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Discreteness criteria in $PU(1, n; C)$

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Abstract. In this paper, we study the discreteness for non-elementary subgroups in $PU(1, n; C)$, and several discreteness criteria are obtained.

Keywords. Discreteness criteria; complex hyperbolic space; subgroup in $PU(1, n; C)$.

1. Introduction

The discreteness of Möbius groups is a fundamental problem, which have been discussed by many authors.

In 1976, Jørgensen [10] proved a necessary condition for a non-elementary two generator subgroup of $SL(2, C)$ to be discrete, which is called Jørgensen's inequality. By using this inequality, Jørgensen established his well-known result [11]:

Theorem J. *A non-elementary subgroup G of $SL(2, C)$ is discrete if and only if all its two-generator subgroups are discrete.*

Furthermore, Gilman [6] and Isachenko [8] showed the discreteness of all two-generator subgroups where each generator is loxodromic, which is enough to secure the discreteness of the group.

In [1, 5, 9, 13], the authors have discussed the generalization of Theorem J to n-dimensional hyperbolic space.

In complex hyperbolic space, Kamiya [12] proved that

Theorem K. *If G is a non-elementary finitely generated subgroup of $PU(1, n; C)$, then G is discrete if and only if $\langle f, g\rangle$ is discrete for any f and g in G.*

In 2001, Dai *et al* [4] generalized Theorem J as follows:

Theorem D. *If G is a non-elementary subgroup of $PU(1, n; C)$ with condition A, then G is discrete if and only if $\langle f, g\rangle$ is discrete for any f and g in G.*

Here, G is said to satisfy condition A if it has no sequence $\{g_i\}$ of distinct elements of finite order such that $\mathrm{Card}(\mathrm{fix}(g_i)) = \infty$ and $g_i \to I$ as $i \to \infty$, where

$$\mathrm{fix}(g_i) = \{x \in \partial H_C^n\colon g_i(x) = x\}.$$

In this paper, we discuss the generalization of Theorem J to the complex hyperbolic space. Our main results are the following:

Theorem 1. *If G is a non-elementary subgroup of $PU(1, n; C)$, then G is discrete if and only if*

(1) $\langle f, g\rangle$ *is discrete for any* f *and* g *in* G;
(2) G_L *is torsion free.*

Theorem 2. *If G is a non-elementary subgroup of $PU(1, n; C)$, then G is discrete if and only if*

(1) $\langle f, g\rangle$ *is discrete for any* f *and* g *in* G;
(2) G_L *is finite.*

Theorem 3. *If G is a non-elementary subgroup of $PU(1, n; C)$, then G is discrete if and only if*

(1) $\langle f, g\rangle$ *is discrete for any two loxodromics* f *and* g *in* G;
(2) G_L *is finite.*

2. Preliminaries

We start by giving some general facts about $PU(1, n; C)$, (for more details, see [7]).

Let C be the field of complex numbers, $V = V^{1,n}(C)$ $(n \geq 1)$ denote the vector space C^{n+1}, together with the unitary structure defined by the Hermitian form

$$\Phi(z^*, w^*) = -\bar{z}_0^* w_0^* + \sum_{j=1}^{n} \bar{z}_j^* w_j^*$$

for $z^* = (z_0^*, z_1^*, \dots, z_n^*)$, $w^* = (w_0^*, w_1^*, \dots, w_n^*) \in V$.

An automorphism g of V, that is a linear bijection such that

$$\Phi(z^*, w^*) = \Phi(g(z^*), g(w^*))$$

for $z^*, w^* \in V$, will be called a unitary transformation. We denote the group consisting of all unitary transformations by $U(1, n; C)$. Let

$$V_0 = \{z^* \in V \colon \Phi(z^*, z^*) = 0\}, \quad V_- = \{z^* \in V \colon \Phi(z^*, z^*) < 0\}.$$

Set

$$PU(1, n; C) = U(1, n; C)/(\text{center}).$$

It is obvious that V_0 and V_- are invariant under $U(1, n; C)$.

An element of $PU(1, n; C)$ acts on H_C^n and its boundary ∂H_C^n. Denote $H_C^n \cup \partial H_C^n$ by $\overline{H_C^n}$.

As in [7], a non-trivial element g in $PU(1, n; C)$ is called

(1) elliptic if it has a fixed point in H_C^n;
(2) parabolic if it has exactly one fixed point and the point lies on ∂H_C^n;

(3) loxodromic if it has exactly two fixed points and the points lie on ∂H^n_C.

