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On qualitative analysis of delay systems and $x^{\triangle} = f(t, x, x^{\sigma})$ on time scales

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Abstract. Here we solve two problems presented in paper [9] (C C Tisdell and A Zaidi, Basic qualitative and quantitative results for solutions to nonlinear, dynamic equations on time scales with an application to economic modelling, *Nonlinear Anal.* **68** (2008) 3504–3524). We study existence and uniqueness of solutions for delay systems and first-order dynamic equations of the form $x^{\triangle} = f(t, x, x^{\sigma})$ on time scales by using the Banach's fixed-point theorem. Some examples are presented to illustrate the efficiency of the proposed results.

Keywords. Existence; uniqueness; delay systems; time scales; fixed-point theorem.

1. Introduction

The theory of dynamical systems on time scales has been developing rapidly and has received a lot of attention in recent years, for example, see [1–8, 10–13] and references therein. Recently, Tisdell and Zaidai [9] investigated both basic qualitative and quantitative properties of solutions to first-order dynamic equations of the forms $x^{\triangle} = f(t, x)$ and $x^{\triangle} = f(t, x^{\sigma})$ on time scales. They gave some open problems. Two of those problems are on obtaining qualitative and quantitative results for the delay systems and first-order dynamic equations of the form $x^{\triangle} = f(t, x, x^{\sigma})$ on time scales, respectively.

In this paper, we solve the two problems, and investigate both basic qualitative and quantitative properties of solutions to delay systems and first-order dynamic equations of the form $x^{\triangle} = f(t, x, x^{\sigma})$ on time scales. Particular focus lies in the existence, uniqueness, and approximation of solutions to the nonlinear initial value problem. Three examples are presented to illustrate the efficiency of the proposed result.

2. Preliminaries

Let T be a time scale (an arbitrary nonempty closed subset of the real numbers) with $t_0 \geq 0$ as minimal element and no maximal element.

DEFINITION 2.1 [1]

The mappings $\sigma, \rho \colon T \to T$ defined as $\sigma(t) = \inf\{s \in T \colon s > t\}$ and $\rho(t) = \sup\{s \in T \colon s < t\}$ are called jump operators.

DEFINITION 2.2 [1]

If $\sigma(t) > t$, we say that t is right-scattered (rs), while if $\rho(t) < t$ we say that t is left-scattered (ls). Also, if $t < \sup T$ and $\sigma(t) = t$, then t is called right-dense (rd), and if $t > \inf T$ and $\rho(t) = t$, then t is called left-dense (ld).

DEFINITION 2.3 [1]

The graininess function μ: $T \to [0, \infty)$ is defined by

$$\mu(t) = \sigma(t) - t.$$

DEFINITION 2.4

We define the interval $[a, b]^*$ in T by

$$[a, b]^* = \{t \in T: a \le t \le b\}.$$

Open intervals and half-open intervals are defined accordingly.

DEFINITION 2.5 [1]

Assume $f: T \to R$ is a function and let $t \in T$. Then we define $f^{\triangle}(t)$ to be the number with the property that given any $\varepsilon > 0$, there is a neighborhood U of t (i.e., $U = (t-\delta, t+\delta) \cap T$ for some $\delta > 0$) such that

$$|[f(\sigma(t)) - f(s)] - f^{\triangle}(t)[\sigma(t) - s]| \le \varepsilon|\sigma(t) - s|,$$

for all $s \in U$ and we call $f^{\triangle}(t)$ the delta derivative of f at t.

DEFINITION 2.6 [1]

A function p: $T \to R$ is called regressive if $1 + \mu(t)p(t) \neq 0$ for all $t \in T$.

If p is a regressive function, then the generalized exponential function e_p is defined by

$$e_p(t, s) = \exp\left\{\int_s^t \xi_{\mu(\tau)}(p(\tau))\triangle\tau\right\}, \quad s, t \in T$$

with the cylinder transformation

$$\xi_h(z) = \begin{cases} \frac{\mathrm{Log}(1+hz)}{h}, & h \neq 0, \\ z, & h = 0, \end{cases}$$

where Log is the principal logarithm function.

