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A$\mathcal{T}$-algebras and extensions of A$\mathbb{T}$-algebras

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Abstract. Lin and Su classified A$\mathcal{T}$-algebras of real rank zero. This class includes all A$\mathbb{T}$-algebras of real rank zero as well as many C^*-algebras which are not stably finite. An A$\mathcal{T}$-algebra often becomes an extension of an A$\mathbb{T}$-algebra by an AF-algebra. In this paper, we show that there is an essential extension of an A$\mathbb{T}$-algebra by an AF-algebra which is not an A$\mathcal{T}$-algebra. We describe a characterization of an extension E of an A$\mathbb{T}$-algebra by an AF-algebra if E is an A$\mathcal{T}$-algebra.

Keywords. AF-algebra; A$\mathbb{T}$-algebra; A$\mathcal{T}$-algebra; extension; index map.

1. Introduction

During the development of classification of nuclear separable C^*-algebras, a special class of inductive limits of finite direct sums of matrix algebras over $\mathcal{T}$-algebras was classified by Lin and Su [6], where $\mathcal{T}$-algebras are unital essential extensions of $C(S^1)$ by compact operators $\mathcal{K}$:

$$0 \longrightarrow \mathcal{K} \longrightarrow E \longrightarrow C(S^1) \longrightarrow 0.$$

Each C^*-algebra in this special class is said to be an A$\mathcal{T}$-algebra. One of the important features which makes A$\mathcal{T}$-algebras essentially different from AH-algebras is that the torsion in K_0 does not arise from the torsion parts of certain metric spaces but from nontrivial extensions of $C(S^1)$ by $\mathcal{K}$. Let A be an A$\mathcal{T}$-algebra. The invariant consists of the abelian semigroup $V(A)$, the Murry–von Neumann equivalence classes of projections in matrices of A, an abelian semigroup $k(A)_+$, some equivalence classes of homotopy classes of hyponormal partial isometries in matrices of A and a homomorphism d from $k(A)_+$ into $V(A)$. The main result of [6] states that the above invariant, together with the class of the identity, is complete for the class of C^*-algebras.

An A$\mathcal{T}$-algebra often becomes an essential extension of an A$\mathbb{T}$-algebra by an AF-algebra. Consequently, a question of whether an essential extension of an A$\mathbb{T}$-algebra by an AF-algebra is an A$\mathcal{T}$-algebra, is raised. In this paper, we show that there is an essential extension of an A$\mathbb{T}$-algebra by an AF-algebra which is not in the class. Recently there have been rapid advances in the study of quasidiagonal extensions of C^*-algebras (c.f. [2]). Tracially quasidiagonal extensions are studied by Lin in [4]. In [1], Brown and Dadarlat show that the index maps δ_0 and δ_1 of a quasidiagonal extension of C^*-algebras are zero.

In [5], Lin and Rørdam show that if E is an extension of an A$\mathbb{T}$-algebra by an A$\mathbb{T}$-algebra and E has real rank zero, then E is an A$\mathbb{T}$-algebra if and only if the index maps are both zero. Accordingly, in this paper, we attempt to describe a characterization of an extension E of an A$\mathbb{T}$-algebra by an AF-algebra if E is an A$\mathcal{T}$-algebra via the index maps.

2. A$\mathcal{T}$-algebra as an essential extension of an A$\mathbb{T}$-algebra

Let $C(S^1)$ be the continuous functions on the unit circle and let $\mathcal{K}$ be the compact operators on an infinite dimensional separable Hilbert space. $\mathcal{T}_k$ is an essential unital extension of $C(S^1)$ by $\mathcal{K}$ with index $-k \in \mathbb{Z}$:

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T}_k \longrightarrow C(S^1) \longrightarrow 0.$$

It is well-known that two extensions with the same index are isomorphic as C^*-algebras. We call these algebras $\mathcal{T}$-algebras. It is obvious that $\mathcal{T}_k$ is isomorphic to $\mathcal{T}_{-k}$. So we consider only those $\mathcal{T}_k$ with $k \geq 0$. We now give another description of $\mathcal{T}_k$ (for $k \geq 0$). Let S_0 be an unitary in $\mathbb{B}(H)$ with essential spectrum S^1. Then $\mathcal{T}_0$ is isomorphic to the C^*-subalgebra of $\mathbb{B}(H)$ generated by S_0 and $\mathcal{K}(H)$. Let S_1 be the standard unilateral shift operator acting on the Hilbert space $H = l^2$. Then $\mathcal{T}_k$ ($k > 0$) is isomorphic to the C^*-subalgebra of $\mathbb{B}(H)$ generated by $(S_1)^k$ and $\mathcal{K}(H)$.

