) \in T_y E_G$ of ξ at y. It is easy to see that μ is a C^∞ isomorphism of vector bundles.

Let

$$\mathrm{ad}(E_G) \longrightarrow M \tag{2.8}$$

be the adjoint vector bundle of E_G. We recall that $\mathrm{ad}(E_G)$ is the quotient of $E_G \times \mathfrak{g}$ constructed using the adjoint action of G on its own Lie algebra $\mathfrak{g}$. More precisely, two points (z, v) and (z', v') of $E_G \times \mathfrak{g}$ are identified in $\mathrm{ad}(E_G)$ if there is some $g_0 \in G$ such that $z' = zg_0$ and $v' = \mathrm{Ad}(g_0^{-1})(v)$ (we recall that $\mathrm{Ad}(g_0^{-1})(v)$ is the Lie algebra automorphism of $\mathfrak{g}$ corresponding to the automorphism of G defined by $\zeta \longmapsto g_0^{-1}\zeta g_0$). Since $\mathrm{Ad}(h)$ is a Lie algebra automorphism of $\mathfrak{g}$ for all $h \in G$, each fiber of $\mathrm{ad}(E_G)$ is a Lie algebra isomorphic to $\mathfrak{g}$.

Lemma 2.1. *The vector bundle* $\mathrm{ad}(E_G) \longrightarrow M$ *is identified with the subbundle of the Atiyah bundle* $\mathrm{At}(E_G)$ *defined by the sheaf of* G*-invariant vector fields on* E_G *that lie in the kernel of the differential*

$$\mathrm{d}p\colon TE_G \longrightarrow p^*TM \tag{2.9}$$

of the projection p *in* (2.1).

Proof. Take any point $(z, v) \in E_G \times \mathfrak{g}$. Let

$$\phi(z, v)\colon p^{-1}(p(z)) \longrightarrow Tp^{-1}(p(z))$$

be the smooth vector field on the fiber $p^{-1}(p(z)) \subset E_G$ that sends any

$$z' = zg \in p^{-1}(p(z)),$$

where $g \in G$, to the element in $T_{z'}p^{-1}(p(z))$ given by the curve $\mathbb{R} \longrightarrow E_G$, based at the point z', defined by $t \longmapsto z\exp(tv)g$.

It is straightforward to check that for any $g_0 \in G$, the two vector fields $\phi(z, v)$ and $\phi(zg_0, \mathrm{Ad}(g_0^{-1})(v))$ on $p^{-1}(p(z))$ coincide. Consequently, for any $x \in M$, we get a homomorphism

$$\tilde{\phi}(x)\colon \mathrm{ad}(E_G)_x \longrightarrow \Gamma(p^{-1}(x), Tp^{-1}(x))^G \subset \Gamma(p^{-1}(x), (TE_G)|_{p^{-1}(x)})^G$$

from the fiber $\mathrm{ad}(E_G)_x$ of $\mathrm{ad}(E_G)$ over x to the space of all G-invariant vector fields on the fiber $p^{-1}(x)$. This homomorphism $\tilde{\phi}(x)$ is clearly injective. Consequently, we get a C^∞ injective homomorphism of vector bundles

$$\tilde{\phi}\colon \mathrm{ad}(E_G) \longrightarrow \mathrm{At}(E_G)$$

defined by

$$x \longmapsto \tilde{\phi}(x) \in \Gamma(p^{-1}(x), (TE_G)|_{p^{-1}(x)})^G = \mathrm{At}(E_G)_x.$$

Clearly, we have $dp \circ \tilde{\phi} = 0$. Also,

$$\mathrm{rank}(\mathrm{ad}(E_G)) = \dim \mathfrak{g} = (\dim M + \dim \mathfrak{g}) - \dim M = \mathrm{rank}(\mathrm{At}(E_G)) - \mathrm{rank}(TM).$$

Hence $\tilde{\phi}$ identifies $\mathrm{ad}(E_G)$ with the kernel of the differential $\mathrm{d}p$ in the statement of the lemma. This completes the proof of the lemma. ■

Using Lemma 2.1, we have a short exact sequence of vector bundles over M,

$$0 \longrightarrow \mathrm{ad}(E_G) \xrightarrow{\iota_0} \mathrm{At}(E_G) \xrightarrow{\eta} TM \longrightarrow 0, \tag{2.10}$$

where the projection η is given by dp in (2.9). This exact sequence of vector bundles is known as the *Atiyah exact sequence* for E_G.

A *connection* on E_G is a C^∞ splitting of the Atiyah exact sequence for E_G. In other words, a connection on E_G is a C^∞ homomorphism of vector bundles

$$D\colon TM \longrightarrow \mathrm{At}(E_G) \tag{2.11}$$

such that $\eta \circ D = \mathrm{Id}_{TM}$, where η is the projection in (2.10).

Let

$$D\colon TM \longrightarrow \mathrm{At}(E_G) \tag{2.12}$$

be a homomorphism defining a connection on E_G. Consider the composition homomorphism

$$p^*TM \xrightarrow{D} p^*\mathrm{At}(E_G) \xrightarrow{\mu} TE_G,$$

where μ is the isomorphism in (2.7). Its image

$$\mathcal{H}(D) := (\mu \circ D)(p^*TM) \subset TE_G \tag{2.13}$$

is known as the *horizontal subbundle* of TE_G for the connection D. Since μ is an isomorphism, and the splitting homomorphism D in (2.12) is uniquely determined by its image $D(TM) \subset \mathrm{At}(E_G)$, it follows that the horizontal subbundle $\mathcal{H}(D)$ determines the connection D uniquely.

The quotient bundle

$$Q := TE_G/\mathcal{H}(D) \longrightarrow E_G$$

is the trivial vector bundle $E_G \times \mathfrak{g}$. The identification of Q with $E_G \times \mathfrak{g}$ is given by the action of G on E_G. Note that the projection to Q from the kernel of $\mathrm{d}p$ in (2.9) is an isomorphism. The natural projection

$$TE_G \longrightarrow TE_G/\mathcal{H}(D) = Q = E_G \times \mathfrak{g} \tag{2.14}$$

defines a $\mathfrak{g}$-valued smooth one-form on E_G. This $\mathfrak{g}$-valued one-form on E_G determines D uniquely because it determines $\mathcal{H}(D)$ uniquely.

