Proc. Indian Acad. Sci. (Math. Sci.) Vol. 120, No. 3, June 2010, pp. 267–284.

Theta function identities associated with Ramanujan's modular equations of degree 15

RUPAM BARMAN and NAYANDEEP DEKA BARUAH

Department of Mathematical Sciences, Tezpur University, Napaam 784 028, India
E-mail: rupamb@tezu.ernet.in; nayan@tezu.ernet.in

MS received 10 June 2008; revised 23 March 2010

Abstract. We present alternative proofs of some of Ramanujan's theta function identities associated with the modular equations of composite degree 15. Along the way we also find some new theta-function identities. We also give simple proofs of his modular equations of degree 15.

Keywords. Theta function; elliptic integral; modular equation; multiplier.

1. Introduction

Ramanujan's general theta function $f(a,b)$ is defined by

$$\begin{aligned} f(a,b) &:= \sum_{k=-\infty}^{\infty} a^{k(k+1)/2} b^{k(k-1)/2} \\ &= a^{n(n+1)/2} b^{n(n-1)/2} f(a(ab)^n, b(ab)^{-n}), \end{aligned} \tag{1.1}$$

where $|ab| < 1$ and n is an integer. If we set $a = q\mathrm{e}^{2iz}$, $b = q\mathrm{e}^{-2iz}$, and $q = \mathrm{e}^{\pi i \tau}$, where z is complex and $\mathrm{Im}(\tau) > 0$, then $f(a,b) = \vartheta_3(z,\tau)$, where $\vartheta_3(z,\tau)$ denotes one of the classical theta functions in its standard notation (p. 464 of [7]). It is assumed throughout the paper that $|q| < 1$. For any complex number a, we define

$$(a;q)_0 = 1, \quad (a;q)_n := \prod_{k=1}^{n}(1 - aq^{k-1}) \quad \text{for } n \geq 1, \quad \text{and}$$

$$(a;q)_\infty := \prod_{k=1}^{\infty}(1 - aq^{k-1}). \tag{1.2}$$

Three special cases of $f(a,b)$ are

$$\phi(q) := f(q,q) = 1 + 2\sum_{k=1}^{\infty} q^{k^2} = \frac{(-q;\ q^2)_\infty (q^2;q^2)_\infty}{(q;q^2)_\infty(-q^2;q^2)_\infty}, \tag{1.3}$$

$$\psi(q) := f(q,q^3) = \frac{1}{2} f(1,q) = \sum_{k=0}^{\infty} q^{k(k+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.4}$$

$$f(-q) := f(-q, -q^2) = \sum_{k=0}^{\infty} (-1)^k q^{k(3k-1)/2} + \sum_{k=1}^{\infty} (-1)^k q^{k(3k+1)/2}$$
$$= (q; q)_\infty. \tag{1.5}$$

If we write $q = e^{2\pi i\tau}$ with $\mathrm{Im}(\tau) > 0$, then $f(-q) = e^{-\pi i\tau/12}\eta(\tau)$, where $\eta(\tau)$ is the classical Dedekind eta-function. The complete elliptic integral of the first kind $K(k)$ is defined by

$$K(k) := \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^2}{(n!)^2} k^{2n} = \frac{\pi}{2} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right), \tag{1.6}$$

where $0 < k < 1$ and where ${}_2F_1(a, b; c; z)$, $|z| < 1$, denotes the ordinary or Gaussian hypergeometric series. The number k is called the modulus of K, and $k' := \sqrt{1 - k^2}$ is called the complementary modulus. Let K, K', L, and L' denote complete elliptic integrals of the first kind associated with the moduli k, k', l and l', respectively. A modular equation of degree n is a relation between the moduli k and l which is implied by $n\frac{K'}{K} = \frac{L'}{L}$. Ramanujan recorded his modular equations in terms of α and β, where $\alpha = k^2$ and $\beta = l^2$. We say that β has degree n over α. The multiplier m is defined by $m = \frac{K}{L}$. Let K, K', L_1, L_1', L_2, L_2', L_3, and L_3' denote complete elliptic integrals of the first kind corresponding, in pairs, to the moduli $\sqrt{\alpha}$, $\sqrt{\beta}$, $\sqrt{\gamma}$, $\sqrt{\delta}$, and their complementary moduli, respectively. Let n_1, n_2 and n_3 be positive integers such that $n_3 = n_1 n_2$. Suppose that the equalities

$$n_1 \frac{K'}{K} = \frac{L_1'}{L_1}, \quad n_2 \frac{K'}{K} = \frac{L_2'}{L_2} \quad \text{and} \quad n_3 \frac{K'}{K} = \frac{L_3'}{L_3} \tag{1.7}$$

hold. Then a 'mixed' modular equation or a modular equation of composite degree n_3 is a relation between the moduli $\sqrt{\alpha}$, $\sqrt{\beta}$, $\sqrt{\gamma}$ and $\sqrt{\delta}$, that is induced by (1.7). Then we say that β, γ, and δ have degrees n_1, n_2 and n_3, respectively, over α. Recalling from Entry 6 (p. 101 of [3]) that $z_r = \phi^2(q^r)$, we define the multipliers m and m' by $m = \frac{z_1}{z_{n_1}}$ and $m' = \frac{z_{n_2}}{z_{n_3}}$.

Ramanujan recorded several modular equations of composite degree 15 in Chapter 20 of his second notebook [6]. All those equations have been proved by Berndt (see pp. 383–395, Entry 11 of [3] and p. 370, Entry 27 of [4]). Recently, using modular equations of degree 15, Baruah and Berndt (Theorems 8.1–8.3 of [1]) have derived many elegant partition identities. It is clear from Chapter 17, p. 101, Entry 6; pp. 122–124, Entries 10–12 of [3] that modular equations can be expressed as identities involving the theta functions ϕ, ψ and f. Therefore, often one first tries to derive a theta function identity and then transcribes it into an equivalent modular equation. But, proofs of some modular equations of composite degree 15 given by Berndt are quite unlike this method. In this paper, we find alternative proofs of some of the theta function identities. Earlier these identities were proved by Berndt using modular equations and a method of parametrizations, and therefore he could not apply these theta function identities to prove those modular equations. After having alternative proofs of the theta function identities, one can use them to derive modular equations. Berndt (p. 388 of [3]) remarked that his proofs of Entries 11(x) and 11(xii) (p. 384 of [3]) were somewhat difficult. It is worthwhile to mention that the modular equations in Entries 11(xi) and 11(xiii) are reciprocals of those in Entries 11(x) and 11(xii), respectively. We have succeeded in finding a less tedious proof of Entry 11(xii).

2. Preliminary results

In this section, we state some results which will be used to derive theta-function identities in the next section.