Lemma 2.1. *Let A be a C^*-algebra with an approximate unit of projections, $\{I_\lambda\}_{\lambda\in\Lambda}$ a set of ideals of A. If quotient A/I_λ is a finite C^*-algebra for each $\lambda \in \Lambda$, then $A/\bigcap_{\lambda\in\Lambda} I_\lambda$ is a finite C^*-algebra.*

Proof. Let $\{p_i\}$ be an approximate unit of projections in A and π be the quotient map from A to $A/\bigcap_{\lambda\in\Lambda} I_\lambda$. Then $\pi(p_i)$ becomes an approximate unit of projections in $A/\bigcap_{\lambda\in\Lambda} I_\lambda$. For any i, we assume that $v^*v = \pi(p_i)$. There is $w \in p_i A p_i$ such that $\pi(w) = v$. Since $\pi(w^*w) = \pi(p_i)$, $w^*w \in p_i + \bigcap_{\lambda\in\Lambda} I_\lambda$. By the hypothesis of the lemma, $ww^* \in p_i + I_\lambda$ for all λ, so $ww^* \in p_i + \bigcap_{\lambda\in\Lambda} I_\lambda$, $vv^* = \pi(ww^*) = \pi(p_i)$. Therefore $A/\bigcap_{\lambda\in\Lambda} I_\lambda$ is a finite C^*-algebra. □

DEFINITION 2.2

Let A be a C^*-algebra with an approximate unit of projections. By the above lemma, there exists the smallest ideal I of A such that A/I is a finite C^*-algebra. We denote this ideal by $I(A)$, and denote A/I by $Q(A)$.

Lemma 2.3. *Let $E = \varinjlim(E_n, \phi_n)$ be an inductive limit C^*-algebra, where each E_n is a finite direct sum of matrix algebras over $\mathcal{T}$-algebras, and each connecting map from E_n to E_{n+1} satisfies the following: if $M_k(\mathcal{T}_0)$ is a summand of E_n, then the connecting map restricted to this block vanishes on $M_k(\mathcal{K})$. Then $I(E) = \varinjlim(I(E_n), \phi_n)$ and $Q(E) = \varinjlim(Q(E_n), \bar{\phi}_n)$, where $\bar{\phi}_n$ is the *-homomorphism which is induced by ϕ_n.*

Proof. It is easy to see that $\phi_n(I(E_n)) \subset I(E_{n+1})$ for each n. Since $\varinjlim(Q(E_n), \bar{\phi}_n)$ is a finite C^*-algebra, $I(E) \subset \varinjlim(I(E_n), \phi_n)$. It remains to show that $\phi(I(E_n)) \subset I(E)$. We may assume that $E_n = M_n(\mathcal{T}_k)$ and $\phi_{n,\infty}(M_n(\mathcal{K})) \neq 0$. Note that $M_n(\mathcal{K})$ is the minimal ideal of $M_n(\mathcal{T}_k)$, $\phi_{n,\infty}$ is injective on $M_n(\mathcal{T}_k)$. There exists $v \in M_n(\mathcal{T}_k)$

such that $v^*v = 1_{M_n(\mathcal{T}_k)}$, $vv^* \neq 1_{M_n(\mathcal{T}_k)}$. Put $w = \phi_{n,\infty}(v)$ and $p = \phi_{n,\infty}(1_{M_n(\mathcal{K})})$, then $w^*w = p$, $ww^* \neq p$, so p is an infinite projection. Since $w^*w - ww^* \in I(E)$, $I(E) \cap \phi(M_n(\mathcal{K})) \neq \varnothing$. Since $M_n(\mathcal{K})$ is a simple C^*-algebra, $\phi(M_n(\mathcal{K})) \subset I(E)$. So $\varinjlim(I(E_n), \phi_n) \subset I(E)$. □

PROPOSITION 2.4 (Proposition 5.7 of [6])

Let $E = \varinjlim(E_n, \phi_n)$ be an inductive limit C^*-algebra, where each E_n is a finite direct sum of matrix algebras over $\mathcal{T}$-algebras. Then E is isomorphic to $E' = \varinjlim(E'_n, \phi'_n)$ where each E'_n is a finite direct sum of matrix algebras over $\mathcal{T}$-algebras and where each connecting map from E'_n to E'_{n+1} satisfies the following: If $M_k(\mathcal{T}_0)$ is a summand of E'_n, then the connecting map restricted to this block vanishes on $M_k(\mathcal{K})$.