2.1 *The curvature*

Let s and t be two G-invariant smooth vector fields on $p^{-1}(U)$, where $U \subset M$ is an open subset, and p is the projection in (2.1). Then the Lie bracket $[s, t]$ is also a G-invariant vector field on $p^{-1}(U)$. Consequently, the locally defined smooth sections of the Atiyah bundle $\mathrm{At}(E_G)$ are equipped with the Lie bracket operation. The Lie bracket of the locally defined sections s_1 and t_1 of $\mathrm{At}(E_G)$ will be denoted by $[s_1, t_1]$.

Let $D\colon TM \longrightarrow \mathrm{At}(E_G)$ be a homomorphism defining a connection on E_G. The *curvature* of D, which we will denote by $\mathcal{K}(D)$, measures the failure of D to be Lie bracket preserving.

More precisely, let v_1 and v_2 be two smooth vector fields on some open subset $U \subset M$. Therefore, $D(v_1)$ and $D(v_2)$ are G-invariant smooth vector fields on $p^{-1}(U)$. Now consider

$$\tilde{D}(v_1, v_2) := [D(v_1), D(v_2)] - D([v_1, v_2]), \tag{2.15}$$

which is also a G-invariant vector field on $p^{-1}(U)$. Therefore, $\tilde{D}(v_1, v_2)$ is a smooth section of $\mathrm{At}(E_G)$ over U. Let

$$\tilde{D}'(v_1, v_2)\colon U \longrightarrow \mathrm{At}(E_G)|_U \tag{2.16}$$

be the section of $\mathrm{At}(E_G)$ over U given by $\tilde{D}(v_1, v_2)$ in (2.15).

Consider the projection η in (2.10). Note that for any two smooth sections s and t of $\mathrm{At}(E_G)$ over U, the equality

$$\eta([s, t]) = [\eta(s), \eta(t)]$$

holds. Therefore, for the section $\tilde{D}'(v_1, v_2)$ in (2.16), we have

$$\begin{aligned}\eta(\tilde{D}'(v_1, v_2)) &= \eta([D(v_1), D(v_2)]) - \eta(D([v_1, v_2]))\\ &= [\eta(D(v_1)), \eta(D(v_2))] - \eta(D([v_1, v_2]))\\ &= [v_1, v_2] - [v_1, v_2] = 0 \end{aligned}\tag{2.17}$$

because $\eta \circ D = \mathrm{Id}_{TM}$. Using the Atiyah exact sequence (see (2.10)), from (2.17) we conclude that $\tilde{D}'(v_1, v_2)$ is a section of $\mathrm{ad}(E_G)$ over U. Let

$$\tilde{D}''(v_1, v_2)\colon U \longrightarrow \mathrm{ad}(E_G)|_U \tag{2.18}$$

be the smooth section given by $\tilde{D}'(v_1, v_2)$.

Let f be any smooth function defined on U. We have

$$\begin{aligned}&[D(fv_1), D(v_2)] - D([fv_1, v_2])\\ &\quad = [fD(v_1), D(v_2)] - D(f[v_1, v_2] - v_2(f)\cdot v_1)\\ &\quad = f[D(v_1), D(v_2)] - \eta\circ D(v_2)(f)\cdot D(v_1)\\ &\qquad - f\cdot D([v_1, v_2]) + v_2(f)\cdot D(v_1). \end{aligned}\tag{2.19}$$

Since $\eta \circ D = \mathrm{Id}_{TM}$, the identity in (2.19) implies that

$$\tilde{D}''(fv_1, v_2) = f\cdot \tilde{D}''(v_1, v_2),$$

where $\tilde{D}''$ is constructed in (2.18).

Also, note that $\tilde{D}(v_1, v_2) = -\tilde{D}(v_2, v_1)$, where $\tilde{D}$ is constructed in (2.15). Hence we have

$$\tilde{D}''(v_1, v_2) = -\tilde{D}''(v_2, v_1).$$

Consequently, $\tilde{D}''$ defines a smooth section

$$\mathcal{K}(D)\colon M \longrightarrow \left(\bigwedge^2 T^*M\right)\otimes \mathrm{ad}(E_G), \tag{2.20}$$

which is called the *curvature* of D.

Remark 2.2. We note that $\mathcal{K}(D) = 0$ if and only if D takes the Lie bracket operation on the sections of TM to the bracket operation on the sections of $\mathrm{At}(E_G)$ (see [1, 3].)

3. Sheaf of connections

Tensoring the Atiyah exact sequence (see (2.10)) with the cotangent bundle

$$T^*M \longrightarrow M$$

we get the short exact sequence of vector bundles

$$\begin{aligned} 0 \longrightarrow \mathrm{ad}(E_G) \otimes T^*M \longrightarrow \mathrm{At}(E_G) \otimes T^*M &\stackrel{\eta\otimes \mathrm{Id}_{T^*M}}{\longrightarrow} TM \otimes T^*M \\ &= \mathrm{End}(TM) \longrightarrow 0 \end{aligned} \tag{3.1}$$

over M. Let Id_{TM} denote the identity automorphism of TM. Let

$$\mathcal{C}(E_G) := (\eta \otimes \mathrm{Id}_{T^*M})^{-1}(\mathrm{Id}_{TM}) \subset \mathrm{At}(E_G) \otimes T^*M \longrightarrow M \tag{3.2}$$

be the fiber bundle over M, where $\eta \otimes \mathrm{Id}_{T^*M}$ is the surjective homomorphism in (3.1) (see also [2]).

We recall that a connection on E_G is a splitting of the Atiyah exact sequence. A homomorphism D as in (2.11) with $\eta \circ D = \mathrm{Id}_{TM}$ gives a smooth section of the fiber bundle $\mathcal{C}(E_G) \longrightarrow M$ in (3.2). Conversely, any smooth section of $\mathcal{C}(E_G)$ gives a homomorphism D as in (2.11) with $\eta \circ D = \mathrm{Id}_{TM}$.