3. Delay systems

In this section we consider following delay systems

$$x^{\triangle} = f(t, x_t), t \in [a, b]^* \tag{3.1}$$

subject to the initial condition

$$x_a = \psi, \tag{3.2}$$

where $f\colon [a,b]^* \times C([-r,0]^*, R^n) \to R^n$ be continuous and $x_t \in C$ is defined by $x_t(s) = x(t+s)$, $s \in [-r,0]^*$ which implies that $t+s \in [a-r, \sigma(b)]^*$, $\psi \in C$. We also equip the space C with the norm $\|\cdot\|$ defined by $\|x_t\| = \sup_{s\in[-r,0]^*} |x(t+s)|$.

Based on the ideas in [3, 9], we present following basic lemmas:

Lemma 3.1. *Let $f\colon [a,b]^* \times C \to R^n$ be continuous.*

(i) *If $x(t) \in C([a-r,\sigma(b)]^*; R^n)$ is a solution of* (3.1) *and* (3.2), *then*

$$\begin{cases} x(t) = \int_a^t f(\tau, x_\tau)\triangle\tau + \psi(0), \ t \in [a,\sigma(b)]^*, \\ x(t) = \psi(t-a), \ t \in [a-r,a]^*, \\ x_a = \psi. \end{cases} \tag{3.3}$$

(ii) *If $x(t) \in C([a-r,\sigma(b)]^*; R^n)$ satisfies* (3.3) *then $x^{\triangle}(t) \in C([a,b]^*; R^n)$ and $x(t)$ is a solution of* (3.1) *and* (3.2).

Proof. If $x(t)$ satisfies (3.1) and (3.2), then

$$\int_a^t x^{\triangle}(\tau)\triangle\tau = \int_a^t f(\tau, x_\tau)\triangle\tau, \ t \in [a,\sigma(b)]^*,$$

so that

$$x(t) = \int_a^t f(\tau, x_\tau)\triangle\tau + \psi(0), \ t \in [a,\sigma(b)]^*.$$

Conversely, if $x(t) \in C([a-r,\sigma(b)]^*; R^n)$ satisfies (3.3), then $x^{\triangle}(t) = f(t, x_t)$, $t \in [a,b]^*$. Therefore, $x(t)$ is a solution of (3.1) and (3.2). □

Let $\beta > 0$ be a constant and let $|\cdot|$ denote the Euclidean norm on R^n. We will consider the space of continuous functions $C([a-r,\sigma(b)]^*; R^n)$ coupled with a suitable metric, either

$$d_\beta(x,y) = \sup_{t\in[a-r,\sigma(b)]^*} \frac{|x(t)-y(t)|}{e_\beta(t,a-r)},$$

or

$$d_0 = \sup_{t\in[a-r,\sigma(b)]^*} |x(t)-y(t)|.$$

We will also consider $C([a-r,\sigma(b)]^*; R^n)$ coupled with a suitable norm, either

$$\|x\|_\beta = \sup_{t\in[a-r,\sigma(b)]^*} \frac{|x(t)|}{e_\beta(t,a-r)},$$

or

$$\|x\|_0 = \sup_{t\in[a-r,\sigma(b)]^*} |x(t)|.$$

Lemma 3.2 [9]. *Some important properties of* d_β *and* $\|\cdot\|_\beta$ *are listed*:

(i) d_β *is a metric.*
(ii) $(C([a-r,\sigma(b)]^*; R^n), d_\beta)$ *is a complete metric space.*
(iii) $\|\cdot\|_\beta$ *is a norm and is equivalent to* $\|\cdot\|_0$.
(iv) $(C([a-r,\sigma(b)]^*; R^n), \|\cdot\|_\beta)$ *is a Banach space.*

Lemma 3.3 [9]. *Let* (Y,d) *be a complete metric space and let* $F\colon Y\to Y$ *be contractive. Then* F *has a unique fixed-point* u *and* $F^i(y)\to u$ *for each* $y\in Y$, *where the sequence* $\{F^i(y)\}$ *is defined by* $F^0(y)=y$ *and* $F^{i+1}(y)=F(F^i(y))$.