Theorem 2.5. *Let E be an $A\mathcal{T}$-algebra and let $\delta_1\colon K_1(Q(E)) \to K_0(I(E))$ be the index map. Then for all $g \in K_0(I(E))$, there are a natural number n and $f \in K_1(Q(E))$ such that $\delta_1(f) = ng$.*

Proof. Let $E = \varinjlim(E_n, \phi_n)$ be an A$\mathcal{T}$-algebra. By the above proposition, we may assume that each connecting map ϕ_n from E_n to E_{n+1} satisfies the following: If $M_k(\mathcal{T}_0)$ is a summand of E_n, then ϕ_n restricted to this block vanishes on $M_k(\mathcal{K})$. We have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B_1 & \xrightarrow{l_1} & E_1 & \xrightarrow{\varphi_1} & A_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \phi_1} & & \downarrow & & \\
0 & \longrightarrow & B_2 & \xrightarrow{l_2} & E_2 & \xrightarrow{\varphi_2} & A_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \phi_2} & & \downarrow & & \\
0 & \longrightarrow & B_3 & \xrightarrow{l_3} & E_3 & \xrightarrow{\varphi_3} & A_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \phi_3} & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
0 & \longrightarrow & B & \xrightarrow[l]{} & E & \xrightarrow[\varphi]{} & A & \longrightarrow & 0
\end{array}$$

where each $B_i = I(E_i)$ is a direct sum of $\mathcal{K}$. By Lemma 2.3, $I(E) = B$. Without loss of generality, we may further assume that each ϕ_n is injective, each E_n is of the following C^*-algebra form:

$$M_{n_1}(\mathcal{T}_{k_1}) \oplus \cdots \oplus M_{n_s}(\mathcal{T}_{k_s}) \oplus M_{n_{s+1}}(C(X_1)) \oplus \cdots \oplus M_{n_{s+m}}(C(X_m)),$$

where $X_1, \ldots, X_m$ are compact subsets of S^1, and $k_1, k_2, \ldots, k_s$ are positive. Note that $I(E) = B$, for each $g \in K_0(I(E))^+$, there exists a projection $p \in B_m$ such that $[p]_0 = g$. Therefore there exist $S \in E_m$ and a natural number n such that $[S^*S - SS^*]_0 = n(l_{m*}[p]_0)$ in $K_0(E_m)$, so $\delta_1([\pi(S)]_1) = [(S^*S - SS^*)]_0 = n[p]_0 = ng$. It is easy to see that for each $g \in K_0(I(E))$, there is a natural number n and $f \in K_1(Q(E))$ such that $\delta_1(f) = ng$. □

3. An example

DEFINITION 3.1

Let $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$, a C^*-algebra

$$\mathcal{T}_\Lambda = C^*(\underbrace{\mathcal{K} \oplus \cdots \oplus \mathcal{K}}_{k}, S_{\lambda_1} \oplus S_{\lambda_2} \oplus \cdots \oplus S_{\lambda_k}) \subset \underbrace{\mathbb{B}(H) \oplus \cdots \oplus \mathbb{B}(H)}_{k},$$

where S_λ ($\lambda \neq 0$) is an unilateral shift operator with index $-\lambda$. If $\lambda = 0$, S_λ is an unitary in $\mathbb{B}(H)$ with essential spectrum S^1. Then $\mathcal{T}_\Lambda$ is an essential unital extension of $C(S^1)$ by $\mathcal{K} \oplus \cdots \oplus \mathcal{K}$, with index $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$:

$$0 \longrightarrow \mathcal{K} \oplus \cdots \oplus \mathcal{K} \longrightarrow \mathcal{T}_\Lambda \xrightarrow{\pi} C(S^1) \longrightarrow 0\cdot$$

We call the C^*-algebras of this kind G$\mathcal{T}$-algebras.

The class of inductive limits of finite direct sums of matrix algebras over G$\mathcal{T}$-algebras will be denoted by Σ. By the following lemma, if a C^*-algebra $A = \varinjlim(A_n, \phi_n)$, where each $A_n = \oplus_{i=1}^{l_n} M_{n_i}(E_{n_i})$, E_{n_i} is a unial essential extension of $C(S^1)$ by $\mathcal{K} \oplus \cdots \oplus \mathcal{K}$, then A is contained in Σ.