Lemma 2.1 (*Entry* 24, *p.* 39 *of* [3]). *Let* $\chi(q) := (-q; q^2)_\infty$. *We have*

$$\frac{\psi(q)}{\psi(-q)} = \sqrt{\frac{\phi(q)}{\phi(-q)}}, \tag{2.1}$$

$$\chi(q) = \frac{f(q)}{f(-q^2)} = \sqrt[3]{\frac{\phi(q)}{\psi(-q)}} = \frac{f(-q^2)}{\psi(-q)}, \tag{2.2}$$

$$f^3(-q^2) = \phi(-q)\psi^2(q). \tag{2.3}$$

Lemma 2.2 (*Corollary*, *p.* 49 *of* [3]). *We have*

$$\phi(q) = \phi(q^9) + 2qf(q^3, q^{15}), \tag{2.4}$$

$$\psi(q) = f(q^3, q^6) + q\psi(q^9), \tag{2.5}$$

$$= f(q^6, q^{10}) + qf(q^2, q^{14}). \tag{2.6}$$

Lemma 2.3 (*Entry* 25, *p.* 40 *of* [3]). *We have*

$$\phi(q) + \phi(-q) = 2\phi(q^4), \tag{2.7}$$

$$\phi(q) - \phi(-q) = 4q\psi(q^8), \tag{2.8}$$

$$\phi(q)\phi(-q) = \phi^2(-q^2), \tag{2.9}$$

$$\psi(q)\psi(-q) = \psi(q^2)\phi(-q^2), \tag{2.10}$$

$$\phi(q)\psi(q^2) = \psi^2(q). \tag{2.11}$$

Lemma 2.4 (*Entries* 10–12, *pp.* 122–124 *of* [3]). *If*

$$z = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right) = \frac{2}{\pi}K(\sqrt{x}) \quad \textit{and} \quad y = \pi\frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-x\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right)},$$

then

$$\phi(\mathrm{e}^{-y}) = \sqrt{z}, \tag{2.12}$$

$$\phi(-\mathrm{e}^{-y}) = \sqrt{z}(1-x)^{1/4}, \tag{2.13}$$

$$\phi(-\mathrm{e}^{-2y}) = \sqrt{z}(1-x)^{1/8}, \tag{2.14}$$

$$\phi(\mathrm{e}^{-4y}) = \frac{1}{2}\sqrt{z}(1+(1-x)^{1/4}), \tag{2.15}$$

$$\psi(\mathrm{e}^{-y}) = \sqrt{z/2}(xe^{y})^{1/8}, \tag{2.16}$$

$$\psi(-\mathrm{e}^{-y}) = \sqrt{z/2}\{x(1-x)\mathrm{e}^{y}\}^{1/8}, \tag{2.17}$$

$$\psi(\mathrm{e}^{-2y}) = \frac{1}{2}\sqrt{z}(x\mathrm{e}^{y})^{1/4}, \tag{2.18}$$

$$\psi(\mathrm{e}^{-8y}) = \frac{1}{4}\sqrt{z}\{1-(1-x)^{1/4}\}\mathrm{e}^{y}, \tag{2.19}$$

$$\psi(\mathrm{e}^{-y/2}) = \sqrt{z}\{(1+\sqrt{x})/2\}^{1/4}(x\mathrm{e}^{y})^{1/16}, \tag{2.20}$$

$$\psi(-\mathrm{e}^{-y/2}) = \sqrt{z}\{(1-\sqrt{x})/2\}^{1/4}(x\mathrm{e}^{y})^{1/16}. \tag{2.21}$$

Lemma 2.5 ((36.2), *p.* 68 *of* [3]). *If* $\mu > \nu \geq 0$, *then*

$$\begin{aligned}
&\frac{1}{2}\{f(Aq^{\mu+\nu}, q^{\mu+\nu}/A)f(Bq^{\mu-\nu}, q^{\mu-\nu}/B)\\
&\quad - f(-Aq^{\mu+\nu}, -q^{\mu+\nu}/A)f(-Bq^{\mu-\nu}, -q^{\mu-\nu}/B)\}\\
&\quad = A\sum_{m=0}^{\mu-1}(AB)^{m}q^{(2m+1)(\mu+\nu)+2\mu m^{2}} f\left(\frac{A}{B}\, q^{4\mu+2\nu+4\nu m}, \frac{B}{A}\, q^{-2\nu-4\nu m}\right)\\
&\quad \times f(A^{\mu-\nu}B^{\mu+\nu}q^{(2\mu+4m+2)(\mu^{2}-\nu^{2})}, q^{(2\mu-4m-2)(\mu^{2}-\nu^{2})}/(A^{\mu-\nu}B^{\mu+\nu})).
\end{aligned} \tag{2.22}$$

Lemma 2.6 ((36.3), *p.* 68 *of* [3]). *If* $\mu > \nu \geq 0$, *then*

$$\begin{aligned}
&\frac{1}{2}\{\phi(q^{\mu+\nu})\phi(q^{\mu-\nu}) + \phi(-q^{\mu+\nu})\phi(-q^{\mu-\nu})\}\\
&\quad = \sum_{m=0}^{\mu-1} q^{2\mu m^{2}} f(q^{(2\mu+4m)(\mu^{2}-\nu^{2})}, q^{(2\mu-4m)(\mu^{2}-\nu^{2})}) f(q^{2\mu+4\nu m}, q^{2\mu-4\nu m}).
\end{aligned} \tag{2.23}$$

Lemma 2.7 ((36.8), *p.* 69 *of* [3]). *If* $\mu > \nu \geq 0$ *and* μ *is even*, *then*

$$\begin{aligned}
\psi(q^{\mu+\nu})\psi(q^{\mu-\nu}) &= \phi(q^{\mu(\mu^{2}-\nu^{2})})\psi(q^{2\mu}) + q^{\mu^{3}/4-\mu\nu/2}\psi(q^{2\mu(\mu^{2}-\nu^{2})})f(q^{\mu\nu}, q^{2\mu-\mu\nu})\\
&\quad + \sum_{m=1}^{\mu/2-1} q^{\mu m^{2}-\nu m} f(q^{(\mu+2m)(\mu^{2}-\nu^{2})}, q^{(\mu-2m)(\mu^{2}-\nu^{2})})\\
&\quad \times f(q^{2\nu m}, q^{2\mu-2\nu m})
\end{aligned} \tag{2.24}$$

3. Theta function identities

In Chapter 20 of his second notebook [6], Ramanujan recorded several theta function identities associated with modular equations of composite degree 15. These identities have previously been proved by Berndt in [3]. But he proved most of these theta function identities using modular equations. These identities can also be found in a recent work by Cooper [5]. Here we present proofs of these theta function identities by using the lemmas given in the previous section. We also find some new theta function identities.

Theorem 3.1. *We have*

$$\begin{aligned}&\phi(q^3)\phi(q^5)+\phi(-q^3)\phi(-q^5)\\&\quad=\phi(q^2)\phi(q^{30})+\phi(-q^2)\phi(-q^{30})+4q^8\psi(q^{60})\psi(q^4).\end{aligned}\tag{3.1}$$

Proof. Setting $\mu=4$, $\nu=1$ and $Q=q^{15}$ in (2.23), and then employing (1.1) with $n=1$ and (1.4), we deduce that

$$\begin{aligned}&\phi(q^3)\phi(q^5)+\phi(-q^3)\phi(-q^5)\\&\quad=2\phi(Q^8)\phi(q^8)+4q^8\psi(Q^4)\psi(q^4)+8q^{32}\psi(Q^{16})\psi(q^{16}).\end{aligned}\tag{3.2}$$

Invoking (2.7) and (2.8) in (3.2), we deduce (3.1). □

Theorem 3.2. *We have*

$$\phi(q)\phi(q^{15})-\phi(-q)\phi(-q^{15})=2q\{\psi(q^3)\psi(q^5)+\psi(-q^3)\psi(-q^5)\}.\tag{3.3}$$

Proof. Setting $\mu=4$ and $\nu=1$ in (2.24), we deduce that

$$\psi(q^3)\psi(q^5)=\psi(q^8)\phi(q^{60})+q^3\psi(q^2)\psi(q^{30})+q^{14}\phi(q^4)\psi(q^{120}).\tag{3.4}$$

Replacing q by $-q$ in (3.4) and adding the resulting identity and (3.4), we obtain

$$\psi(q^3)\psi(q^5)+\psi(-q^3)\psi(-q^5)=2\psi(q^8)\phi(q^{60})+2q^{14}\phi(q^4)\psi(q^{120}).\tag{3.5}$$

Employing (2.7) and (2.8) in (3.5), we readily deduce (3.3) to complete the proof. □