We are now ready to present the main result of this section, which will be proved by using Banach's theorem.

Theorem 3.4. *Let* $f\colon [a,b]^*\times C\to R^n$ *be continuous and let* $L>0$ *be a constant. If* $\forall t\in[a,b]^*$, $\phi\in C$, $\varphi\in C$,

$$|f(t,\phi)-f(t,\varphi)|\le L\|\phi-\varphi\|, \tag{3.4}$$

holds, *then the dynamic equations* (3.1) *and* (3.2) *has a unique solution. In addition*, *if a sequence of functions* $\{x_i\}$ *is defined inductively by choosing any* $x_0\in C$ *and setting*

$$x_{i+1}(t)=\int_a^t f(\tau,(x_i)_\tau)\triangle\tau+\psi(0), \tag{3.5}$$

then the sequence $\{x_i\}$ *converges uniformly on* $[a-r,\sigma(b)]^*$ *to the unique solution* $x(t)$ *of* (3.1) *and* (3.2). *Furthermore*, $x^\triangle\in C([a,b]^*; R^n)$.

Proof. Since f is a continuous function, (3.5) is well-defined. Let $\beta=L\gamma$ where $\gamma>1$ is a constant. Consider the complete metric space $(C([a-r,\sigma(b)]^*; R^n), d_\beta)$. Let

$$F\colon C([a-r,\sigma(b)]^*; R^n)\to C([a-r,\sigma(b)]^*; R^n)$$

be defined by

$$[Fx](a+s)=\psi(s),\ s\in[-r,0]^*,$$

$$[Fx](t)=\int_a^t f(\tau,x_\tau)\triangle\tau+\psi(0),\ t\in[a,\sigma(b)]^*.$$

By Lemma 3.1., fixed-points of F will be solutions to the delay systems (3.1) and (3.2). Thus, we want to prove that there exists a unique x such that $Fx=x$. To do this, we show that F is a contractive map and Banach's fixed-point theorem will then apply. For any $u,v\in C([a-r,\sigma(b)]^*; R^n)$, we have

$$Fu-Fv=\begin{cases}0, & t\in[a-r,a]^*\\ \int_a^t[f(\tau,u_\tau)-f(\tau,v_\tau)]\triangle\tau, & t\in[a,\sigma(b)]^*\end{cases}$$