Therefore, we have the following lemma:

Lemma 3.1. *The space of connections on E_G is in bijective correspondence with the space of smooth sections of the fiber bundle*

$$\delta\colon \mathcal{C}(E_G) \longrightarrow M$$

constructed in (3.2).

Consider the projection δ in Lemma 3.1. We will show that the principal G-bundle

$$\delta^* E_G \longrightarrow \mathcal{C}(E_G)$$

has a tautological connection.

Let $T\mathcal{C}(E_G)$ be the tangent bundle of the manifold $\mathcal{C}(E_G)$. Let

$$T_\delta \subset T\mathcal{C}(E_G) \tag{3.3}$$

be the relative tangent bundle for the projection δ in Lemma 3.1. In other words, T_δ is the kernel of the differential

$$\mathrm{d}\delta\colon T\mathcal{C}(E_G) \longrightarrow \delta^* TM \tag{3.4}$$

of the smooth map δ. Let

$$\iota\colon T_\delta \hookrightarrow T\mathcal{C}(E_G) \tag{3.5}$$

be the inclusion map.

Let $\mathrm{At}(\delta^* E_G) \longrightarrow \mathcal{C}(E_G)$ be the Atiyah bundle for the principal G-bundle $\delta^* E_G$ over $\mathcal{C}(E_G)$.

PROPOSITION 3.2

There is a natural short exact sequence of vector bundles

$$0 \longrightarrow T_\delta \longrightarrow \mathrm{At}(\delta^* E_G) \longrightarrow \delta^* \mathrm{At}(E_G) \longrightarrow 0$$

over $\mathcal{C}(E_G)$.

Proof. We recall that the total space of the pull back $\delta^* E_G$ coincides with the submanifold of $\mathcal{C}(E_G) \times E_G$ consisting of all points (y_1, y_2) such that $\delta(y_1) = p(y_2)$, where p is the projection in (2.1). In other words, $\delta^* E_G$ is the fiber product $\mathcal{C}(E_G) \times_M E_G$. Therefore, we have a commutative diagram

$$\begin{array}{ccc} \delta^* E_G & \xrightarrow{\phi_0} & E_G \\ \downarrow q & & \downarrow p \\ \mathcal{C}(E_G) & \xrightarrow{\delta} & M \end{array} \tag{3.6}$$

which is Cartesian. Let

$$\mathrm{d}\phi_0 \colon T\delta^* E_G \longrightarrow \phi_0^* T E_G \tag{3.7}$$

be the differential of the projection ϕ_0 in (3.6). Since δ is a submersion, the map ϕ_0 is also a submersion. Hence the kernel

$$\mathcal{W} := \mathrm{kernel}(\mathrm{d}\phi_0) \subset T\delta^* E_G \tag{3.8}$$

is a C^∞ subbundle of the tangent bundle $T\delta^* E_G$. The rank of $\mathcal{W}$ is $\dim M + \dim G$.

Let

$$\mathrm{d}q \colon T\delta^* E_G \longrightarrow q^* T\mathcal{C}(E_G) \tag{3.9}$$

be the differential of the projection q in (3.6). Consider the restriction of $\mathrm{d}q$ to the subbundle $\mathcal{W}$ defined in (3.8). It is easy to see that the image of the homomorphism

$$\mathrm{d}q|_{\mathcal{W}} \colon \mathcal{W} \longrightarrow q^* T\mathcal{C}(E_G)$$

coincides with the subbundle

$$q^* T_\delta \subset q^* T\mathcal{C}(E_G),$$

where T_δ is the relative tangent bundle defined in (3.3). Furthermore, the homomorphism

$$\tilde{q} := \mathrm{d}q|_{\mathcal{W}} \colon \mathcal{W} \longrightarrow q^* T_\delta \tag{3.10}$$

is an isomorphism.

Let $s \colon U \longrightarrow T_\delta$ be a smooth section of T_δ defined over some open subset $U \subset \mathcal{C}(E_G)$. Let

$$q^* s \colon q^{-1}(U) \longrightarrow q^* T_\delta$$

be the pull back of the section s, where q is the projection in (3.6). Let

$$\tilde{q}^{-1} \circ q^* s \colon q^{-1}(U) \longrightarrow \mathcal{W} \tag{3.11}$$

be the section of $\mathcal{W}$, where $\tilde{q}$ is the isomorphism in (3.10). The section $\tilde{q}^{-1} \circ q^* s$ in (3.11) is clearly left invariant by the action of G on the principal G-bundle $\delta^* E_G$. Consequently, the isomorphism $\tilde{q}$ defines an injective homomorphism of vector bundles

$$f_0 \colon T_\delta \longrightarrow \mathrm{At}(\delta^* E_G) \tag{3.12}$$

that sends any section s to $\tilde{q}^{-1} \circ q^* s$.

On the other hand, the differential $\mathrm{d}\phi_0$ in (3.7) induces a surjective homomorphism of vector bundles

$$g \colon \mathrm{At}(\delta^* E_G) \longrightarrow \delta^* \mathrm{At}(E_G) \tag{3.13}$$

by sending a G-invariant vector field on $\delta^* E_G$ to its image by the homomorphism $\mathrm{d}\phi_0$. The kernel of the homomorphism g in (3.13) evidently coincides with the image of the homomorphism f_0 in (3.12). Indeed, this follows immediately from the construction of f_0. Therefore, we get a short exact sequence of vector bundles

$$0 \longrightarrow T_\delta \xrightarrow{f_0} \mathrm{At}(\delta^* E_G) \xrightarrow{g} \delta^* \mathrm{At}(E_G) \longrightarrow 0$$

over $\mathcal{C}(E_G)$. This completes the proof of the proposition. ■

Let $\mathrm{ad}(\delta^* E_G) \longrightarrow \mathcal{C}(E_G)$ be the adjoint vector bundle of the principal G-bundle $\delta^* E_G \longrightarrow \mathcal{C}(E_G)$; see (2.8) for its definition. Clearly, we have