Theorem 3.3. *We have*

$$\phi(q^3)\phi(q^5)-\phi(-q^3)\phi(-q^5)=2q^2\{\psi(q)\psi(q^{15})-\psi(-q)\psi(-q^{15})\}.\tag{3.6}$$

Proof. Putting $\mu=4$, $\nu=1$, $A=B=1$ and $Q=q^{15}$ in (2.22), we deduce that

$$\begin{aligned}&\phi(q^3)\phi(q^5)-\phi(-q^3)\phi(-q^5)\\&\quad=4q^3f(q^2,q^{14})f(Q^6,Q^{10})+4q^{17}f(q^6,q^{10})f(Q^2,Q^{14}).\end{aligned}\tag{3.7}$$

From (2.6), we find that

$$f(q^6,q^{10})=\frac{1}{2}\{\psi(q)+\psi(-q)\}\tag{3.8}$$

and

$$f(q^2,q^{14})=\frac{1}{2q}\{\psi(q)-\psi(-q)\}.\tag{3.9}$$

Employing (3.8) and (3.9) in (3.7), we deduce (3.6). □

Berndt proved the following two results by using theta function identities. So, we cite only the references for the proofs.

Theorem 3.4 (Entry 9(i), p. 377 of [3]). *We have*

$$\psi(q^3)\psi(q^5) - \psi(-q^3)\psi(-q^5) = 2q^3\psi(q^2)\psi(q^{30}). \tag{3.10}$$

For a proof, see p. 377 in [3].

Theorem 3.5 (Entry 9(iv), p. 377 of [3]). *We have*

$$\psi(q)\psi(q^{15}) + \psi(-q)\psi(-q^{15}) = 2\psi(q^6)\psi(q^{10}). \tag{3.11}$$

For a proof, see p. 378 in [3].

Remark 3.6. Baruah and Berndt (Theorems 6.1– 6.3 of [2]) extracted nice partition identities from Theorems 3.3–3.5.

Berndt proved the following two results using modular equations. Here we prove them by using theta function identities.

Theorem 3.7 (Entry 9(iii), p. 377 of [3]). *We have*

$$\phi(-q^2)\phi(-q^{30}) + 2q^2\psi(q)\psi(q^{15}) = \phi(q^3)\phi(q^5). \tag{3.12}$$

Proof. Replacing q by q^2 in (3.3), we find that

$$\phi(q^2)\phi(q^{30}) - \phi(-q^2)\phi(-q^{30}) = 2q^2\{\psi(q^6)\psi(q^{10}) + \psi(-q^6)\psi(-q^{10})\}. \tag{3.13}$$

Also, we rewrite (3.1) as

$$\begin{aligned}\phi(q^2)\phi(q^{30}) &+ \phi(-q^2)\phi(-q^{30}) \\ &= \phi(q^3)\phi(q^5) + \phi(-q^3)\phi(-q^5) - 4q^8\psi(q^4)\psi(q^{60}).\end{aligned} \tag{3.14}$$

Subtracting (3.13) from (3.14), we deduce that

$$\begin{aligned}2\phi(-q^2)\phi(-q^{30}) = \phi(q^3)\phi(q^5) &+ \phi(-q^3)\phi(-q^5) - 4q^8\psi(q^4)\psi(q^{60}) \\ &- 2q^2\{\psi(q^6)\psi(q^{10}) + \psi(-q^6)\psi(-q^{10})\}.\end{aligned} \tag{3.15}$$

Again, replacing q by q^2 in (3.10), we obtain

$$2q^6\psi(q^4)\psi(q^{60}) = \psi(q^6)\psi(q^{10}) - \psi(-q^6)\psi(-q^{10}). \tag{3.16}$$

From (3.15) and (3.16), we find that

$$\phi(q^3)\phi(q^5) + \phi(-q^3)\phi(-q^5) = 2\phi(-q^2)\phi(-q^{30}) + 4q^2\psi(q^6)\psi(q^{10}). \tag{3.17}$$

Adding (3.17) and (3.6), we arrive at

$$\begin{aligned}\phi(q^3)\phi(q^5) = \phi(-q^2)\phi(-q^{30}) &+ q^2\{\psi(q)\psi(q^{15}) \\ &- \psi(-q)\psi(-q^{15})\} + 2q^2\psi(q^6)\psi(q^{10}).\end{aligned} \tag{3.18}$$

Employing (3.11) in (3.18), we deduce (3.12) to finish the proof. □

Theorem 3.8 (Entry 9(ii), p. 377 of [3]). *We have*

$$\phi(-q^6)\phi(-q^{10}) + 2q\psi(q^3)\psi(q^5) = \phi(q)\phi(q^{15}). \tag{3.19}$$

Proof. Replacing q by $q^{1/3}$ in (3.12), we have

$$\phi(q)\phi(q^{5/3}) = \phi(-q^{2/3})\phi(-q^{10}) + 2q^{2/3}\psi(q^{1/3})\psi(q^5). \tag{3.20}$$

Again, replacing q by $q^{5/3}$ and $-q^{2/3}$ in (2.4), we obtain

$$\phi(q^{5/3}) = \phi(q^{15}) + 2q^{5/3} f(q^5, q^{25}) \tag{3.21}$$

and

$$\phi(-q^{2/3}) = \phi(-q^6) - 2q^{2/3} f(-q^2, -q^{10}), \tag{3.22}$$

respectively.

Also, replacing q by $q^{1/3}$ in (2.5), we find that

$$\psi(q^{1/3}) = f(q, q^2) + q^{1/3}\psi(q^3). \tag{3.23}$$

Employing (3.21)–(3.23) in (3.20), we deduce that

$$\begin{aligned} &\phi(q)\phi(q^{15}) + 2q^{5/3}\phi(q) f(q^5, q^{25}) \\ &\quad = \phi(-q^6)\phi(-q^{10}) - 2q^{2/3} f(-q^2, -q^{10})\phi(-q^{10}) \\ &\qquad + 2q^{2/3} f(q, q^2)\psi(q^5) + 2q\psi(q^3)\psi(q^5). \end{aligned} \tag{3.24}$$

Equating the rational parts of (3.24), we arrive at (3.19) to finish the proof. □

To prove Entries 9(v)–(vii), pp. 395–397 of [3], Berndt used modular equations. In Theorems 3.12–3.14, we present alternative proofs by employing the new theta function identities given in the next three theorems. It would be interesting to see any applications of these identities.

Theorem 3.9. *We have*

$$\begin{aligned} &\phi(q)\phi(q^{15})\{2q\psi(q^3)\psi(q^5) - \phi(-q^6)\phi(-q^{10})\} \\ &\quad = \phi(q^3)\phi(q^5)\{2q^2\psi(q)\psi(q^{15}) - \phi(-q^2)\phi(-q^{30})\}. \end{aligned} \tag{3.25}$$

Proof. From (3.10) and (3.19), we obtain

$$-2q\psi(-q^3)\psi(-q^5) = 4q^4\psi(q^2)\psi(q^{30}) - 2q\psi(q^3)\psi(q^5) \tag{3.26}$$

and

$$-2q\psi(-q^3)\psi(-q^5) = \phi(-q)\phi(-q^{15}) - \phi(-q^6)\phi(-q^{10}), \tag{3.27}$$

respectively.