so

$$\begin{aligned}
&d_\beta(Fu, Fv)\\
&\quad = \sup_{t\in[a-r,\sigma(b)]^*} \frac{|[Fu](t) - [Fv](t)|}{e_\beta(t, a-r)}\\
&\quad \le \sup_{t\in[a,\sigma(b)]^*} \left[\frac{1}{e_\beta(t, a-r)} \int_a^t |f(\tau, u_\tau) - f(\tau, v_\tau)|\triangle\tau\right]\\
&\quad \le \sup_{t\in[a,\sigma(b)]^*} \left[\frac{1}{e_\beta(t, a-r)} \int_a^t L\|u_\tau - v_\tau\|\triangle\tau\right]\\
&\quad = L \sup_{t\in[a,\sigma(b)]^*} \left[\frac{1}{e_\beta(t, a-r)} \int_a^t \sup_{s\in[-r,0]^*} |u(\tau+s) - v(\tau+s)|\triangle\tau\right]\\
&\quad = L \sup_{t\in[a,\sigma(b)]^*} \left[\frac{1}{e_\beta(t, a-r)} \int_a^t e_\beta(\tau, a-r) \sup_{s\in[-r,0]^*} \frac{|u(\tau+s) - v(\tau+s)|}{e_\beta(\tau, a-r)}\triangle\tau\right]\\
&\quad \le L \sup_{t\in[a,\sigma(b)]^*} \left[\frac{1}{e_\beta(t, a-r)} \int_a^t e_\beta(\tau, a-r) \sup_{s\in[-r,0]^*} \frac{|u(\tau+s) - v(\tau+s)|}{e_\beta(\tau+s, a-r)}\triangle\tau\right]\\
&\quad \le L \sup_{t\in[a,\sigma(b)]^*} \left[\frac{1}{e_\beta(t, a-r)} \int_a^t e_\beta(\tau, a-r) \sup_{s\in[a-r,\sigma(b)]^*} \frac{|u(s) - v(s)|}{e_\beta(s, a-r)}\triangle\tau\right]\\
&\quad = Ld_\beta(u, v) \sup_{t\in[a,\sigma(b)]^*} \left[\frac{1}{e_\beta(t, a-r)} \int_a^t e_\beta(\tau, a-r)\triangle\tau\right]\\
&\quad \le Ld_\beta(u_t, v_t) \sup_{t\in[a,\sigma(b)]^*} \left[\frac{1}{e_\beta(t, a-r)} \cdot \frac{e_\beta(t, a-r) - e_\beta(a, a-r)}{\beta}\right]\\
&\quad = \frac{d_\beta(u, v)}{\gamma} \sup_{t\in[a,\sigma(b)]^*} \left[1 - \frac{e_\beta(a, a-r)}{e_\beta(t, a-r)}\right]\\
&\quad = \frac{d_\beta(u, v)}{\gamma} \left[1 - \frac{e_\beta(a, a-r)}{e_\beta(\sigma(b), a-r)}\right]\\
&\quad < d_\beta(u, v).
\end{aligned}$$

We see that F is a contractive map and Banach's fixed-point theorem applies, yielding the existence of an unique fixed-point x of F. In addition, from Banach's theorem, the sequence $\{x_i\}$ defined in (3.5) converges uniformly in the norm $\|\cdot\|_\beta$ and thus the sequence $\{x_i\}$ converges uniformly in the norm $\|\cdot\|_0$ to that fixed-point x. This completes the proof. □

Remark. The existence and uniqueness of the solutions of delay equations $x^\Delta = F(t, x(t), x(\theta(t)))$ have been studied in [8]. While in the present paper, we studied the open problem that is mentioned in [9] and the systems we studied are in the form $x^\Delta = f(t, x_t)$.

Example 3.1. Consider following problem:

$$x^{\Delta} = (2 + \cos t)x(t - 2\pi), \quad t = 0, \pi, \dots, 100\pi.$$
$$x_0 = \psi.$$

We claim that this delay dynamic system has a unique solution $x(t)$, $t = -2\pi, -\pi, 0, \dots, \sigma(100\pi)$.

Since

$$\begin{aligned} |f(t,\phi) - f(t,\varphi)| &= |(2+\cos t)(\phi - \varphi)|, \quad t = 0, \pi, \dots, 100\pi \\ &\le 3|\phi - \varphi| \end{aligned}$$

so that (3.4) holds with respect to $L = 3$. By Theorem 3.4, this delay dynamic system has an unique solution.

Example 3.2. Consider following problem:

$$x^{\Delta} = (x^2(t-\tau) + 3)^{\frac{1}{2}} + t - 1, \quad t \in [a, \sigma(b)]^*.$$
$$x_a = \psi.$$

where $\tau > 0$.

We claim that this delay dynamic system has an unique solution $x(t)$, $t \in [a, \sigma(b)]^*$.

Since

$$\begin{aligned} |f(t,\phi) - f(t,\varphi)| &= |(\phi^2(-\tau) + 3)^{\frac{1}{2}} - (\varphi^2(-\tau) + 3)^{\frac{1}{2}}| \\ &\le \sup_{r \in R} \left| \frac{r}{(r^2+3)^{\frac{1}{2}}} \right| |\phi(-\tau) - \varphi(-\tau)| \\ &\le \|\phi - \varphi\|, \qquad t \in [a, \sigma(b)]^*, \end{aligned}$$

(3.4) holds with respect to $L = 1$. By Theorem 3.4, this delay dynamic system has a unique solution.

