Proc. Indian Acad. Sci. (Math. Sci.) Vol. 120, No. 2, April 2010, pp. 185–198.

On split Lie triple systems II

ANTONIO J CALDERÓN MARTÍN and M FORERO PIULESTÁN

Departamento de Matemáticas, Universidad de Cádiz, 11510 Puerto Real, Cádiz, Spain
E-mail: ajesus.calderon@uca.es; ForeroManuel@hotmail.com

MS received 24 June 2009; revised 11 September 2009

Abstract. In [4] it is studied that the structure of split Lie triple systems with a coherent 0-root space, that is, satisfying $[T_0, T_0, T] = 0$ and $[T_0, T_\alpha, T_0] \neq 0$ for any nonzero root α and where T_0 denotes the 0-root space and T_α the α-root space, by showing that any of such triple systems T with a symmetric root system is of the form $T = \mathcal{U} + \sum_j I_j$ with $\mathcal{U}$ a subspace of the 0-root space T_0 and any I_j a well described ideal of T, satisfying $[I_j, T, I_k] = 0$ if $j \neq k$. It is also shown in [4] that under certain conditions, a split Lie triple system with a coherent 0-root space is the direct sum of the family of its minimal ideals, each one being a simple split Lie triple system, and the simplicity of T is characterized. In the present paper we extend these results to arbitrary split Lie triple systems with no restrictions on their 0-root spaces.

Keywords. Lie triple system; system of roots; root space; split Lie algebra; structure theory.

1. Introduction

Throughout this paper, Lie triple systems T are considered to be of arbitrary dimension and over an arbitrary field $\mathbb{K}$. It is worth mentioning that, unless otherwise stated, there is not any restriction on $\dim T_\alpha$ or $\{k \in \mathbb{K}: k\alpha \in \Lambda^1$ for a fixed $\alpha \in \Lambda^1\}$, where T_α denotes the root space associated to the root α, and Λ^1 the set of nonzero roots of T.

Calderón and Forero [2] introduced the concept of split Lie triple system of arbitrary dimension and studied the locally finite ones. In [3], Calderón introduced techniques of connection of roots in the field of split Lie algebras. These techniques were extended to the setup of split Lie triple systems with a coherent 0-root space, that is, those satisfying $[T_0, T_0, T] = 0$ and $[T_0, T_\alpha, T_0] \neq 0$ for any nonzero root α and where T_0 denotes the 0-root space and T_α the α-root space [4, 5] and [6]. In the present paper we develop these techniques in the framework of split Lie triple systems with no restrictions on their 0-root space so as to obtain a generalization of the results in [3, 4] and, in a sense, [7] and [10].

In §2, we establish the preliminaries on split Lie triple systems theory. In §3, we show that an arbitrary Lie triple system with a symmetric root system T is of the form $T = \mathcal{U} + \sum_j I_j$ with $\mathcal{U}$ a subspace of the 0-root space T_0 and any I_j a well described ideal of T, satisfying $[I_j, T, I_k] = 0$ if $j \neq k$. In §4, and under certain conditions, the simplicity of T is characterized and shown that T is the direct sum of the family of its minimal ideals, each one being a simple split Lie triple system with a symmetric root system and having all its nonzero roots connected. Finally, note that we have opted for writing a self-contained paper in such a way that can be read independently of [4].

2. Preliminaries

Let T be a vector space over an arbitrary field $\mathbb{K}$. We say that T is a *triple system* if it is endowed with a trilinear map $\langle\cdot,\cdot,\cdot\rangle\colon T\times T\times T\to T$, called the *triple product* of T.

A triple system T is called a *Lie triple system* if its triple product, denoted by $[\cdot,\cdot,\cdot]$, satisfies

(1) $[x,x,y]=0$,
(2) $[x,y,z]+[y,z,x]+[z,x,y]=0$ (Jacobi identity),
(3) $[x,y,[a,b,c]]-[a,b,[x,y,c]]=[[x,y,a],b,c]+[a,[x,y,b],c]$,

for any $x,y,z,a,b,c\in T$ (see [8, 9]).