Lemma 3.2. *Let*

$$0 \longrightarrow \mathcal{K} \oplus \cdots \oplus \mathcal{K} \longrightarrow E \longrightarrow C(S^1) \longrightarrow 0$$

be an essential unital extension of $C(S^1)$ by $\mathcal{K} \oplus \cdots \oplus \mathcal{K}$. Then E is an inductive limit of finite direct sums of matrix algebras over G$\mathcal{T}$-algebras.

Proof. Let $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ be the index of this extension E. If each λ_i ($i = 1, 2, \ldots, k$) is not zero, then E is unique under the isomorphism of C^*-algebra. Otherwise, without loss of generality, we can assume that $\lambda_1 = 0, \lambda_2, \ldots, \lambda_k$ are not zero, $E \cong C^*(\mathcal{K} \oplus \cdots \oplus \mathcal{K}, T \oplus S_{\lambda_2} \oplus \cdots \oplus S_{\lambda_k})$. Let $\{p_i : i = 1, 2, \ldots\}$ be mutually orthogonal projections of rank one in $\mathbb{B}(H)$ such that $\sum p_i = 1$. Then E is generated by $\underbrace{\mathcal{K} \oplus \cdots \oplus \mathcal{K}}_{k}$ and a partial isometry

$$S = \left(\sum \xi_i p_i\right) \oplus S_{\lambda_2} \oplus \cdots \oplus S_{\lambda_k},$$

where each $\xi_i \in S^1$. Let $\{e_{ij}\}$ be a matrix unit for $\mathcal{K}$. Put

$$E_n = M_n \oplus C^*(U_n \oplus S_{\lambda_2} \oplus \cdots \oplus S_{\lambda_k}, 0 \oplus \underbrace{\mathcal{K} \oplus \cdots \oplus \mathcal{K}}_{k-1}),$$

where M_n is the finite dimensional C^*-algebra generated by $\{e_{ij}\}_{i,j \leq n}$ and $U_n = \sum_{i>n} \xi_i p_i$. It is obvious that $E_n \subset E_{n+1}$. Note that each E_n is an inductive limit of finite direct sums of matrix algebras over G$\mathcal{T}$-algebras, and $\cup E_n$ is dense in E. Therefore $E = \varinjlim E_n$ is an inductive limit of finite direct sums of matrix algebras over G$\mathcal{T}$-algebras. □

Lemma 3.3. *If $\Lambda = \mathbf{0}$, then $\mathcal{T}_\Lambda$ is an $A\mathbb{T}$-algebra.*

Proof. Fix a separable infinite dimensional Hilbert space $\underbrace{H \oplus \cdots \oplus H}_{n}$. Let $\{p_{ij}\colon i = 1, 2, \ldots,\ j = 1, \ldots, n\}$ be mutually orthogonal projections of rank one in $\mathbb{B}(H) \oplus \cdots \oplus \mathbb{B}(H)$ and $\sum p_{ij} = 1$. Then $\mathcal{T}_{\mathbf{0}}$ is generated by $\underbrace{\mathcal{K} \oplus \cdots \oplus \mathcal{K}}_{n}$ and an unitary $U = \sum \xi_{ij} p_{ij}$, where $\{\xi_{ij}\}_{i=1}^{\infty}$ is dense in S^1 for each $j = 1, 2, \ldots, n$.

Let $\{e_{ij}^{(l)}\}$ be a matrix unit for $\underbrace{\mathcal{K} \oplus \cdots \oplus \mathcal{K}}_{n}$. Put

$$B_k = \underbrace{M_k \oplus \cdots \oplus M_k}_{n} \oplus C^*(U_k),$$

where $\underbrace{M_k \oplus \cdots \oplus M_k}_{n}$ is a finite dimensional C^*-algebra generated by $\{e_{ij}^{(l)}\colon l = 1, \ldots, n\}_{i,j \le k}$ and $U_k = \sum_{i>k, j=1}^{j=n} \xi_{ij} p_{ij}$. It is obvious that $B_k \subset B_{k+1}$. Note that $\mathrm{sp}(U_k) = S^1$ for each k and $U \in B_k$. Since $\mathcal{T}_{\mathbf{0}} = C^*(U, \mathcal{K} \oplus \cdots \oplus \mathcal{K})$ and $\cup B_k$ is dense in $\mathcal{T}_{\mathbf{0}}$, $\mathcal{T}_{\mathbf{0}} = \varinjlim B_k$ is an $A\mathbb{T}$-algebra. □