$$\mathrm{ad}(\delta^* E_G) = \delta^* \mathrm{ad}(E_G),$$

where $\mathrm{ad}(E_G) \longrightarrow M$ is the adjoint vector bundle of E_G. Furthermore, we have the following commutative diagram of homomorphisms of vector bundles on $\mathcal{C}(E_G)$:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & T_\delta & = & T_\delta & & \\
 & & & & \downarrow f_0 & & \downarrow \iota & & \\
0 & \longrightarrow & \mathrm{ad}(\delta^* E_G) & \longrightarrow & \mathrm{At}(\delta^* E_G) & \xrightarrow{I} & T\mathcal{C}(E_G) & \longrightarrow & 0, \\
 & & \| & & \downarrow g & & \downarrow d\delta & & \\
0 & \longrightarrow & \delta^* \mathrm{ad}(E_G) & \xrightarrow{\delta^* \iota_0} & \delta^* \mathrm{At}(E_G) & \xrightarrow{\delta^* \eta} & \delta^* TM & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{3.14}$$

where the top short exact sequence is the Atiyah exact sequence for the principal G-bundle $\delta^* E_G$, the bottom exact sequence is the pull back of the Atiyah exact sequence constructed in (2.10), the projection g is constructed in (3.13), the homomorphism f_0 is constructed in (3.12), the homomorphism ι is defined in (3.5), and $\mathrm{d}\delta$ is defined in (3.4). We note that the vertical sequences in (3.14) are also exact.

Now, from the construction of $\mathcal{C}(E_G)$ (see (3.2)) it follows that we have a tautological C^∞ homomorphism of vector bundles

$$\beta_0 \colon \delta^* TM \longrightarrow \delta^* \mathrm{At}(E_G) \tag{3.15}$$

such that $(\delta^*\eta) \circ \beta_0 = \mathrm{Id}_{\delta^* TM}$, where $\delta^*\eta$ is the homomorphism in (3.14). To explain the homomorphism β_0, take any point

$$y \in \mathcal{C}(E_G).$$

Recall that y corresponds to a homomorphism

$$\lambda_y \colon T_{\delta(y)}M \longrightarrow \mathrm{At}(E_G)_{\delta(y)} \tag{3.16}$$

such that $\eta(\delta(y)) \circ \lambda_y = \mathrm{Id}_{T_{\delta(y)}M}$, where δ is the projection in Lemma 3.1, the homomorphism

$$\eta(\delta(y)) \colon \mathrm{At}(E_G)_{\delta(y)} \longrightarrow T_{\delta(y)}M$$

is the projection in (2.10) with $\mathrm{At}(E_G)_{\delta(y)}$ being the fiber of $\mathrm{At}(E_G)$ over the point $\delta(y)$.

The homomorphism β_0 in (3.15) is defined by the condition that for each point $y \in \mathcal{C}(E_G)$, the restriction of β_0 to the fiber $(\delta^*TM)_y$ coincides with the homomorphism λ_y in (3.16).

Let

$$\mathcal{W}_0 := g^{-1}(\beta_0(\delta^*TM)) \subset \mathrm{At}(\delta^*E_G) \tag{3.17}$$

be the subbundle of $\mathrm{At}(\delta^*E_G)$, where g is the projection in (3.14), and β_0 is the homomorphism in (3.15).

Consider the homomorphism I in (3.14). Using Proposition 3.2 and the diagram (3.14) it follows that the restriction

$$I_0 := I|_{\mathcal{W}_0} \colon \mathcal{W}_0 \longrightarrow T\mathcal{C}(E_G) \tag{3.18}$$

is an isomorphism, where $\mathcal{W}_0$ is constructed in (3.17). Therefore, there is an unique C^∞ homomorphism of vector bundles

$$S_0 \colon T\mathcal{C}(E_G) \longrightarrow \mathrm{At}(\delta^*E_G) \tag{3.19}$$

such that

- $S_0(T\mathcal{C}(E_G)) = \mathcal{W}_0 \subset \mathrm{At}(\delta^*E_G)$, and
- $I \circ S_0 = \mathrm{Id}_{T\mathcal{C}(E_G)}$, where I is the homomorphism in (3.14).

Consequently, the homomorphism S_0 in (3.19) gives a C^∞ splitting of the Atiyah exact sequence for δ^*E_G (which is the top exact sequence in (3.14)). Therefore, S_0 defines a connection on the principal G-bundle δ^*E_G.

The connection on the principal G-bundle δ^*E_G defined by S_0 will be denoted by D_0.

We will investigate the above connection D_0 on δ^*E_G. First we will describe the horizontal subbundle of $T\delta^*E_G$ for the connection D_0.

Take any point

$$z \in \delta^*E_G. \tag{3.20}$$

Let $y = q(z) \in \mathcal{C}(E_G)$ be the image of z, where q is the projection in (3.6). Consider the differential $\mathrm{d}\phi_0$ defined in (3.7). Let

$$\mathrm{d}\phi_0(z) \colon T_z\delta^*E_G \longrightarrow T_{\phi_0(z)}E_G$$

be its restriction over the point z in (3.20). The horizontal subspace of $T_z\delta^* E_G$ for the connection D_0 on $\delta^* E_G$ coincides with the inverse image

$$(d\phi_0(z))^{-1}(\mu(\phi_0(z))(\lambda_y(T_{\delta(y)}M))) \subset T_z\delta^* E_G,$$

where λ_y is constructed in (3.16), and

$$\mu(\phi_0(z))\colon \mathrm{At}(E_G)_{\delta(y)} \longrightarrow T_{\phi_0(z)}E_G$$

is the isomorphism in (2.7).

In Lemma 3.1 we noted that the connections on E_G are in bijective correspondence with the sections of $\mathcal{C}(E_G)$. Take any smooth section

$$\chi\colon M \longrightarrow \mathcal{C}(E_G) \tag{3.21}$$

of the fiber bundle $\mathcal{C}(E_G) \longrightarrow M$. Let D_χ be the corresponding connection on the principal G-bundle E_G. We note that $\chi^*\delta^* E_G = E_G$ because $\delta \circ \chi = \mathrm{Id}_M$.