From (3.26), (3.27), (2.9) and (2.11), we deduce that

$$\begin{aligned}&\frac{4q^4\psi^2(q)\psi^2(q^{15})-\phi^2(-q^2)\phi^2(-q^{30})}{\phi(q)\phi(q^{15})}\\&\quad=2q\psi(q^3)\psi(q^5)-\phi(-q^6)\phi(-q^{10}).\end{aligned}\tag{3.28}$$

We rewrite (3.28) as

$$\begin{aligned}&\phi(q)\phi(q^{15})\{2q\psi(q^3)\psi(q^5)-\phi(-q^6)\phi(-q^{10})\}\\&\quad=\{2q^2\psi(q)\psi(q^{15})+\phi(-q^2)\phi(-q^{30})\}\\&\qquad\times\{2q^2\psi(q)\psi(q^{15})-\phi(-q^2)\phi(-q^{30})\}.\end{aligned}\tag{3.29}$$

Employing (3.12) in (3.29), we arrive at the desired result. □

Theorem 3.10. *We have*

$$\frac{\psi(q^3)\psi(q^5)}{\phi(q^3)\phi(q^5)}-q\frac{\psi(q)\psi(q^{15})}{\phi(q)\phi(q^{15})}-\frac{\psi(-q^3)\psi(-q^5)}{\phi(q)\phi(q^{15})}=q\frac{\psi(-q)\psi(-q^{15})}{\phi(q)\phi(q^{15})}\tag{3.30}$$

and

$$-q\frac{\psi(q)\psi(q^{15})}{\phi(q)\phi(q^{15})}+\frac{\psi(q^3)\psi(q^5)}{\phi(q^3)\phi(q^5)}+q\frac{\psi(-q)\psi(-q^{15})}{\phi(q^3)\phi(q^5)}=\frac{\psi(-q^3)\psi(-q^5)}{\phi(q^3)\phi(q^5)}.\tag{3.31}$$

Proof. We first prove (3.30). We have

$$\begin{aligned}&\frac{\psi(q^3)\psi(q^5)}{\phi(q^3)\phi(q^5)}-\frac{\psi(-q^3)\psi(-q^5)}{\phi(q)\phi(q^{15})}\\&\quad=\frac{\psi(q^3)\psi(q^5)\phi(q)\phi(q^{15})-\psi(-q^3)\psi(-q^5)\phi(q^3)\phi(q^5)}{\phi(q)\phi(q^3)\phi(q^5)\phi(q^{15})}.\end{aligned}\tag{3.32}$$

From (2.10) and (2.11), we note that

$$\psi(q)\phi(-q^2)=\psi(-q)\phi(q).\tag{3.33}$$

Employing (3.33) in (3.32), and then using (3.19), we obtain

$$\frac{\psi(q^3)\psi(q^5)}{\phi(q^3)\phi(q^5)}-\frac{\psi(-q^3)\psi(-q^5)}{\phi(q)\phi(q^{15})}=2q\frac{\psi^2(q^3)\psi^2(q^5)}{\phi(q)\phi(q^3)\phi(q^5)\phi(q^{15})}.\tag{3.34}$$

Now using (2.11) and (3.11) in (3.34), we readily arrive at (3.30).

Next we prove (3.31). Employing (3.12), we have

$$\begin{aligned}-q\frac{\psi(q)\psi(q^{15})}{\phi(q)\phi(q^{15})}&=\frac{-q\psi(q)\psi(q^{15})\phi(q^3)\phi(q^5)}{\phi(q)\phi(q^3)\phi(q^5)\phi(q^{15})}\\&=\frac{-q\psi(q)\psi(q^{15})\phi(-q^2)\phi(-q^{30})-2q^3\psi^2(q)\psi^2(q^{15})}{\phi(q)\phi(q^3)\phi(q^5)\phi(q^{15})}.\end{aligned}\tag{3.35}$$

Employing (3.33) in (3.35), we deduce that

$$-q\frac{\psi(q)\psi(q^{15})}{\phi(q)\phi(q^{15})}=-q\frac{\psi(-q)\psi(-q^{15})}{\phi(q^3)\phi(q^5)}-2q^3\frac{\psi^2(q)\psi^2(q^{15})}{\phi(q)\phi(q^3)\phi(q^5)\phi(q^{15})}. \tag{3.36}$$

Using (2.11) and (3.10) in (3.36), we easily arrive at (3.31). □

Theorem 3.11. *We have*

$$\frac{\phi^2(q^3)\phi^2(q^5)}{\phi^2(q)\phi^2(q^{15})}+\frac{\phi(q)\phi(q^{15})}{\phi(q^3)\phi(q^5)}-\frac{\phi(q^3)\phi(q^5)}{\phi(q)\phi(q^{15})}-1=8q^2\frac{\psi(-q)\psi(-q^{15})}{\phi(q)\phi(q^{15})} \tag{3.37}$$

and

$$\frac{\phi^2(q)\phi^2(q^{15})}{\phi^2(q^3)\phi^2(q^5)}+\frac{\phi(q)\phi(q^{15})}{\phi(q^3)\phi(q^5)}-\frac{\phi(q^3)\phi(q^5)}{\phi(q)\phi(q^{15})}-1=8q\frac{\psi(-q^3)\psi(-q^5)}{\phi(q^3)\phi(q^5)}. \tag{3.38}$$

Proof. We first prove (3.37). Squaring both sides of (3.12), we obtain

$$\begin{aligned}\phi^2(q^3)\phi^2(q^5)&=\phi^2(-q^2)\phi^2(-q^{30})+4q^2\phi(-q^2)\phi(-q^{30})\psi(q)\psi(q^{15})\\&\quad+4q^4\psi^2(q)\psi^2(q^{15}).\end{aligned} \tag{3.39}$$

From (3.39), (3.19) and (3.12), we obtain

$$\begin{aligned}&\frac{\phi^2(q^3)\phi^2(q^5)}{\phi^2(q)\phi^2(q^{15})}+\frac{\phi(q)\phi(q^{15})}{\phi(q^3)\phi(q^5)}-\frac{\phi(q^3)\phi(q^5)}{\phi(q)\phi(q^{15})}-1\\&=\frac{\phi^2(-q^2)\phi^2(-q^{30})}{\phi^2(q)\phi^2(q^{15})}+4q^2\frac{\phi(-q^2)\phi(-q^{30})\psi(q)\psi(q^{15})}{\phi^2(q)\phi^2(q^{15})}\\&\quad+4q^4\frac{\psi^2(q)\psi^2(q^{15})}{\phi^2(q)\phi^2(q^{15})}+\frac{\phi(-q^6)\phi(-q^{10})}{\phi(q^3)\phi(q^5)}\\&\quad+2q\frac{\psi(q^3)\psi(q^5)}{\phi(q^3)\phi(q^5)}-\frac{\phi(-q^2)\phi(-q^{30})}{\phi(q)\phi(q^{15})}-2q^2\frac{\psi(q)\psi(q^{15})}{\phi(q)\phi(q^{15})}-1.\end{aligned} \tag{3.40}$$

Now, from (3.25), we find that

$$\begin{aligned}&\frac{\phi(-q^6)\phi(-q^{10})}{\phi(q^3)\phi(q^5)}-\frac{\phi(-q^2)\phi(-q^{30})}{\phi(q)\phi(q^{15})}\\&=2q\frac{\psi(q^3)\psi(q^5)}{\phi(q^3)\phi(q^5)}-2q^2\frac{\psi(q)\psi(q^{15})}{\phi(q)\phi(q^{15})}.\end{aligned} \tag{3.41}$$

Next, from (3.33), we have

$$\frac{\phi(-q^2)\phi(-q^{30})\psi(q)\psi(q^{15})}{\phi^2(q)\phi^2(q^{15})}=\frac{\psi(-q)\psi(-q^{15})}{\phi(q)\phi(q^{15})}. \tag{3.42}$$