Theorem 3.4. *Let $E = \varinjlim(E_n, \phi_n)$ be an inductive limit, where each E_n is a finite direct sum of matrix algebras over $G\mathcal{T}$-algebras. Then E is isomorphic to $E' = \varinjlim(E'_n, \phi'_n)$, where E'_n is a finite direct sum of C^*-algebras with the form of $M_n(\mathcal{T}_0)$ or $M_n(\mathcal{T}_\Lambda)$ $(\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n))$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are not zero, where each connecting map ϕ'_n satisfies the following: If $M_n(\mathcal{T}_0)$ is a summand of E'_n, then ϕ'_n restricted to this block vanishes on $M_n(\mathcal{K})$. Further, $I(E) = \varinjlim(I(E'_n), \phi'_n)$ and $Q(E) = \varinjlim(Q(E'_n), \bar{\phi}'_n)$, where $\bar{\phi}'_n$ is the quotient map which is induced by ϕ'_n.*

Proof. We first assume that $E = \mathcal{T}_\Lambda$, where $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are non-zero, then the theorem is true. If $\Lambda = \mathbf{0}$, by Lemma 3.3, E is an $A\mathbb{T}$-algebra. We may assume that $E = \varinjlim(A_n, \phi_n)$, where

$$A_n = \bigoplus_{j=1}^{r_n} M_{k_j}(C(S^1)).$$

Set

$$E_n = \bigoplus_{j=1}^{r_n} M_{k_j}(\mathcal{T}_0).$$

Let $\pi_n\colon E_n \to A_n$ be the quotient map. There is a unital monomorphism $j_n\colon A_n \to E_n$ such that $j_n \circ \pi_n$ is the identity map on A_n. So $E = \varinjlim(E_n, j_{n+1} \circ \phi_n \circ \pi_n)$ satisfies the condition of theorem. So without loss of generality, we assume that $\lambda_1 = 0$ and $\lambda_2, \ldots, \lambda_n$ are not zero. Fix a separable infinite dimensional Hilbert space $\underbrace{H \oplus \cdots \oplus H}_{n}$. Let $\{p_i\colon i = 1, 2, \ldots\}$ be mutually orthogonal projections of rank one in $\mathbb{B}(H)$ and $\sum p_i = 1$. Then $\mathcal{T}_\Lambda$ is generated by $\underbrace{\mathcal{K} \oplus \cdots \oplus \mathcal{K}}_{n}$ and a partial isometry

$$S = \left(\sum \xi_i p_i\right) \oplus S_{\lambda_2} \oplus \cdots \oplus S_{\lambda_n},$$

where $\{\xi_i\}_{i=k}^{\infty}$ is dense in S^1 for each positive integer k.

Let $\{e_{ij}\}$ be a matrix unit of $\mathcal{K}$. Put

$$C_k = M_k \oplus C^*(U_k \oplus S_{\lambda_2} \oplus \cdots \oplus S_{\lambda_n}, 0 \oplus \underbrace{\mathcal{K} \oplus \cdots \oplus \mathcal{K}}_{n-1}),$$

where M_k is the finite dimensional C^*-algebra generated by $\{e_{ij}\}_{i,j \le k}$ and $U_k = \sum_{i>k} \xi_i p_i$. Note that $C_k \subset C_{k+1}$ and $\cup C_k$ is dense in $\mathcal{T}_\Lambda$, $\mathcal{T}_\Lambda = \varinjlim C_k$ satisfy the condition in the theorem. Therefore, if E is a finite direct sum of matrix algebras on G$\mathcal{T}$-algebras, this theorem is true.

Generally, $E = \varinjlim(E_n, \phi_n)$, where E_n is a finite direct sum of matrix algebras on G$\mathcal{T}$-algebras. Without loss of generality, we may assume that E_n is a finite direct sum of C^*-algebras with the form of $M_n(\mathcal{T}_\Lambda)$ ($\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$), where $\Lambda = 0$ or $\lambda_1, \lambda_2, \ldots, \lambda_n$ are not zero.

$$E_n = E_n^{(1)} \oplus \varinjlim D_n^{(m)},$$

where $D_n^{(m)}$ is a finite direct sum of matrix algebras over $C(S^1)$ or $\mathbb{C}$, and $E_n^{(1)}$ is a finite direct sum of C^*-algebras with the form of $M_n(\mathcal{T}_\Lambda)$ ($\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ and $\lambda_1, \lambda_2, \ldots, \lambda_k$ are not zero). Denote $\varinjlim D_n^{(m)}$ by $E_n^{(2)}$.