An *ideal* of a Lie triple system T is a subspace I for which $[I,T,T]\subseteq I$. Let us observe that $[I,T,T]\subseteq I$ implies $[T,I,T]\subseteq I$ and $[T,T,I]\subseteq I$. A Lie triple system T is called *simple* if the product is nonzero and its only ideals are $\{0\}$ and T.

We recall that the *Annihilator* of a Lie triple system T is defined as the set of elements in T such that $[x,T,T]=0$. This is an ideal of T denoted by $\mathrm{Ann}(T)$.

A *two-graded* $\mathbb{K}$*-algebra* A is a $\mathbb{K}$-algebra which splits into the direct sum $A=A^0\oplus A^1$ of subspaces (called the even and the odd part respectively) satisfying $A^\alpha A^\beta\subset A^{\alpha+\beta}$ for any α,β in $\mathbb{Z}_2$.

The *standard embedding* of a Lie triple system T is the two-graded Lie algebra $L=L^0\oplus L^1$, L^0 being the $\mathbb{K}$-span of $\{\mathcal{L}(x,y)\colon x,y\in T\}$, where $\mathcal{L}(x,y)$ denotes the left multiplication operator in T, $\mathcal{L}(x,y)(z):=[x,y,z]$; $L^1:=T$ and where the product is given by

$$[(\mathcal{L}(x,y),z),(\mathcal{L}(u,v),w)]:=(\mathcal{L}([u,v,y],x)-\mathcal{L}([u,v,x],y)\\ +\mathcal{L}(z,w),[x,y,w]-[u,v,z]).$$

Let us observe that L^0 with the product induced by the one in $L=L^0\oplus L^1$ becomes a Lie algebra.

Given an element x of a Lie algebra L, we denote by $\mathrm{ad}(x)$ the *adjoint mapping* defined as $\mathrm{ad}(x)(y)=[x,y]$ for any $y\in L$.

A *splitting Cartan subalgebra* H *of a Lie algebra* L is defined as a maximal abelian subalgebra (MASA) of L satisfying the adjoint mappings $\mathrm{ad}(h)$ for $h\in H$ are simultaneously diagonalizable. If L contains a splitting Cartan subalgebra H, then L is called a *split Lie algebra*. This means that we have a *root decomposition* $L=H\oplus(\bigoplus_{\alpha\in\Lambda}L_\alpha)$ where $L_\alpha=\{v_\alpha\in L\colon[h,v_\alpha]=\alpha(h)v_\alpha$ for any $h\in H\}$ for a linear functional $\alpha\in H^*$ and $\Lambda:=\{\alpha\in H^*\backslash\{0\}\colon L_\alpha\neq 0\}$. The subspaces L_α for $\alpha\in H^*$ are called *root spaces* of L (with respect to H) and the elements $\alpha\in\Lambda\cup\{0\}$ are called *roots* of L with respect to H. Clearly $L_0=H$.

DEFINITION 2.1

Let T be a Lie triple system, $L=L^0\oplus L^1$ be its standard embedding, and H^0 be a MASA of L^0. For a linear functional $\alpha\in(H^0)^*$ we define the root space of T (with respect to H^0) associated to α as the subspace $T_\alpha:=\{t_\alpha\in T\colon[h,t_\alpha]=\alpha(h)t_\alpha$ for any $h\in H^0\}$. The elements $\alpha\in(H^0)^*$ satisfying $T_\alpha\neq 0$ are called roots of T with respect to H^0 and we denote $\Lambda^1:=\{\alpha\in(H^0)^*\backslash\{0\}\colon T_\alpha\neq 0\}$.