Lemma 3.3. *The connection D_χ on E_G coincides with the pulled back connection $\chi^* D_0$ on the principal G-bundle $\chi^*\delta^* E_G = E_G$.*

Proof. This follows immediately from the construction of the connection D_0 on $\delta^* E_G$. ■

4. The curvature of D_0

Let

$$\mathcal{K}(D_0)\colon \mathcal{C}(E_G) \longrightarrow \left(\bigwedge^2 T^*\mathcal{C}(E_G)\right) \otimes \delta^*\mathrm{ad}(E_G) \tag{4.1}$$

be the curvature of the connection D_0 on the principal G-bundle $\delta^* E_G$. (Note that $\delta^*\mathrm{ad}(E_G) = \mathrm{ad}(\delta^* E_G)$.)

Take any point $x \in M$. Let

$$\mathcal{K}(D_0)^x\colon Z^x := \delta^{-1}(x) \longrightarrow \left(\bigwedge^2 T^* Z^x\right) \otimes \delta^*\mathrm{ad}(E_G) \tag{4.2}$$

be the $\delta^*\mathrm{ad}(E_G)$-valued two form on Z^x obtained by restricting the two-form $\mathcal{K}(D_0)$ in (4.1).

Lemma 4.1. *The two-form $\mathcal{K}(D_0)^x$ in* (4.2) *vanishes identically.*

Proof. Consider the restriction

$$(\delta^* E_G)^x := (\delta^* E_G)|_{\delta^{-1}(x)} \longrightarrow \delta^{-1}(x)$$

of the principal G-bundle $\delta^* E_G$ to the submanifold $\delta^{-1}(x) \subset \mathcal{C}(E_G)$. The connection D_0 on $\delta^* E_G$ defines a connection on $(\delta^* E_G)^x$; this connection on $(\delta^* E_G)^x$ will be denoted by D_0^x. The curvature of D_0^x is $\mathcal{K}(D_0)^x$ defined in (4.2).

From the construction of D_0 it follows that the horizontal subbundle of $T(\delta^* E_G)^x$ for the connection D_0^x on $(\delta^* E_G)^x$ coincides with the subbundle $\mathcal{W}|_{(\delta^* E_G)^x}$, where $\mathcal{W}$ is constructed in (3.8). The subbundle $\mathcal{W} \subset T\delta^* E_G$ is clearly integrable. Therefore, the curvature $\mathcal{K}(D_0)^x$ vanishes; see Remark 2.2. This completes the proof of the lemma. ■

As in (3.21), let $\chi\colon M \longrightarrow \mathcal{C}(E_G)$ be a section of the fiber bundle $\mathcal{C}(E_G)$ over M. As before, let D_χ be the corresponding connection on the principal G-bundle E_G. Then Lemma 3.3 has the following corollary:

COROLLARY 4.2

The curvature of the connection D_χ coincides with the pull back $\chi^*\mathcal{K}(D_0)$.

For any $g_0 \in G$, let

$$\mathrm{Ad}(g_0)\colon G \longrightarrow G$$

be the automorphism defined by $h \longmapsto g_0 h g_0^{-1}$. This defines the adjoint action of G on itself.

Let

$$\varphi\colon \mathrm{Ad}(E_G) := E_G(G) \longrightarrow M \tag{4.3}$$

be the fiber bundle associated to E_G for the adjoint action of G on itself. Therefore, $\mathrm{Ad}(E_G)$ is a quotient of $E_G \times G$, and two points (z, h) and (z', h') of $E_G \times G$ are identified in $\mathrm{Ad}(E_G)$ if there is some $g_0 \in G$ such that $z' = z g_0$ and $h' = \mathrm{Ad}(g_0^{-1})(h)$.

Since the adjoint action of G on itself preserves the group structure of G, each fiber of $\mathrm{Ad}(E_G)$ is a Lie group isomorphic to G. It is easy to see that the Lie algebra bundle over M corresponding to $\mathrm{Ad}(E_G)$ coincides with the adjoint vector bundle $\mathrm{ad}(E_G)$.

Take any point $x \in M$. The fiber $\mathrm{Ad}(E_G)_x$ of $\mathrm{Ad}(E_G)$ over x acts on the fiber $(E_G)_x$ of E_G over x. To explain this action, take any point $(z_0, g_0) \in (E_G)_x \times G$. Let

$$f_{(z_0,g_0)}\colon (E_G)_x \longrightarrow (E_G)_x$$

be the map defined by $z_0 h \longmapsto z_0 g_0 h$, $h \in G$. Note that

$$f_{(z_0,g_0)} = f_{(z_0 h_0, h_0^{-1} g_0 h_0)}$$

for all $h_0 \in G$. Therefore, the map $f_{(z_0,g_0)}$ depends only on the image of (z_0, g_0) in $\mathrm{Ad}(E_G)_x$. In this way we get an action of the group $\mathrm{Ad}(E_G)_x$ on $(E_G)_x$. Note that the diffeomorphism $f_{(z_0,g_0)}$ of $(E_G)_x$ commutes with the action of G on $(E_G)_x$.

Consequently, a smooth section of $\mathrm{Ad}(E_G)$ gives a diffeomorphism of E_G which commutes with the projection p in (2.1). Furthermore, such a diffeomorphism of E_G commutes with the action of G on E_G.

An automorphism of the principal G-bundle E_G is a diffeomorphism

$$f\colon E_G \longrightarrow E_G$$

such that

- $p \circ f = p$, where p is the projection in (2.1), and
- f commutes with the action of G on E_G.

Note that any diffeomorphism of G that commutes with all the right translations of G must be a left translation. Using this it follows that all the automorphisms of the principal G-bundle E_G are given by the smooth sections of $\mathrm{Ad}(E_G)$.

Let

$$\beta\colon M \longrightarrow \mathrm{Ad}(E_G) \tag{4.4}$$

be a smooth section. Let

$$\beta^{-1}\colon M \longrightarrow \mathrm{Ad}(E_G) \tag{4.5}$$

be the section defined by $y \longmapsto \beta(y)^{-1}$. As we noted above, the section β gives an automorphism

$$\tilde{\beta}'\colon E_G \longrightarrow E_G \tag{4.6}$$

of the principal G-bundle E_G.