4. First-order dynamic equations of the form $x^{\Delta} = f(t, x, x^{\sigma})$

In this section we consider first-order dynamic equations of the type

$$x^{\Delta} = f(t, x, x^{\sigma}), \quad t \in [a, b]^* \tag{4.1}$$

subject to the initial condition

$$x(a) = A, \tag{4.2}$$

where $f \colon [a, b]^* \times R^n \times R^n \to R^n$ be rd-continuous and $x^{\sigma} = x \circ \sigma$.

Now, we present lemmas as follows:

Lemma 4.1. *Let* $f\colon [a,b]^* \times R^n \times R^n \to R^n$ *be continuous.*

(i) *If* $x \in C([a,\sigma(b)]^*; R^n)$ *is a solution of* (4.1) *and* (4.2), *then*

$$x(t) = \int_a^t f(\tau, x(\tau), x^\sigma(\tau))\Delta\tau + A, \quad t \in [a,\sigma(b)]^*. \tag{4.3}$$

(ii) *If* $x \in C([a,\sigma(b)]^*; R^n)$ *satisfies* (4.3), *then* $x^\Delta \in C_{\mathrm{rd}}([a,b]^*; R^n)$ *and* x *is a solution of* (4.1) *and* (4.2).

Lemma 4.2. *Let* $f\colon [a,b]^* \times R^n \times R^n \to R^n$ *be* rd-*continuous. Then* (i) *and* (ii) *of Lemma* 4.1 *hold.*

The proofs of the above results are similar to the ideas in [3, 9] and is thus omitted.

Theorem 4.3. *Let* $f\colon [a,b]^* \times R^n \times R^n \to R^n$ *be* rd-*continuous and let* $L > 0$ *be a constant. If* $\forall t \in T$, $p_1, p_2, q_1, q_2 \in R^n$,

$$\|f(t,p_1,p_2) - f(t,q_1,q_2)\| \le L(\|p_1 - q_1\| + \|p_2 - q_2\|), \tag{4.4}$$

$$L \sup_{t\in[a,\sigma(b)]^*} \mu(t) < 1 \tag{4.5}$$

hold. Then the dynamic equations (4.1) *and* (4.2) *have an unique solution. In addition, if a sequence of functions* $\{x_i\}$ *is defined inductively by choosing any* $x_0 \in C([a,\sigma(b)]^*; R^n)$ *and setting*

$$x_{i+1}(t) = \int_a^t f(\tau, x_i(\tau), x_i^\sigma(\tau))\Delta\tau + A, \tag{4.6}$$

then the sequence $\{x_i\}$ *converges uniformly on* $[a,\sigma(b)]^*$ *to the unique solution* x *of* (4.1) *and* (4.2). *Furthermore,* $x^\Delta \in C_{\mathrm{rd}}([a,\sigma(b)]^*; R^n)$.

Proof. Since f is a rd-continuous function, (4.6) is well-defined. Let $\beta = L\gamma$ where $\gamma > 2$ is a constant chosen such that $L|\mu|_0 = 1 - 2/\gamma$. Consider the complete metric space $(C([a,\sigma(b)]^*; R^n), d_\beta)$. Let

$$F\colon C([a,\sigma(b)]^*; R^n) \to C([a,\sigma(b)]^*; R^n)$$

be defined by

$$[Fx](t) = \int_a^t f(\tau, x_i(\tau), x_i^\sigma(\tau))\Delta\tau + A, t \in [a,\sigma(b)]^*.$$