Let us observe that $T_0 = \{t_0 \in T : [h, t_0] = 0$ for any $h \in H^0\}$. In the following, we shall denote by Λ^0 the set of all nonzero $\alpha \in (H^0)^*$ such that $L^0_\alpha := \{v^0_\alpha \in L^0 : [h, v^0_\alpha] = \alpha(h)v^0_\alpha$ for any $h \in H^0\} \neq 0$. As an immediate consequence of Jacobi identity we have:

(i) If $[T_\alpha, T_\beta] \neq 0$, then $\alpha + \beta \in \Lambda^0 \cup \{0\}$ and $[T_\alpha, T_\beta] \subseteq L^0_{\alpha+\beta}$,
(ii) If $[L^0_\delta, T_\alpha] \neq 0$, then $\delta + \alpha \in \Lambda^1 \cup \{0\}$ and $[L^0_\delta, T_\alpha] \subseteq T_{\delta+\alpha}$,
(iii) If $[T_\alpha, T_\beta, T_\gamma] \neq 0$, then $\alpha + \beta + \gamma \in \Lambda^1 \cup \{0\}$ and $[T_\alpha, T_\beta, T_\gamma] \subseteq T_{\alpha+\beta+\gamma}$,

for any $\alpha, \beta, \gamma \in \Lambda^1 \cup \{0\}$ and $\delta \in \Lambda^0 \cup \{0\}$.

DEFINITION 2.2

Let T be a Lie triple system, let $L = L^0 \oplus L^1$ be its standard embedding, and let H^0 be a MASA of L^0. We shall call that T is a split Lie triple system (with respect to H^0) if $T = T_0 \oplus (\bigoplus_{\alpha\in\Lambda^1} T_\alpha)$. We say that Λ^1 is the root system of T.

We observe that it is straightforward to prove that if T is a split Lie triple system with respect to H^0, then L^0 is a split Lie algebra with respect to the splitting Cartan subalgebra H^0, with set of nonzero roots Λ^0 (see Proposition 3.1 of [2]). We also note that the facts $H^0 \subset L^0 = [T, T]$ and $T = T_0 \oplus \bigoplus_{\alpha\in\Lambda^1} T_\alpha$ imply

$$H^0 = [T_0, T_0] + \sum_{\alpha\in\Lambda^1} [T_\alpha, T_{-\alpha}]. \tag{1}$$

Finally, as $[T_0, T_0] \subset L^0_0 = H^0$ we have

$$[T_0, T_0, T_0] = 0. \tag{2}$$

We finally note that $\alpha \in \Lambda^1$ does not imply $\alpha \in \Lambda^0$.

DEFINITION 2.3

A root system Λ^1 of a split Lie triple system T is called symmetric if it satisfies that $\alpha \in \Lambda^1$ implies $-\alpha \in \Lambda^1$.

A similar concept applies to the set Λ^0 of nonzero roots of L^0.

In the following, T denotes a split Lie triple system with a symmetric root system Λ^1, and $T = T_0 \oplus (\bigoplus_{\alpha\in\Lambda^1} T_\alpha)$ the corresponding root decomposition. Our main tool, introduced in [4], in the study of split Lie triple systems is the concept of connection of roots.

DEFINITION 2.4

Let α and β be two nonzero roots, we shall say that α and β are connected if there exists a family $\{\alpha_1, \alpha_2, \ldots, \alpha_{2n}, \alpha_{2n+1}\} \subset \Lambda^1 \cup \{0\}$ of roots of T such that

(1) $\{\alpha_1, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, \ldots, \alpha_1 + \cdots + \alpha_{2n} + \alpha_{2n+1}\} \subset \Lambda^1$;
(2) $\{\alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \ldots, \alpha_1 + \cdots + \alpha_{2n}\} \subset \Lambda^0$;
(3) $\alpha_1 = \alpha$ and $\alpha_1 + \cdots + \alpha_{2n} + \alpha_{2n+1} \in \pm\beta$.

We shall also say that $\{\alpha_1, \alpha_2, \ldots, \alpha_{2n}, \alpha_{2n+1}\}$ is a connection from α to β.