Given any G-invariant vector field ω on E_G, the pull back of ω by the diffeomorphism $\tilde{\beta}'$ in (4.6) remains G-invariant. Indeed, this follows from the fact that $\tilde{\beta}'$ commutes with the action of G on E_G. Therefore, $\tilde{\beta}'$ gives an automorphism

$$\tilde{\beta}\colon \mathrm{At}(E_G) \longrightarrow \mathrm{At}(E_G) \tag{4.7}$$

of the Atiyah vector bundle.

The automorphism $\tilde{\beta}$ in (4.7) preserves the subbundle $\mathrm{ad}(E_G) \subset \mathrm{At}(E_G)$ in (2.10). The automorphism of $\mathrm{ad}(E_G)$ obtained by restricting $\tilde{\beta}$ coincides with the adjoint action of β on $\mathrm{ad}(E_G)$ (recall that $\mathrm{ad}(E_G)$ is the Lie algebra bundle for the bundle $\mathrm{Ad}(E_G)$ of Lie groups). Also, note that the action of $\tilde{\beta}$ on the quotient bundle $TM = \mathrm{At}(E_G)/\mathrm{ad}(E_G)$ (see (2.10)) is the trivial one.

The automorphism $\tilde{\beta}$ in (4.7) gives a diffeomorphism

$$\beta_0\colon \mathcal{C}(E_G) \longrightarrow \mathcal{C}(E_G) \tag{4.8}$$

of the fiber bundle $\mathcal{C}(E_G)$ in (3.2). More precisely, the automorphism $\tilde{\beta}$ of $\mathrm{At}(E_G)$ and the identity map of T^*M together define an automorphism of $\mathrm{At}(E_G) \otimes T^*M$. This automorphism of $\mathrm{At}(E_G) \otimes T^*M$ clearly preserve the submanifold $\mathcal{C}(E_G)$ in (3.2). The automorphism β_0 in (4.8) is defined to be the restriction of this automorphism of $\mathrm{At}(E_G) \otimes T^*M$.

PROPOSITION 4.3

Let $\beta_0^*\mathcal{K}(D_0)$ be the $\delta^*\mathrm{ad}(E_G)$-valued two-form on $\mathcal{C}(E_G)$ obtained by pulling back $\mathcal{K}(D_0)$ (defined in (4.1)) *by the map β_0 constructed in* (4.8). *Then the following equality holds*:

$$\beta_0^*\mathcal{K}(D_0) = \mathrm{Ad}(\beta^{-1})(\mathcal{K}(D_0)),$$

where β^{-1} is the section in (4.5), *and* $\mathrm{Ad}(\beta^{-1})$ *is the adjoint action of β^{-1} on* $\mathrm{ad}(E_G)$.

Proof. Consider the automorphism $\tilde{\beta}'$ of E_G constructed in (4.6). It gives a diffeomorphism of δ^*E_G which we will describe. Take any point

$$z \in \delta^*E_G.$$

So z is a pair (y, t), where $y \in \mathcal{C}(E_G)$ and $t \in (E_G)_{\delta(y)}$. Note that $y = q(z)$, where q is the projection in (3.6). Let

$$t' := \tilde{\beta}'(t) \in (E_G)_{\delta(y)} \tag{4.9}$$

be the image of t, where $\tilde{\beta}'$ is the automorphism in (4.6).

Now we have a diffeomorphism

$$\hat{\beta}: \delta^* E_G \longrightarrow \delta^* E_G \tag{4.10}$$

defined by $z \longmapsto (\beta_0(y), t')$, where β_0 and t' are constructed in (4.8) and (4.9) respectively. From the construction of $\hat{\beta}$ in (4.10) we now conclude that the following diagram

$$\begin{array}{ccc} \delta^* E_G & \stackrel{\hat{\beta}}{\longrightarrow} & \delta^* E_G \\ \Big\downarrow q & & \Big\downarrow q \\ \mathcal{C}(E_G) & \stackrel{\beta_0}{\longrightarrow} & \mathcal{C}(E_G) \end{array}$$

is commutative, where q is the projection in (3.6). Furthermore, $\hat{\beta}$ commutes with the action of G on the principal G-bundle $\delta^* E_G$. In other words, $\hat{\beta}$ is an isomorphism of the principal G-bundle $\delta^* E_G$ with its pull back $(\beta_0^{-1})^* \delta^* E_G$.

It is easy to see that the isomorphism $\hat{\beta}$ of $\delta^* E_G$ with $(\beta_0^{-1})^* \delta^* E_G$ preserves the connection D_0 on the principal G-bundle $\delta^* E_G$. This means that $\hat{\beta}$ takes the connection D_0 on $\delta^* E_G$ to the connection on $(\beta_0^{-1})^* \delta^* E_G$ defined by D_0. Consequently, $\hat{\beta}$ pulls back the curvature of D_0 to itself. Therefore, from the action of $\hat{\beta}$ on the fibers of $\delta^* E_G$ — see (4.9) — we conclude that

$$\beta_0^* \mathcal{K}(D_0) = \mathrm{Ad}(\beta^{-1})(\mathcal{K}(D_0)).$$

This completes the proof of the proposition. ■

Let

$$\tilde{\mathcal{C}}(E_G) := \mathcal{C}(E_G) \times_M \mathcal{C}(E_G) \subset \mathcal{C}(E_G) \times \mathcal{C}(E_G) \tag{4.11}$$

be the submanifold consisting of all $(y, z) \in \mathcal{C}(E_G) \times \mathcal{C}(E_G)$ such that

$$\delta(y) = \delta(z),$$

where δ is the projection in Lemma 3.1. Therefore, $\tilde{\mathcal{C}}(E_G)$ has a natural projection

$$\tilde{\delta}: \tilde{\mathcal{C}}(E_G) \longrightarrow M \tag{4.12}$$

defined by $(y, z) \longmapsto \delta(y) = \delta(z)$. This projection $\tilde{\delta}$ makes $\tilde{\mathcal{C}}(E_G)$ a fiber bundle over M.