Employing (3.41) and (3.42) in (3.40), we find that

$$\begin{aligned}
&\frac{\phi^2(q^3)\phi^2(q^5)}{\phi^2(q)\phi^2(q^{15})}+\frac{\phi(q)\phi(q^{15})}{\phi(q^3)\phi(q^5)}-\frac{\phi(q^3)\phi(q^5)}{\phi(q)\phi(q^{15})}-1\\
&\quad=\frac{\phi^2(-q^2)\phi^2(-q^{30})}{\phi^2(q)\phi^2(q^{15})}+4q^2\frac{\psi(-q)\psi(-q^{15})}{\phi(q)\phi(q^{15})}+4q^4\frac{\psi^2(q)\psi^2(q^{15})}{\phi^2(q)\phi^2(q^{15})}\\
&\qquad+4q\frac{\psi(q^3)\psi(q^5)}{\phi(q^3)\phi(q^5)}-4q^2\frac{\psi(q)\psi(q^{15})}{\phi(q)\phi(q^{15})}-1. \qquad (3.43)
\end{aligned}$$

Again, from (2.11) and (3.10), we note that

$$\begin{aligned}
4q^4\frac{\psi^2(q)\psi^2(q^{15})}{\phi^2(q)\phi^2(q^{15})}&=4q^4\frac{\psi(q^2)\psi(q^{30})}{\phi(q)\phi(q^{15})}\\
&=2q\frac{\psi(q^3)\psi(q^5)-\psi(-q^3)\psi(-q^5)}{\phi(q)\phi(q^{15})}. \qquad (3.44)
\end{aligned}$$

Also, from (2.9) and (3.19), we obtain

$$\begin{aligned}
\frac{\phi^2(-q^2)\phi^2(-q^{30})}{\phi^2(q)\phi^2(q^{15})}&=\frac{\phi(-q)\phi(-q^{15})}{\phi(q)\phi(q^{15})}\\
&=\frac{\phi(-q^6)\phi(-q^{10})-2q\psi(-q^3)\psi(-q^5)}{\phi(q)\phi(q^{15})}. \qquad (3.45)
\end{aligned}$$

Employing (3.44) and (3.45) in (3.43), we deduce that

$$\begin{aligned}
&\frac{\phi^2(q^3)\phi^2(q^5)}{\phi^2(q)\phi^2(q^{15})}+\frac{\phi(q)\phi(q^{15})}{\phi(q^3)\phi(q^5)}-\frac{\phi(q^3)\phi(q^5)}{\phi(q)\phi(q^{15})}-1\\
&\quad=4q^2\frac{\psi(-q)\psi(-q^{15})}{\phi(q)\phi(q^{15})}+4q\left(\frac{\psi(q^3)\psi(q^5)}{\phi(q^3)\phi(q^5)}-q\frac{\psi(q)\psi(q^{15})}{\phi(q)\phi(q^{15})}-\frac{\psi(-q^3)\psi(-q^5)}{\phi(q)\phi(q^{15})}\right)\\
&\qquad\times\frac{\phi(-q^6)\phi(-q^{10})+2q\phi(q^3)\phi(q^5)}{\phi(q)\phi(q^{15})}-1. \qquad (3.46)
\end{aligned}$$

Using (3.30) and (3.19) in (3.46), we arrive at (3.37).

Again, as in the proof of (3.37), from (3.19), (3.12), (3.41) and (3.33), we deduce that

$$\begin{aligned}
&\frac{\phi^2(q)\phi^2(q^{15})}{\phi^2(q^3)\phi^2(q^5)}+\frac{\phi(q)\phi(q^{15})}{\phi(q^3)\phi(q^5)}-\frac{\phi(q^3)\phi(q^5)}{\phi(q)\phi(q^{15})}-1\\
&\quad=\frac{\phi^2(-q^6)\phi^2(-q^{10})}{\phi^2(q^3)\phi^2(q^5)}+4q\frac{\psi(-q^3)\psi(-q^5)}{\phi(q^3)\phi(q^5)}+4q^2\frac{\psi^2(q^3)\psi^2(q^5)}{\phi^2(q^3)\phi^2(q^5)}\\
&\qquad+4q\frac{\psi(q^3)\psi(q^5)}{\phi(q^3)\phi(q^5)}-4q^2\frac{\psi(q)\psi(q^{15})}{\phi(q)\phi(q^{15})}-1. \qquad (3.47)
\end{aligned}$$

Using (2.9), (2.11), (3.12), (3.11) and (3.31) as shown in the proof of (3.37), we readily deduce (3.38). □

Theorem 3.12 (Entry 9(v), p. 377 of [3]). *We have*

$$\phi(q)\phi(q^{15}) - \phi(q^3)\phi(q^5) = 2qf(-q^2)f(-q^{30})\chi(q^3)\chi(q^5). \tag{3.48}$$

Proof. Replacing q by $-q$ in (3.19), and multiplying with (3.10), we deduce that

$$\begin{aligned}&\psi(q^3)\psi(q^5)\phi(-q^6)\phi(-q^{10}) - \psi(-q^3)\psi(-q^5)\phi(q)\phi(q^{15}) + 2q\psi^2(-q^3)\psi^2(-q^5)\\ &\quad = 2q^3\psi(q^2)\psi(q^{30})\phi(-q)\phi(-q^{15}).\end{aligned} \tag{3.49}$$

Thus,

$$\begin{aligned}&1 - \frac{\psi(-q^3)\psi(-q^5)\phi(q)\phi(q^{15})}{\psi(q^3)\psi(q^5)\phi(-q^6)\phi(-q^{10})} + 2q\frac{\psi^2(-q^3)\psi^2(-q^5)}{\psi(q^3)\psi(q^5)\phi(-q^6)\phi(-q^{10})}\\ &\quad = 2q^3\frac{\psi(q^2)\psi(q^{30})\phi(-q)\phi(-q^{15})}{\psi(q^3)\psi(q^5)\phi(-q^6)\phi(-q^{10})}.\end{aligned} \tag{3.50}$$

Employing (2.9)–(2.11) in (3.50), we deduce that

$$1 - \frac{\phi(q)\phi(q^{15})}{\phi(q^3)\phi(q^5)} + 2q\frac{\psi(-q^3)\psi(-q^5)}{\phi(q^3)\phi(q^5)} = 2q^3\frac{\psi^2(-q)\psi^2(-q^{15})}{\psi(-q^3)\psi(-q^5)\phi(q^3)\phi(q^5)}. \tag{3.51}$$

Multiplying both sides of (3.51) by $4\{\phi^2(q^3)\phi^2(q^5)\}/\{\phi^2(q)\phi^2(q^{15})\}$ and employing (2.2) and (2.3), we deduce that

$$\begin{aligned}&4\left(\frac{\phi^2(q^3)\phi^2(q^5)}{\phi^2(q)\phi^2(q^{15})} - \frac{\phi(q^3)\phi(q^5)}{\phi(q)\phi(q^{15})} + 2q\frac{\psi(-q^3)\psi(-q^5)\phi(q^3)\phi(q^5)}{\phi^2(q)\phi^2(q^{15})}\right)\\ &\quad = \left(\frac{2qf(-q^2)f(-q^{30})\chi(q^3)\chi(q^5)}{\phi(q)\phi(q^{15})}\right)^3.\end{aligned} \tag{3.52}$$

Again, multiplying both sides of (3.38) by $\{\phi^2(q^3)\phi^2(q^5)\}/\{\phi^2(q)\phi^2(q^{15})\}$, we obtain

$$\begin{aligned}&\frac{\phi^2(q^3)\phi^2(q^5)}{\phi^2(q)\phi^2(q^{15})}\left(\frac{\phi^2(q)\phi^2(q^{15})}{\phi^2(q^3)\phi^2(q^5)} + \frac{\phi(q)\phi(q^{15})}{\phi(q^3)\phi(q^5)} - \frac{\phi(q^3)\phi(q^5)}{\phi(q)\phi(q^{15})} - 1\right)\\ &\quad = 8q\frac{\psi(-q^3)\psi(-q^5)\phi(q^3)\phi(q^5)}{\phi^2(q)\phi^2(q^{15})}.\end{aligned} \tag{3.53}$$