Let $\{f_k^{(n)}\}$ be dense in E_n and $\varepsilon_n = 1/2^n$. Let $G_1 = \{f_1^{(1)}\} \subset E_1$. There is $E_1^{(1)} \oplus D_{n_1}$ such that $\operatorname{dist}(f_1^{(1)}, E_1^{(1)} \oplus D_{n_1}) < \varepsilon_1$. Put $E_1' = E_1^{(1)} \oplus D_{n_1}$. Let $\{h_i^{(1)}\}$ be dense in E_1' and H_1 be a finite set which contain the generators of E_1' and $h_1^{(1)}$. Let p be the projection in E_2 corresponding to the summand $E_2^{(2)}$. Then $p \cdot \phi_1$ vanishes on $I(E_1^{(1)})$. Then there is a map ρ_1: $Q(E_1^{(1)}) \oplus D_{n_1} \to E_2^{(2)}$ such that $p \cdot \phi_1 = \rho_1 \circ \pi_1$. By Lemma 4.2 [3], there is a C^*-subalgebra D_{n_2} in $E_2^{(2)}$ and *-homomorphism φ_1: $Q(E_1^{(1)}) \oplus D_{n_1} \to D_{n_2}$ such that the following diagram commutes on $\pi_1(H_1)$ to within ε_1:

$$\begin{array}{ccc} D_{n_2} & \searrow & \\ \uparrow{\scriptstyle \varphi_1} & & \\ Q(E_1^{(1)}) \oplus D_{n_1} & \xrightarrow[\rho_1]{} & E_2^{(2)}. \end{array}$$

Put $E_2' = E_2^{(1)} \oplus D_{n_2}$, $\psi_1 = (1-p) \cdot \phi_1 \oplus \varphi_1 \circ \pi_1$. Then the following diagram commutes on H_1 to within ε_1:

$$\begin{array}{ccc} E_1 & \xrightarrow{\phi_1} & E_2 \\ \uparrow & & \uparrow \\ E_1' & \xrightarrow[\psi_1]{} & E_2' \end{array}.$$

Write $E_n' = T_n \oplus L_n$, where T_n is a finite direct sums of matrix algebras over $\mathcal{T}_\Lambda$ or $C(S^1)$, L_n is a finite dimensional C^*-algebra. Define j_n: $T_n \oplus L_n \to T_n \oplus L_n \otimes C(S^1)$ by $j_n|_{T_n} = \mathrm{id}$ and $j_n|_{L_n}$ is the nature map from L_n to $L_n \otimes C(S^1)$. At the same time we define d_n: $T_n \oplus L_n \otimes C(S^1) \to T_n \oplus L_n$ by $d_n|_{T_n} = \mathrm{id}$ and $d_n|_{L_n \otimes C(S^1)}$ is a map from $L_n \otimes C(S^1)$ to L_n with $d_n|_{L_n \otimes C(S^1)}(f) = f(1)$. So we have the following commutative diagram:

$$\begin{array}{ccccc} T_n \oplus L_n & \xrightarrow{\psi_n} & T_{n+1} \oplus L_{n+1} & \xrightarrow{\psi_{n+1}} & T_{n+2} \oplus L_{n+2} \\ d_n \uparrow & & j_{n+1} \downarrow & & \uparrow d_{n+2} \\ T_n \oplus L_n \otimes C(S^1) & \xrightarrow[\beta_n]{} & T_{n+1} \oplus L_{n+1} \otimes C(S^1) & \xrightarrow[\beta_{n+1}]{} & T_{n+2} \oplus L_{n+2} \otimes C(S^1) \end{array}.$$

Therefore

$$\varinjlim (T_n \oplus L_n \otimes C(S^1), \beta_n) = E.$$

We denote $R_n \oplus K_n = T_n \oplus L_n \otimes C(S^1)$, where R_n is a finite direct sum of matrix algebras on G$\mathcal{T}$-algebras, $K_n = L' \otimes C(S^1)$, L'_n is a finite dimensional C^*-algebra. Put $E''_n = R_n \oplus L'_n \otimes \mathcal{T}_0$. We have a trivial extension:

$$0 \longrightarrow L'_n \otimes \mathcal{K} \longrightarrow L'_n \otimes \mathcal{T}_0 \xrightarrow{\pi'_n} K_n \longrightarrow 0.$$