Consider the vector bundle

$$\nu: T^*M \otimes \mathrm{ad}(E_G) \longrightarrow M.$$

Let

$$\mathcal{B} \subset \tilde{\mathcal{C}}(E_G) \times (T^*M \otimes \mathrm{ad}(E_G))$$

be the submanifold defined by all $(v, w) \in \mathcal{C}(E_G) \times (T^*M \otimes \mathrm{ad}(E_G))$, where $v \in \mathcal{C}(E_G)$ and

$$w \in T^*M \otimes \mathrm{ad}(E_G),$$

such that

$$\delta(v) = \nu(w).$$

Therefore,

$$\tilde{\nu}\colon \mathcal{B} \longrightarrow M \tag{4.13}$$

is a fiber bundle, where the projection $\tilde{\nu}$ is defined by $(v, w) \longmapsto \delta(v) = \nu(w)$.

We will show that there is a natural isomorphism between the fiber bundles $\tilde{\mathcal{C}}(E_G)$ and $\mathcal{B}$ constructed in (4.12) and (4.13) respectively.

Take any

$$(z, w) \in \mathcal{B} \subset \tilde{\mathcal{C}}(E_G) \times (T^*M \otimes \mathrm{ad}(E_G)). \tag{4.14}$$

Let

$$x := \tilde{\nu}((z, w)) \in M$$

be the image, where $\tilde{\nu}$ is the projection in (4.13). Therefore, z gives a homomorphism

$$\lambda_z\colon T_xM \longrightarrow \mathrm{At}(E_G)_x \tag{4.15}$$

(see (3.16)) such that $\eta(x) \circ \lambda_z = \mathrm{Id}_{T_xM}$, where

$$\eta(x)\colon \mathrm{At}(E_G)_x \longrightarrow T_xM \tag{4.16}$$

is the projection in (2.10).

Now consider the section w in (4.14). Its evaluation at x gives a homomorphism

$$\tilde{w}_x\colon T_xM \longrightarrow \mathrm{ad}(E_G)_x \tag{4.17}$$

defined by $\tilde{w}_x(\alpha) = i_\alpha w(x) \in \mathrm{ad}(E_G)_x$, where i_α is the contraction of T_x^*M by $\alpha \in T_xM$. Let

$$\zeta_{(z,w)} := \lambda_z + \iota_0(x) \circ \tilde{w}_x\colon T_xM \longrightarrow \mathrm{At}(E_G)_x \tag{4.18}$$

be the homomorphism, where the homomorphism

$$\iota_0(x)\colon \mathrm{ad}(E_G)_x \longrightarrow \mathrm{At}(E_G)_x$$

is the one in (2.10); the homomorphisms λ_z and $\tilde{w}_x$ are constructed in (4.15) and (4.17) respectively. It is easy to see that

$$\eta(x) \circ \zeta_{(z,w)} = \mathrm{Id}_{T_xM}, \tag{4.19}$$

where $\eta(x)$ is the homomorphism in (4.16). In view of the identity in (4.19), we conclude that

$$\zeta_{(z,w)} \in \delta^{-1}(x) \subset \mathcal{C}(E_G),$$

where δ is the projection in Lemma 3.1.

Consider the fiber bundles $\mathcal{B}$ and $\tilde{\mathcal{C}}(E_G)$ constructed in (4.13) and (4.12) respectively. Let

$$F\colon \mathcal{B} \longrightarrow \tilde{\mathcal{C}}(E_G) \tag{4.20}$$

be the map defined by $(z, w) \longmapsto (z, \zeta_{(z,w)})$, where $\zeta_{(z,w)}$ is constructed in (4.18) from (z, w) in (4.14). It is straight forward to check that F is an isomorphism of fiber bundles over M.

Let

$$\theta\colon M \longrightarrow T^*M \otimes \mathrm{ad}(E_G) \tag{4.21}$$

be a smooth one-form on M with values in the adjoint vector bundle $\mathrm{ad}(E_G)$. Consider the fiber bundle $\delta\colon \mathcal{C}(E_G) \longrightarrow M$ in Lemma 3.1. Let

$$\mathcal{T}_\theta\colon \mathcal{C}(E_G) \longrightarrow \mathcal{C}(E_G) \tag{4.22}$$

be the automorphism of it defined by

$$z \longmapsto (p_2 \circ F)(z, \theta),$$

where F is constructed in (4.20), and

$$p_2\colon \tilde{\mathcal{C}}(E_G) \longrightarrow \mathcal{C}(E_G) \tag{4.23}$$

is the restriction of the projection of $\mathcal{C}(E_G) \times \mathcal{C}(E_G)$ to the second factor (recall that $\tilde{\mathcal{C}}(E_G) \subset \mathcal{C}(E_G) \times \mathcal{C}(E_G)$).

A connection on a principal bundle induces a connection on each associated vector bundle. In particular, a connection on a principal bundle induces a connection on the adjoint vector bundle. The connection on the adjoint vector bundle $\delta^*\mathrm{ad}(E_G)$ induced by the connection D_0 on δ^*E_G will be denoted by D_0'. So for each nonnegative integer i, we have a first order differential operator

$$\begin{aligned} D_0'\colon \Gamma\Big(\mathcal{C}(E_G), \bigwedge^i T^*\mathcal{C}(E_G) \otimes \delta^*\mathrm{ad}(E_G)\Big) \\ \longrightarrow \Gamma\Big(\mathcal{C}(E_G), \bigwedge^{i+1} T^*\mathcal{C}(E_G) \otimes \delta^*\mathrm{ad}(E_G)\Big) \end{aligned} \tag{4.24}$$

given by the connection D_0' on $\delta^*\mathrm{ad}(E_G)$ (here $\Gamma(\mathcal{C}(E_G), V)$ stands for the space of C^∞ sections of $V \longrightarrow \mathcal{C}(E_G)$).