For a subgroup $G \subset PU(1,n;C)$, the limit set $L(G)$ of G is defined as

$$L(G) = \overline{G(p)} \cap \partial H^n_C \quad (p \in H^n_C).$$

The fixed point sets of $f \in G$ and of G are

$$\mathrm{fix}(f) = \{x \in \overline{H^n_C}\colon f(x) = x\},$$

$$\mathrm{fix}(G) = \bigcap_{f \in G} \mathrm{fix}(f).$$

DEFINITION 2.1 [4]

A subgroup $G \subset PU(1,n;C)$, is said to be non-elementary, if G contains two non-elliptic elements of infinite order with distinct fixed points, or G is said to be elementary.

DEFINITION 2.2

$G_L = \{g \in G\colon g(x) = x$, for any $x \in L(G)\}$.

DEFINITION 2.3 [4]

A subgroup $G \subset PU(1,n;C)$, is said to be bounded torsion if there exists an integer number m such that for each $g \in G$ has $\mathrm{ord}(g) \le m$ or $\mathrm{ord}(g) = \infty$.

DEFINITION 2.4 [2]

Let X be a subset of the vector space V. The span of X denoted as $\langle X \rangle$ is the smallest C-subspace containing X. If X is a subset of H^n_C, the span $\langle X \rangle$ is defined by $\langle X \rangle = P(\langle P^{-1}X \rangle \cap V_-)$.

Lemma 2.1. *Suppose G be a non-elementary subgroup of $PU(1,n;C)$, then G_L only contains elliptic elements but for identity.*

Proof. Since G is a non-elementary subgroup of $PU(1,n;C)$, we have $\mathrm{card}(L(G)) \ge 3$.

Let $g \in G_L$. Suppose g is loxodromic, for arbitrary three distinct points $x_1, x_2, x_3 \in L(G)$, $g(x_1) = x_1, g(x_2) = x_2, g(x_3) = x_3$ i.e., g has three fixed points in ∂H^n_C. So g is not loxodromic. Similarly, we can prove g is not parabolic.

Hence G_L only contains elliptic elements but for identity. This proves Lemma 2.1.

Lemma 2.2. *Suppose a non-elementary subgroup G of $PU(1,n;C)$ be discrete, then G_L is a bounded torsion.*

Proof. Suppose for some $g \in G$, $G_g = \{f \in G, \mathrm{fix}(g) \subset \mathrm{fix}(f)\}$ is not bounded torsion. Since G is a non-elementary subgroup of $PU(1,n;C)$, there exist infinite loxodromic elements. Since the properties of discreteness and bounded torsion are invariant under conjugation, we may assume that 0 and ∞ are the fixed points of a loxodromic element $g \in G$. Then for every non-trivial element $f \in G_g$, we have

$$f = \begin{pmatrix} \mu_f & 0 & 0 \\ 0 & \lambda_f & 0 \\ 0 & 0 & A_f \end{pmatrix},$$

where $\bar{\mu}_f \lambda_f = 1$, $|\lambda_f| = 1$, $A_f \in U(1, n-1; C)$. Since G_g is not a bounded torsion, there exists infinite elements in G_g, and $PU(1, n; C)$ is a compact set, so there is a distinct sequence $\{h_j\}$ converging to some element. This is a contradiction. Hence G_g is a bounded torsion. However $G_L \subset G_g$, therefore G_L is a bounded torsion. This proves Lemma 2.2.

Lemma 2.3. *G_L is bounded torsion if and only if G_L is finite.*

Proof. Since for every element $g \in G_L$, for each $k \in N$, g^k is also an element of G_L. However G_L is finite. Hence for each g, there exists an integer number m such that $g^m = I$. By Definition 2.3, we know the subgroup G_L of G is a bounded torsion.

On the other hand, if G_L is a bounded torsion, by Lemma of [2], we know that for every element $g \in G_L$, there exists a constant number $\delta(M) > 0$ such that $\|g - I\| > \delta(M)$. Hence G_L is discrete. Since the properties of discreteness and bounded torsion are invariant under conjugation, we may assume that 0 and ∞ are the fixed points of a loxodromic element $f \in G$. Then for every non-trivial element $g \in G_L$, we have $g(0) = 0$, therefore we have $\|g\|^2 = n + 1$, hence G_L is finite. This proves Lemma 2.3.

3. Proofs of theorems

Proof of Theorem 1. In order to prove necessity, it suffices to show that G_L has bounded torsion if G is discrete. By Lemma 2.2, we know G_L has bounded torsion.

Now we prove sufficiency. Suppose that G_L has bounded torsion and every two-generator subgroup is discrete and yet G itself is not discrete. Then there is a distinct sequence $\{f_i\} \subset G$ converging to the identity. Since every two-generator subgroup is discrete and G_L is an elliptic group, every element of G_L has finite order by Definition 2.3. Noticing that G_L has bounded torsion, we may assume every non-trivial element $g \in G_L$ has order less than $M \in N$.