From (3.52) and (3.53), we find that

$$\left(1 - \frac{\phi(q^3)\phi(q^5)}{\phi(q)\phi(q^{15})}\right)^3 = \left(\frac{2qf(-q^2)f(-q^{30})\chi(q^3)\chi(q^5)}{\phi(q)\phi(q^{15})}\right)^3. \tag{3.54}$$

Taking cube roots on both sides, we readily arrive at (3.48). □

Theorem 3.13 (Entry 9(vi), p. 377 of [3]). *We have*

$$\phi(q)\phi(q^{15}) + \phi(q^3)\phi(q^5) = 2f(-q^6)f(-q^{10})\chi(q)\chi(q^{15}). \tag{3.55}$$

Proof. Replacing q by $-q$ in (3.12), and multiplying with (3.11), we deduce that

$$\psi(q)\psi(q^{15})\phi(-q^2)\phi(-q^{30}) + \psi(-q)\psi(-q^{15})\phi(q^3)\phi(q^5) + 2q^2\psi^2(-q)\psi^2(-q^{15})$$
$$= 2\psi(q^6)\psi(q^{10})\phi(-q^3)\phi(-q^5). \tag{3.56}$$

Employing (2.9)–(2.11), we find that

$$1 + \frac{\phi(q^3)\phi(q^5)}{\phi(q)\phi(q^{15})} + 2q^2\frac{\psi(-q)\psi(-q^{15})}{\phi(q)\phi(q^{15})} = 2\frac{\psi^2(-q^3)\psi^2(-q^5)}{\psi(-q)\psi(-q^{15})\phi(q)\phi(q^{15})}. \tag{3.57}$$

As shown in the proof of Entry 9(v), multiplying both sides of (3.57) by $4\{\phi^2(q)\phi^2(q^{15})\}/\{\phi^2(q^3)\phi^2(q^5)\}$ and multiplying both sides of (3.37) by $\{\phi^2(q)\phi^2(q^{15})\}/\{\phi^2(q^3)\phi^2(q^5)\}$, we obtain

$$\left(1 + \frac{\phi(q)\phi(q^{15})}{\phi(q^3)\phi(q^5)}\right)^3 = \left(\frac{2f(-q^6)f(-q^{10})\chi(q)\chi(q^{15})}{\phi(q^3)\phi(q^5)}\right)^3. \tag{3.58}$$

Taking cube roots on both sides, we arrive at (3.55) to finish the proof of the theorem. □

Theorem 3.14 (Entry 9(vii), p. 377 of [3]). *We have*

$$\{\psi(q^3)\psi(q^5) - q\psi(q)\psi(q^{15})\}\phi(-q^3)\phi(-q^5)$$
$$= \{\psi(q^3)\psi(q^5) + q\psi(q)\psi(q^{15})\}\phi(-q)\phi(-q^{15}) \tag{3.59}$$
$$= f(-q)f(-q^3)f(-q^5)f(-q^{15}). \tag{3.60}$$

Proof. Adding (3.48) and (3.55), and then employing (2.2) and replacing q by $-q$, we obtain

$$\phi(-q)\phi(-q^{15})$$
$$= \chi(-q)\chi(-q^3)\chi(-q^5)\chi(-q^{15})\{\psi(q^3)\psi(q^5) - q\psi(q)\psi(q^{15})\}. \tag{3.61}$$

Similarly, subtracting (3.48) from (3.55), we find that

$$\phi(-q^3)\phi(-q^5)$$
$$= \chi(-q)\chi(-q^3)\chi(-q^5)\chi(-q^{15})\{\psi(q^3)\psi(q^5) + q\psi(q)\psi(q^{15})\}. \tag{3.62}$$

From (3.61) and (3.62), we easily arrive at (3.59).

To complete the proof of the theorem, it is enough to show that

$$\{\psi(q^3)\psi(q^5) - q\psi(q)\psi(q^{15})\}\phi(-q^3)\phi(-q^5)$$
$$= f(-q)f(-q^3)f(-q^5)f(-q^{15}). \tag{3.63}$$

Now, employing (2.9) in (3.19), we obtain

$$\phi(q)\phi(q^{15}) - \{\phi(q^3)\phi(-q^3)\phi(q^5)\phi(-q^5)\}^{1/2} = 2q\psi(q^3)\psi(q^5). \tag{3.64}$$

Again, using (2.1), (3.10) and (2.11), we deduce that

$$\phi(q)\phi(q^{15}) - \phi(q^3)\phi(q^5) = 2q\psi(q^3)\psi(q^5)\left\{1 - q^2\frac{\psi(q^2)\psi(q^{30})}{\psi(q^6)\psi(q^{10})}\right\}. \tag{3.65}$$

Employing (3.48) in (3.65), and using (2.2) and (2.10), we find that

$$\begin{aligned}&\phi(-q^6)\phi(-q^{10})\{\psi(q^6)\psi(q^{10}) - q^2\psi(q^2)\psi(q^{30})\}\\ &\quad= f(-q^2)f(-q^6)f(-q^{10})f(-q^{30}).\end{aligned} \tag{3.66}$$

Replacing q by $q^{1/2}$, we arrive at (3.63) to complete the proof of the theorem. □

4. Modular equations

In this section, we present proofs of Ramanujan's modular equations of composite degree 15 by using theta function identities. Throughout this section, we suppose that β, γ and δ are of the third, fifth and fifteenth degrees over α. Also, m and m' are the multipliers associated with the pairs α, β and γ, δ, respectively. Modular equations in Entries 11(i)–11(v), Entry 11(xiv), and 11(v) of [3] can be easily obtained transcribing some of the theta function identities in the previous section.

Remark 4.1. Baruah and Berndt (Theorems 8.1, 8.2 and 8.3 of [1]) have shown that the modular equations in Entry 11(iv), Entry 11(v) and Entry 11(xiv) of [3] imply nice partition identities.

Theorem 4.2 (Entry 11(vi), p. 384 of [3]). *We have*

$$\begin{aligned}&(\alpha\delta)^{1/16}(\{(1+\sqrt{\alpha})(1+\sqrt{\delta})\}^{1/4} + \{(1-\sqrt{\alpha})(1-\sqrt{\delta})\}^{1/4})\\ &\quad+ \{(1-\alpha)(1-\delta)\}^{1/16}(\{(1+\sqrt{1-\alpha})(1+\sqrt{1-\delta})\}^{1/4}\\ &\quad+ \{(1-\sqrt{1-\alpha})(1-\sqrt{1-\delta})\}^{1/4}) = \sqrt{2}.\end{aligned} \tag{4.1}$$

Proof. Adding (3.2) and (3.6), and then putting $Q = q^{15}$, we find that

$$\begin{aligned}\phi(q^3)\phi(q^5) &= \phi(Q^8)\phi(q^8) + 2q^8\psi(Q^4)\psi(q^4)\\ &\quad+ 4q^{32}\psi(Q^{16})\psi(q^{16}) + q^2\{\psi(q)\psi(Q) - \psi(-q)\psi(-Q)\}.\end{aligned} \tag{4.2}$$

From (3.12) and (4.2), and replacing q by $q^{1/2}$, we deduce that

$$\begin{aligned}&\phi(-q)\phi(-Q) + q\{\psi(q^{1/2})\psi(Q^{1/2}) + \psi(-q^{1/2})\psi(-Q^{1/2})\}\\ &\quad= \phi(q^4)\phi(Q^4) + 2q^4\psi(Q^2)\psi(q^2) + 4q^{16}\psi(Q^8)\psi(q^8).\end{aligned} \tag{4.3}$$