Also, the Lie algebra structure of the fibers of $\mathrm{ad}(E_G)$ and the exterior algebra structure of the fibers of $\oplus_{i\geq 0} \bigwedge^i T^*\mathcal{C}(E_G)$ together define a homomorphism

$$\begin{aligned} \Gamma(\mathcal{C}(E_G), T^*\mathcal{C}(E_G) \otimes \delta^*\mathrm{ad}(E_G)) \times \Gamma(\mathcal{C}(E_G), T^*\mathcal{C}(E_G) \otimes \delta^*\mathrm{ad}(E_G)) \\ \longrightarrow \Gamma\Big(\mathcal{C}(E_G), \bigwedge^2 T^*\mathcal{C}(E_G) \otimes \delta^*\mathrm{ad}(E_G)\Big). \end{aligned} \tag{4.25}$$

For $\theta_1, \theta_2 \in \Gamma(\mathcal{C}(E_G), T^*\mathcal{C}(E_G) \otimes \delta^*\mathrm{ad}(E_G))$, the image of (θ_1, θ_2) in

$$\Gamma\Big(\mathcal{C}(E_G), \bigwedge^2 T^*\mathcal{C}(E_G) \otimes \delta^*\mathrm{ad}(E_G)\Big)$$

by the above pairing will be denoted by $[\theta_1, \theta_2]$.

PROPOSITION 4.4

The pulled back form $\mathcal{T}_\theta^* \mathcal{K}(D_0)$, *where* $\mathcal{T}_\theta$ *is constructed in* (4.22) *and* $\mathcal{K}(D_0)$ *is the curvature form in* (4.1), *satisfies the identity*

$$\mathcal{T}_\theta^* \mathcal{K}(D_0) = \mathcal{K}(D_0) + D_0'(\delta^*\theta) + \frac{1}{2}[\theta, \theta]$$

(*see* (4.24) *and* (4.25) *for* $D_0'(\delta^*\theta)$ *and* $[\theta, \theta]$ *respectively*).

The proof of the above proposition is a straight forward computation.

5. The case of abelian groups

In this section we assume the group G to be abelian.

First we assume G to be a connected abelian of dimension one. So G is either $\mathbb{R}$ or $S^1 = \mathrm{U}(1)$.

We first take E_G to be the trivial principal G-bundle $M \times G$. For $E_G = M \times G$, we have

$$\mathrm{At}(E_G) = T^*M \oplus (M \times \mathbb{R})$$

(note that the Lie algebra of G is $\mathbb{R}$). Hence $\mathcal{C}(E_G)$ constructed in (3.2) is the cotangent bundle $T^*M \longrightarrow M$. The identification of T^*M with $\mathcal{C}(E_G)$ can also seen using the map F in (4.20). For this, first note that the trivial connection on the trivial principal G-bundle E_G defines a section

$$\tau_0 \colon M \longrightarrow \mathcal{C}(E_G)$$

of the fiber bundle $\mathcal{C}(E_G) \longrightarrow M$. Now we have a map

$$\tilde{F} \colon T^*M \longrightarrow \mathcal{C}(E_G) \tag{5.1}$$

that sends any $\alpha \in T_x^*M$ to $p_2 \circ F(\tau_0(x), \alpha)$, where p_2 is the projection in (4.23), and F is the map in (4.20). It is easy to see that $\tilde{F}$ in (5.1) is an isomorphism of fiber bundles over M.

Since $\mathrm{ad}(E_G)$ is the trivial line bundle $M \times \mathbb{R}$, the curvature $\mathcal{K}(D_0)$ in (4.1) is a usual two-form on $\mathcal{C}(E_G)$. Now from the properties of $\mathcal{K}(D_0)$ (see Lemma 4.1, Corollary 4.2 and Proposition 4.4) it follows that the diffeomorphism $\tilde{F}$ in (5.1) takes $\mathcal{K}(D_0)$ to the canonical symplectic form on T^*M.

Now take E_G to be an arbitrary principal G-bundle over M (as before, G is either $\mathbb{R}$ or S^1). We can locally trivialize E_G, hence locally we reduce to the above situation. Therefore, the above observation has the following corollary:

COROLLARY 5.1

The curvature two-form $\mathcal{K}(D_0)$ *on* $\mathcal{C}(E_G)$ *is symplectic. The fibers of the projection* $\delta \colon \mathcal{C}(E_G) \longrightarrow M$ *are Lagrangian.*

Now assume that

$$G = \prod_{i=1}^{n} G_i,$$

where each G_i is either $\mathbb{R}$ or S^1. Then any principal G-bundle E_G over M is of the form

$$E_G = E_{G_1} \times_M E_{G_2} \times_M \cdots \times_M E_{G_n},$$

where each $f_i\colon E_{G_i} \longrightarrow M$ is a principal G_i bundle, and

$$E_{G_1} \times_M E_{G_2} \times_M \cdots \times_M E_{G_n} \subset \prod_{i=1}^{n} E_{G_i}$$

is the submanifold consisting of all $(z_1, \ldots, z_n)$ such that

$$f_1(z_1) = f_2(z_2) = \cdots = f_n(z_n).$$

Clearly, we have

$$\mathcal{C}(E_G) = \mathcal{C}(E_{G_1}) \times_M \mathcal{C}(E_{G_2}) \times_M \cdots \times_M \mathcal{C}(E_{G_n}) \subset \prod_{i=1}^{n} \mathcal{C}(E_{G_i}),$$

where $\mathcal{C}(E_{G_1}) \times_M \mathcal{C}(E_{G_2}) \times_M \cdots \times_M \mathcal{C}(E_{G_n})$ is the submanifold consisting of all

$$(z_1, z_2, \ldots, z_n) \in \prod_{i=1}^{n} \mathcal{C}(E_{G_i})$$

that project to some common point in M. It is now straight forward to check that the curvature form $\mathcal{K}(D_0)$ on $\mathcal{C}(E_G)$ is the direct sum of the pull backs of the curvature forms on $\mathcal{C}(E_{G_i})$, $1 \le i \le n$.

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Note added in the Proof. This has been achieved in a paper entitled “A construction of a universal connection” by I Biswas, J Hurtubise and J D Stasheff, to appear in *Forum Mathematicum.*