Conjugating by an element of $PU(1, n; C)$ if necessary, we may assume that $g_0 \in G$ is a loxodromic element fixing 0 and ∞, then both 0 and ∞ are in $L(G)$. Since $L(G)$ is the closure of the set of all fixed points of loxodromic elements of G, we may find s loxodromic elements of G, $g_1, g_2, \ldots, g_s$ $(s \leq n)$, where the fixed points of g_j are denoted by μ_{g_j} and ν_{g_j} for $1 \leq j \leq s$. By Definition 2.4 and Lemma 3.1.1 of [2], there exists some $m \leq n$ such that

$$L(G) \subset \overline{H_C^m} = \overline{\langle \{\mu_{g_1}, \mu_{g_2}, \ldots, \mu_{g_s}\} \cup \{\infty\} \rangle}\,.$$

Since $\{f_i\} \subset G$ converges to the identity, for every g_j $(0 \leq j \leq s)$, if f_i is parabolic or loxodromic, for large enough i,

$$N(f_i) + N([f_i, g_j]) < 2 - \sqrt{3}.$$

Hence, all of the groups $\langle f_i, g_1\rangle, \langle f_i, g_2\rangle, \ldots, \langle f_i, g_s\rangle$ are elementary for large enough i. Since g_j is a loxodromic element, f_i fixes or exchanges two fixed points of g_j, $j = 1, 2, \ldots, s$, so f_i is not a parabolic element. Considering the group $\langle f_i^2, g_j\rangle$, we have that the group $\langle f_i^2, g_j\rangle$ is elementary and f_i^2 fixes each fixed point of g_j, so f_i^2 fixes every point in $\{0, \infty, \mu_{g_1}, \mu_{g_2}, \ldots, \mu_{g_s}\}$. By Lemma 3.1.1 of [2], we know that f_i^2 fixes every point in H_C^m, which implies that $f_i^2 \in G_L$. Since $f_i \to I$ as $(i \to \infty)$, we have $f_i^2 \to I$ as $(i \to \infty)$.

Since $f_i^2 \in G_L$, every non-trivial element of G_L has order less than M. From $f_i^2 \to I$ as $(i \to \infty)$, we have $f_i^2 = I$ for large enough m. Using Lemma of [3], we obtain $\|f_i - I\| > \delta(M)$ for large enough i, a contradiction. This completes the proof of Theorem 1.

Remark. Suppose G is a finitely generated non-elementary subgroup of $PU(1, n; C)$. Then G has bounded torsion, which means that G_L as a subgroup of G has bounded torsion too. Hence we have the following.

COROLLARY 1

Theorem K.

COROLLARY 2

Theorem D.

Proof of Theorem 2. By Lemma 2.3, we have G_L is a bounded torsion if and only if G_L is finite. Hence we complete the proof of Theorem 2 by using Theorem 1.

Proof of Theorem 3. We only need to prove the sufficiency. Suppose that G is not discrete. Then there is a sequence $\{f_i\} \in G$ such that $f_i \to I$ as $i \to \infty$. Since G is non-elementary, as in the proof of Theorem 1, one may find loxodromic elements $g_j \in G$ $(j = 0, 1, 2, \ldots, s)$ with the property that no two loxodromic elements of which have a common fixed point and

$$L(G) \subset \overline{H_C^m} = \overline{\langle\{\mu_{g_1}, \mu_{g_2}, \ldots, \mu_{g_s}\} \cup \{\infty\}\rangle},$$

where 0 and ∞ are the two fixed points of g_0 and μ_{g_j} is a fixed point of g_j for $1 \leq j \leq s$.

We may assume that for large enough i, f_i does not interchange the fixed points of g_j, $(j = 1, 2, \ldots, s)$. Since $f_i \to I$ $(i \to \infty)$ and g_j is a loxodromic element, therefore by Lemma 2.2 of [3] for every g_j and large enough i, $f_i g_j$ is a loxodromic element too. Without loss of generality, we may assume that every $f_i g_j$ is loxodromic.

Now since both $f_i g_j$ and g_j are loxodromic, the group $\langle f_i g_j, g_j\rangle$ is discrete $(j = 1, 2, \ldots, s)$. Noticing that $\langle f_i g_j, g_j\rangle = \langle f_i, g_j\rangle$, we have that the group $\langle f_i, g_j\rangle$ is discrete and elementary for large enough m. Arguing as in the proof of Theorem 1, we know f_i^2 fixes every point in $\{0, \infty, \mu_{g_1}, \mu_{g_2}, \ldots, \mu_{g_s}\}$. By Lemma 3.1.1 of [2], we know that f_i^2 fixes every point in H_C^m, which implies that $f_i^2 \in G_L$ for large enough i, which is a contradiction to G_L being finite. This finishes the proof of Theorem 3.

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