Put $i_n\colon K_n \to L'_n \otimes \mathcal{T}_0$ such that $\pi'_n \circ i'_n = 1$. Put $\pi_n = \mathrm{id}_{R_n} \oplus \pi'_n$, $i_n = \mathrm{id}_{R_n} \oplus i'_n$. Then the following diagraph is commutative:

$$\begin{array}{ccccc} R_n \oplus K_n & \xrightarrow{\beta_n} & R_{n+1} \oplus K_{n+1} & \xrightarrow{\beta_{n+1}} & R_{n+2} \oplus K_{n+2} \\ \pi_n \uparrow & & \downarrow i_{n+1} & & \uparrow \pi_{n+2} \\ R_n \oplus L'_n \otimes \mathcal{T}_0 & \longrightarrow & R_{n+1} \oplus L'_{n+1} \otimes \mathcal{T}_0 & \longrightarrow & R_{n+2} \oplus L'_{n+2} \otimes \mathcal{T}_0 \end{array}.$$

So

$$E = \varinjlim (R_n \oplus L'_n \otimes \mathcal{T}_0, i_{n+1} \circ \beta_n \circ \pi_n).$$

By now all the above complete the proof of the first part. The proof of the last part is similar to that of Lemma 2.3. □

Theorem 3.5. *Let E be a C^*-algebra in Σ and let $\delta_1\colon K_1(Q(E)) \to K_0(I(E))$ be the index map. Then E is an $A\mathcal{T}$-algebra if and only if, for each $g \in K_0(I(E))$, there is a natural number n and $f \in K_1(Q(E))$ such that $\delta_1(f) = ng$.*

Proof. The 'only if' part of the theorem follows from Theorem 2.5. To prove the converse, let $E = \varinjlim(E_n, \phi_n)$, where each E_n is a finite direct sum of matrix algebras over G$\mathcal{T}$-algebras. Let $Q(E) = \varinjlim(Q(E_n), \psi_n)$. For any given $x_1, x_2, \ldots, x_n \in E$ and $\varepsilon > 0$, there exist a positive number m_1 and $z'_1, z'_2, \ldots, z'_n \in Q(E_{m_1})$ such that

$$\|\psi_{m_1,\infty}(z'_i) - \pi(x_i)\| < \varepsilon \ (i = 1, 2, \ldots, n).$$

So there exists $y'_1, y'_2, \ldots, y'_n \in E_{m_1}$ such that $\pi(y'_i) = z'_i$ $(i = 1, 2, \ldots, n)$ and

$$\begin{aligned} \|\pi(\phi_{m_1,\infty}(y'_i) - x_i)\| &= \|\psi_{m_1,\infty}(\pi_{m_1}(y'_i)) - \pi(x_i)\| \\ &= \|\psi_{m_1,\infty}(z'_i) - \pi(x_i)\| \\ &< \varepsilon. \end{aligned}$$

There exists $a_1, a_2, \ldots, a_n \in I(E)$ such that $\|\phi_{m_1,\infty}(y_i') - x_i - a_i\| < \varepsilon$, so there exist a positive number $m_2 \geq m_1$ and $b_1, b_2, \ldots, b_n \in I(E_{m_2})$, such that $\|\phi_{m_2,\infty}(b_i) - a_i\| < \varepsilon$ $(i = 1, 2, \ldots, n)$. We write $y_i = \phi_{m_1,m_2}(y_i')$, $z_i = \psi_{m_1,m_2}(z_i')$. Note that $x_1, x_2, \ldots, x_n \in_\varepsilon \phi_{m_2,\infty}(E_{m_2})$.