For any $u, v \in C([a, \sigma(b)]^*; R^n)$, one has

$$
\begin{aligned}
& d_\beta(Fu, Fv) \\
& \quad \le \sup_{t\in[a,\sigma(b)]^*} \left[\frac{1}{e_\beta(t,a)} \int_a^t \| f(\tau, u(\tau), u^\sigma(\tau)) - f(\tau, v(\tau), v^\sigma(\tau)) \| \triangle\tau \right] \\
& \quad \le \sup_{t\in[a,\sigma(b)]^*} \left[\frac{1}{e_\beta(t,a)} \int_a^t L(\|u(\tau) - v(\tau)\| + \|u^\sigma(\tau) - v^\sigma(\tau)\|) \triangle\tau \right] \\
& \quad = L \sup_{t\in[a,\sigma(b)]^*} \left[\frac{1}{e_\beta(t,a)} \int_a^t e_\beta(\tau,a) \frac{\|u(\tau) - v(\tau)\|}{e_\beta(\tau,a)} \triangle\tau \right. \\
& \qquad \left. + \frac{1}{e_\beta(t,a)} \int_a^t e_\beta(\sigma(\tau),a) \frac{\|u^\sigma(\tau) - v^\sigma(\tau)\|}{e_\beta(\sigma(\tau),a)} \triangle\tau \right] \\
& \quad \le L d_\beta(u,v) \sup_{t\in[a,\sigma(b)]^*} \left[\frac{1}{e_\beta(t,a)} \int_a^t e_\beta(\tau,a) \triangle\tau \right] \\
& \qquad + L d_\beta(u,v) \sup_{t\in[a,\sigma(b)]^*} \left[\frac{1}{e_\beta(t,a)} \int_a^t (1 + \beta\mu(\tau)) e_\beta(\tau,a) \triangle\tau \right] \\
& \quad \le L d_\beta(u,v) \sup_{t\in[a,\sigma(b)]^*} \left[\frac{1}{e_\beta(t,a)} \cdot \frac{e_\beta(t,a) - 1}{\beta} \right] \\
& \qquad + L(1 + \beta\|\mu\|_0) d_\beta(u,v) \sup_{t\in[a,\sigma(b)]^*} \left[\frac{1}{e_\beta(t,a)} \cdot \frac{e_\beta(t,a) - 1}{\beta} \right] \\
& \quad = \frac{d_\beta(u,v)}{\gamma} \sup_{t\in[a,\sigma(b)]^*} \left[1 - \frac{1}{e_\beta(t,a)} \right] \\
& \qquad + \left(\frac{1}{\gamma} + L\|\mu\|_0 \right) d_\beta(u,v) \sup_{t\in[a,\sigma(b)]^*} \left[1 - \frac{1}{e_\beta(t,a)} \right] \\
& \quad = d_\beta(u,v) \left(\frac{2}{\gamma} + L\|\mu\|_0 \right) \left[1 - \frac{1}{e_\beta(\sigma(b),a)} \right] \\
& \quad < d_\beta(u,v).
\end{aligned}
$$

Hence, we can see that F is a contractive map, and the results of Theorem 4.3 hold. This completes the proof. □

Example 4.1. Consider the problem as follows:

$$
\begin{aligned}
& x^\triangle = (x^2 + 5)^{1/2} + [(x^\sigma)^2 + 5]^{1/2} + t, \\
& x(a) = A.
\end{aligned}
$$

We claim that this dynamic system has an unique solution if $\mu(t) < M < 1$ for $t \in [a, \sigma(b)]^*$.

One can note that

$$
\begin{aligned}
&|f(t, p_1, p_2) - f(t, q_1, q_2)| \\
&\quad = |(p_1^2 + 5)^{1/2} + (q_1^2 + 5)^{1/2} - (p_2^2 + 5)^{1/2} - (q_2^2 + 5)^{1/2}| \\
&\quad \leq |(p_1^2 + 5)^{1/2} - (p_2^2 + 5)^{1/2}| + |(q_1^2 + 5)^{1/2} - (q_2^2 + 5)^{1/2}| \\
&\quad \leq \sup_{r \in R} \left| \frac{r}{(r^2 + 5)^{1/2}} \right| \cdot [|p_1 - q_1| + |p_2 - q_2|] \\
&\quad \leq |p_1 - q_1| + |p_2 - q_2|
\end{aligned}
$$

so that (4.4) and (4.5) hold with respect to $L = 1$. By Theorem 4.3, this dynamic system has an unique solution.

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