Transcribing (4.3) by employing (2.13), (2.15), (2.18) and (2.19)–(2.21), we find that

$$\begin{aligned}&\sqrt{2}(\alpha\delta)^{1/16}(\{(1+\sqrt{\alpha})(1+\sqrt{\delta})\}^{1/4} + \{(1-\sqrt{\alpha})(1-\sqrt{\delta})\}^{1/4})\\ &\quad- (\alpha\delta)^{1/4} + \{(1-\alpha)(1-\delta)\}^{1/4} = 1.\end{aligned} \tag{4.4}$$

The reciprocal, in the sense of (Entry 24(v), p. 216 of [3]), of (4.4) is given by

$$\begin{aligned}\sqrt{2}\{(1-\alpha)(1-\delta)\}^{1/16}&(\{(1+\sqrt{1-\alpha})(1+\sqrt{1-\delta})\}^{1/4}\\&+\{(1-\sqrt{1-\alpha})(1-\sqrt{1-\delta})\}^{1/4})\\&-\{(1-\alpha)(1-\delta)\}^{1/4}+(\alpha\delta)^{1/4}=1.\end{aligned}\tag{4.5}$$

Adding (4.4) and (4.5), we deduce (4.1). □

Theorem 4.3 (Entry 11(vii), p. 384 of [3]). *We have*

$$\begin{aligned}(\beta\gamma)^{1/16}&(\{(1+\sqrt{\beta})(1+\sqrt{\gamma})\}^{1/4}-\{(1-\sqrt{\beta})(1-\sqrt{\gamma})\}^{1/4})\\&+\{(1-\beta)(1-\gamma)\}^{1/16}(\{(1+\sqrt{1-\beta})(1+\sqrt{1-\gamma})\}^{1/4}\\&-\{(1-\sqrt{1-\beta})(1-\sqrt{1-\gamma})\}^{1/4})=\sqrt{2}.\end{aligned}\tag{4.6}$$

Proof. From (3.6), we note that

$$\psi(q)\psi(q^{15})-\psi(-q)\psi(-q^{15})=\frac{1}{2q^2}\{\phi(q^3)\phi(q^5)-\phi(-q^3)\phi(-q^5)\}.\tag{4.7}$$

Adding (4.7) and (3.11), we find that

$$2\psi(q)\psi(q^{15})=2\psi(q^6)\psi(q^{10})+\frac{1}{2q^2}\{\phi(q^3)\phi(q^5)-\phi(-q^3)\phi(-q^5)\}.\tag{4.8}$$

Now, replacing q by $q^{1/2}$ in (3.10), we obtain

$$2q^{3/2}\psi(q)\psi(q^{15})=\psi(q^{3/2})\psi(q^{5/2})-\psi(-q^{3/2})\psi(-q^{5/2}).\tag{4.9}$$

From (4.8) and (4.9), we deduce that

$$\begin{aligned}2q^{1/2}&\{\psi(q^{3/2})\psi(q^{5/2})-\psi(-q^{3/2})\psi(-q^{5/2})\}\\&=4q^2\psi(q^6)\psi(q^{10})+\{\phi(q^3)\phi(q^5)-\phi(-q^3)\phi(-q^5)\}.\end{aligned}\tag{4.10}$$

Transcribing (4.10) by using (2.12), (2.13), (2.18), (2.20) and (2.21), we find that

$$\begin{aligned}\sqrt{2}(\beta\gamma)^{1/16}&(\{(1+\sqrt{\beta})(1+\sqrt{\gamma})\}^{1/4}-\{(1-\sqrt{\beta})(1-\sqrt{\gamma})\}^{1/4})\\&=(\beta\gamma)^{1/4}-\{(1-\beta)(1-\gamma)\}^{1/4}+1.\end{aligned}\tag{4.11}$$

The reciprocal of the above modular equation is given by

$$\begin{aligned}\sqrt{2}\{(1-\beta)(1-\gamma)\}^{1/16}&(\{(1+\sqrt{1-\beta})(1+\sqrt{1-\gamma})\}^{1/4}\\&\quad-\{(1-\sqrt{1-\beta})(1-\sqrt{1-\gamma})\}^{1/4})\\&=\{(1-\beta)(1-\gamma)\}^{1/4}-(\beta\gamma)^{1/4}+1.\end{aligned}\tag{4.12}$$

Adding (4.11) and (4.12), we arrive at (4.6) to complete the proof. □

Theorem 4.4 (Entry 11(viii), p. 384 of [3]). *We have*

$$\left(\frac{\alpha\delta}{\beta\gamma}\right)^{1/8}+\left(\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}\right)^{1/8}-\left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/8}=\sqrt{\frac{m'}{m}}. \tag{4.13}$$

Proof. Transcribing (3.11) by using (2.16)–(2.18), we deduce that

$$\left(\frac{\alpha\delta}{\beta\gamma}\right)^{1/8}+\left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma}\right)^{1/8}=(\beta\gamma)^{1/8}\sqrt{\frac{m'}{m}}. \tag{4.14}$$

The reciprocal modular equation of (4.14) is given by

$$\left(\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}\right)^{1/8}+\left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}\right)^{1/8}$$
$$=\{(1-\beta)(1-\gamma)\}^{1/8}\sqrt{\frac{m'}{m}}. \tag{4.15}$$

Adding (4.14) and (4.15), and employing modular equations in Entries 11(i), 11(ii), 11(iv) and 11(v) of [3], we finish the proof. □

Theorem 4.5 (Entry 11(ix), p. 384 of [3]). *We have*

$$\left(\frac{\beta\gamma}{\alpha\delta}\right)^{1/8}+\left(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right)^{1/8}-\left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/8}=-\sqrt{\frac{m}{m'}}. \tag{4.16}$$

Proof. One can arrive at the above modular equation transcribing (3.10) by means of (2.16)–(2.18) and proceeding as shown in the previous theorem. □

In the next theorem we present two new modular equations.

Theorem 4.6. *We have*

$$(\beta\gamma)^{1/8}-(\alpha\delta)^{1/8}$$
$$=1-(\alpha\beta\gamma\delta)^{1/8}-\{(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{1/8}, \tag{4.17}$$
$$=\{\alpha\delta(1-\beta)(1-\gamma)\}^{1/8}+\{\beta\gamma(1-\alpha)(1-\delta)\}^{1/8}. \tag{4.18}$$

Proof. Transcribing (3.11) and (3.10) by using (2.16)–(2.18), and then multiplying, we deduce that

$$\{(1-\beta)(1-\gamma)\}^{1/8}-\{(1-\alpha)(1-\delta)\}^{1/8}$$
$$=1-(\alpha\beta\gamma\delta)^{1/8}-\{(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{1/8}. \tag{4.19}$$

Again transcribing (3.25) by employing (2.12), (2.14) and (2.16), we find that

$$(\alpha\delta)^{1/8}-\{(1-\alpha)(1-\delta)\}^{1/8}=(\beta\gamma)^{1/8}-\{(1-\beta)(1-\gamma)\}^{1/8}. \tag{4.20}$$

Employing (4.20) in (4.19), we arrive at (4.17).