Without loss of generality, we may assume that $E_{m_2} = M_m(\mathcal{T}_\Lambda)$, where $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ and $\mathcal{T}_\Lambda = C^*(\underbrace{\mathcal{K} \oplus \cdots \oplus \mathcal{K}}_{k}, S)$. Then $I(E_{m_2}) = \underbrace{\mathcal{K} \oplus \cdots \oplus \mathcal{K}}_{k}$. By the hypothesis of the theorem, there exists a natural number $m_3 \geq m_2$ satisfying the following: For any $f \in K_0(\phi_{m_2,m_3})(K_0(I(E_{m_2})))$, there exists a positive integer k and $g \in K_1(Q(E_{m_3}))$ such that $\delta_1^{m_3}(g) = kf$. Similarly, without loss of generality, we may assume that $E_{m_3} = M_{m'}(\mathcal{T}_{\Lambda'})$, where $\Lambda' = (\lambda_1', \lambda_2', \ldots, \lambda_{k'}')$. From the above theorem, it follows that $\Lambda' = 0$ or $\lambda_1', \lambda_2', \ldots, \lambda_{k'}'$ are not zero. If $\Lambda' = 0$, by Lemma 3.3, $\phi_{m_2,m_3}(E_{m_2})$ is an A$\mathbb{T}$-algebra. If $\lambda_1', \lambda_2', \ldots, \lambda_{k'}'$ are not zero, we define a map $l_i \colon \mathcal{K} \oplus \cdots \oplus \mathcal{K} \to \mathcal{K}$ by $l_i((a_1, \ldots, a_{k'})) = a_i$, then $l_i \circ \phi_{m_2,m_3}|_{I(E_{m_2})}$ $(i = 1, 2, \ldots, k')$ are injective. Therefore there is a C^*-subalgebra $C \cong \mathcal{K}$ of $I(E_{m_3})$ such that $\phi_{m_2,m_3}(I(E_{m_2})) \subset C$. Put $D = C^*(C, \phi_{m_2,m_3}(S))$, then $\phi_{m_2,m_3}(E_{m_2}) \subset D$, D is a matrix algebra over $\mathcal{T}$-algebra. By Theorem 8 of [7], E is an A$\mathcal{T}$-algebra. □

Example 3.6. Let S_1 be an unilateral shift operator on a separable infinite dimensional Hilbert space H, $A = C^*(S_1 \oplus S_1, \mathcal{K} \oplus \mathcal{K}) \subset \mathbb{B}(H) \oplus \mathbb{B}(H)$ is a C^*-algebra. Then we have the following exact sequence:

$$0 \longrightarrow \mathcal{K} \oplus \mathcal{K} \longrightarrow A \xrightarrow{\ \pi\ } C(S^1) \longrightarrow 0,$$

where π is the quotient map. Induced by K-theory we have

$$\begin{array}{ccccc} 0 & \longrightarrow & K_1(A) & \longrightarrow & \mathbb{Z} \\ {\scriptstyle\delta_0}\uparrow & & & & \downarrow{\scriptstyle\delta_1} \\ \mathbb{Z} & \longleftarrow & K_0(A) & \longleftarrow & \mathbb{Z} \oplus \mathbb{Z} \end{array},$$

where $\delta_1(1) = 1 \oplus 1$. By Theorem 3.5, A is not an A$\mathcal{T}$-algebra.

Let $E = \varinjlim(M_{n!}(A), \phi_n)$, where for each $a \in M_{n!}(A)$,

$$\phi_n(a) = \operatorname{diag}(a, \pi(a)(1), \pi(a)(e^{\frac{1}{n}2\pi i}), \ldots, \pi(a)(e^{\frac{n-1}{n}2\pi i})).$$

Then E is an unital essential extension of an A$\mathbb{T}$-algebra by an AF-algebra, $RR(E) = 0$ and $Q(E)$ is simple, but E is not an A$\mathcal{T}$-algebra. In fact, for each self-adjoint element $a \in E$ and $\varepsilon > 0$, there exist a natural number n and $b \in M_{n!}(A)$ such that $\|a - b\| < \varepsilon/2$ and $\{\pi(b)(1), \pi(b)(e^{\frac{1}{n}2\pi i}), \ldots, \pi(b)(e^{\frac{n-1}{n}2\pi i})\}$ is $\varepsilon/16$-dense in $\mathrm{sp}(b)$. By Lemma 8 of [5], there exists a self-adjoint element $c \in E_{n+1}$ with finite spectrum such that $\|\phi_n(b) - c\| < \varepsilon/2$. So $RR(E) = 0$. The construction of E ensures that $Q(E)$ is a simple C^*-algebra. Note that $K_1(Q(E)) \cong \mathbb{Z}$, $K_0(I(E)) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $\delta_1(1) = 1 \oplus 1$. For $g = 1 \oplus 0$ in $K_0(I(E))$, there are not a natural number n and $f \in K_1(Q(E))$ such that $\delta_1(f) = ng$. By theorem 3.5, E is not an A$\mathcal{T}$-algebra.

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