Similarly, transcribing (3.12) and (3.19) by means of (2.12), (2.14) and (2.16), and then multiplying them we readily obtain (4.18). □

Theorem 4.7 (Entries 11(xii)–11(xiii), p. 384 of [3]). *We have*

$$\left(\frac{\gamma\delta}{\alpha\beta}\right)^{1/4}+\left(\frac{(1-\gamma)(1-\delta)}{(1-\alpha)(1-\beta)}\right)^{1/4}+\left(\frac{\gamma\delta(1-\gamma)(1-\delta)}{\alpha\beta(1-\alpha)(1-\beta)}\right)^{1/4}$$
$$-2\left(\frac{\gamma\delta(1-\gamma)(1-\delta)}{\alpha\beta(1-\alpha)(1-\beta)}\right)^{1/8}\left\{1+\left(\frac{\gamma\delta}{\alpha\beta}\right)^{1/8}+\left(\frac{(1-\gamma)(1-\delta)}{(1-\alpha)(1-\beta)}\right)^{1/8}\right\}=\frac{z_1z_3}{z_5z_{15}}, \tag{4.21}$$

$$\left(\frac{\alpha\beta}{\gamma\delta}\right)^{1/4}+\left(\frac{(1-\alpha)(1-\beta)}{(1-\gamma)(1-\delta)}\right)^{1/4}+\left(\frac{\alpha\beta(1-\alpha)(1-\beta)}{\gamma\delta(1-\gamma)(1-\delta)}\right)^{1/4}$$
$$-2\left(\frac{\alpha\beta(1-\alpha)(1-\beta)}{\gamma\delta(1-\gamma)(1-\delta)}\right)^{1/8}\left\{1+\left(\frac{\alpha\beta}{\gamma\delta}\right)^{1/8}+\left(\frac{(1-\alpha)(1-\beta)}{(1-\gamma)(1-\delta)}\right)^{1/8}\right\}=25\frac{z_5z_{15}}{z_1z_3}. \tag{4.22}$$

The modular equation (4.22) is the reciprocal of (4.21). So, we prove only (4.21).

Proof of (4.21). Since γ and δ have degree 5 over α and β respectively, from Entry 13(xii), p. 281 of [3], we have

$$\frac{z_1}{z_5}=\left(\frac{\gamma}{\alpha}\right)^{1/4}+\left(\frac{1-\gamma}{1-\alpha}\right)^{1/4}-\left(\frac{\gamma(1-\gamma)}{\alpha(1-\alpha)}\right)^{1/4}, \tag{4.23}$$

$$\frac{z_3}{z_{15}}=\left(\frac{\delta}{\beta}\right)^{1/4}+\left(\frac{1-\delta}{1-\beta}\right)^{1/4}-\left(\frac{\delta(1-\delta)}{\beta(1-\beta)}\right)^{1/4}. \tag{4.24}$$

Multiplying (4.23) and (4.24), we obtain

$$\frac{z_1z_3}{z_5z_{15}}=\left(\left(\frac{\gamma}{\alpha}\right)^{1/4}+\left(\frac{1-\gamma}{1-\alpha}\right)^{1/4}-\left(\frac{\gamma(1-\gamma)}{\alpha(1-\alpha)}\right)^{1/4}\right)$$
$$\times\left(\left(\frac{\delta}{\beta}\right)^{1/4}+\left(\frac{1-\delta}{1-\beta}\right)^{1/4}-\left(\frac{\delta(1-\delta)}{\beta(1-\beta)}\right)^{1/4}\right). \tag{4.25}$$

We can rewrite (4.25) as

$$\frac{z_1z_3}{z_5z_{15}}=\left(\frac{\gamma\delta}{\alpha\beta}\right)^{1/4}+\left(\frac{(1-\gamma)(1-\delta)}{(1-\alpha)(1-\beta)}\right)^{1/4}+\left(\frac{\gamma\delta(1-\gamma)(1-\delta)}{\alpha\beta(1-\alpha)(1-\beta)}\right)^{1/4}$$
$$-\frac{A}{\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/4}}, \tag{4.26}$$

where

$$A=(\alpha\delta)^{1/4}(\{(1-\gamma)(1-\delta)\}^{1/4}-\{(1-\beta)(1-\gamma)\}^{1/4})$$
$$+(\beta\gamma)^{1/4}(\{(1-\gamma)(1-\delta)\}^{1/4}-\{(1-\alpha)(1-\delta)\}^{1/4})$$
$$+(\gamma\delta)^{1/4}(\{(1-\alpha)(1-\delta)\}^{1/4}+\{(1-\beta)(1-\gamma)\}^{1/4}). \tag{4.27}$$

Now, we recall from (4.18) that

$$(\beta\gamma)^{1/8} - (\alpha\delta)^{1/8} = \{\alpha\delta(1-\beta)(1-\gamma)\}^{1/8} + \{\beta\gamma(1-\alpha)(1-\delta)\}^{1/8}. \tag{4.28}$$

Squaring both sides of (4.28), we obtain

$$\begin{aligned}\{\alpha\delta(1-\beta)(1-\gamma)\}^{1/4} &+ \{\beta\gamma(1-\alpha)(1-\delta)\}^{1/4}\\ &= (\beta\gamma)^{1/4} + (\alpha\delta)^{1/4} - 2(\alpha\beta\gamma\delta)^{1/8}\\ &\quad - 2\{\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{1/8}.\end{aligned} \tag{4.29}$$

Again, from (4.20), we have

$$\{(1-\alpha)(1-\delta)\}^{1/8} - \{(1-\beta)(1-\gamma)\}^{1/8} = (\alpha\delta)^{1/8} - (\beta\gamma)^{1/8}. \tag{4.30}$$

Squaring both sides of (4.30), we find that

$$\begin{aligned}\{(1-\alpha)(1-\delta)\}^{1/4} &+ \{(1-\beta)(1-\gamma)\}^{1/4}\\ &= (\alpha\delta)^{1/4} + (\beta\gamma)^{1/4} - 2(\alpha\beta\gamma\delta)^{1/8}\\ &\quad + 2\{(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{1/8}.\end{aligned} \tag{4.31}$$

Employing (4.29) and (4.31) in (4.27), we deduce that

$$\begin{aligned}A = &\, 2\{\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{1/8}\\ &+ 2(\gamma\delta)^{1/4}\{(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{1/8}\\ &+ 2(\alpha\beta\gamma\delta)^{1/8}\{1-(\gamma\delta)^{1/4}\} + ((\gamma\delta)^{1/4}\\ &+ \{(1-\gamma)(1-\delta)\}^{1/4} - 1)\{(\alpha\delta)^{1/4} + (\beta\gamma)^{1/4}\}\end{aligned} \tag{4.32}$$

Since δ has degree 3 over γ, from Entry 5(ii), p. 230 of [3], we have

$$(\gamma\delta)^{1/4} + \{(1-\gamma)(1-\delta)\}^{1/4} = 1. \tag{4.33}$$

Employing (4.33) in (4.32), and then using (4.26), we finally arrive at (4.21) to complete the proof. □

Acknowledgment

The authors are grateful to the anonymous referee for his/her helpful comments.

References

[1] Baruah N D and Berndt B C, Partition identities and Ramanujan's modular equations, *J. Combin. Theory* **A114** (2007) 1024–1045

[2] Baruah N D and Berndt B C, Partition identities arising from theta function identities, *Acta Math. Sin. (Engl. Ser.)* **24** (2008) 955–970

[3] Berndt B C, Ramanujan's Notebooks, Part III (New York: Springer-Verlag) (1991)
[4] Berndt B C, Ramanujan's Notebooks, Part V (New York: Springer-Verlag) (1998)
[5] Cooper S, Theta function identities for level 15, Ramanujan Rediscovered (eds) N D Baruah *et al*, Lecture Note Series, Ramanujan Mathematical Society, to appear
[6] Ramanujan S, Notebooks (2 volumes) (Bombay: Tata Institute of Fundamental Research) (1957)
[7] Whittaker E T and Watson G N, A Course of Modern Analysis (Cambridge: University Press) (1966) Indian edition (New Delhi: Universal Book Stall) (1991)