We denote by

$$\Lambda^1_\alpha := \{\beta \in \Lambda^1 \colon \alpha \text{ and } \beta \text{ are connected}\}$$

Let us observe that $\{\alpha\}$ is a connection from α to itself and to $-\alpha$. Therefore $\pm\alpha \in \Lambda^1_\alpha$.

DEFINITION 2.5

A subset Ω^1 of a root system Λ^1, associated to a split Lie triple system T, is called a root subsystem if it is symmetric, and given $\alpha, \beta, \gamma \in \Omega^1 \cup \{0\}$ such that $\alpha + \beta \in \Lambda^0$ and $\alpha + \beta + \gamma \in \Lambda^1$ then $\alpha + \beta + \gamma \in \Omega^1$.

Let Ω^1 be a root subsystem of Λ^1. We define

$$T_{0,\Omega^1} := \mathrm{span}_{\mathbb{K}}[[T_\alpha, T_\beta, T_\gamma] \colon \alpha + \beta + \gamma = 0;\ \alpha, \beta, \gamma \in \Omega^1 \cup \{0\}\} \subset T_0$$

and $V_{\Omega^1} := \bigoplus_{\alpha \in \Omega^1} T_\alpha$. Taking into account eq. (2), it is straightforward to verify that

$$T_{\Omega^1} := T_{0,\Omega^1} \oplus V_{\Omega^1}$$

is a Lie triple subsystem of T. We will say that T_{Ω^1} is the *Lie subtriple associated* to the root subsystem Ω^1.

We recall the following results from [4]:

PROPOSITION 2.1 (Proposition 3.1, Lemma 3.1 of [4])

The following assertions hold:

(1) *If Λ^0 is symmetric, then the relation $\sim$ in Λ^1, defined by $\alpha \sim \beta$ if and only if $\beta \in \Lambda^1_\alpha$, is of equivalence*;
(2) *Let α be a nonzero root and suppose Λ^0 is symmetric. Then Λ^1_α is a root subsystem.*

3. Decompositions

We are going to state a series of previous results in order to show that for a fixed $\alpha_0 \in \Lambda^1$, the Lie subtriple $T_{\Lambda^1_{\alpha_0}}$ associated to the root subsystem $\Lambda^1_{\alpha_0}$ is an ideal of T.

Lemma 3.1. *The following assertions hold*:

1. *If $\alpha, \beta \in \Lambda^1$ with $[T_\alpha, T_\beta] \neq 0$, then α is connected with β*;
2. *If $\alpha, \beta \in \Lambda^1$, $\alpha \in \Lambda^0$ and $[L^0_\alpha, T_\beta] \neq 0$, then α is connected with β*;
3. *If $\alpha, \beta \in \Lambda^1$, $\alpha, \beta \in \Lambda^0$ and $[L^0_\alpha, L^0_\beta] \neq 0$, then α is connected with β*;
4. *If $\alpha, \bar{\beta} \in \Lambda^1$ such that α is not connected with $\bar{\beta}$, then $[T_\alpha, T_{\bar{\beta}}] = 0$, $[L^0_\alpha, T_{\bar{\beta}}] = 0$ if furthermore $\alpha \in \Lambda^0$ and $[L^0_\alpha, L^0_{\bar{\beta}}] = 0$ if furthermore $\alpha, \bar{\beta} \in \Lambda^0$.*

Proof.

(1) Suppose $[T_\alpha, T_\beta] \neq 0$ and so $\alpha + \beta \in \Lambda^0 \cup \{0\}$. If $\alpha + \beta = 0$, then $\beta = -\alpha$ and so α is connected with β. Suppose $\alpha + \beta \neq 0$. Since $\alpha + \beta \in \Lambda^0$, we have $\{\alpha, \beta, -\alpha\}$ is a connection from α to β.

(2) and (3) We can argue similarly with the connections $\{\alpha, 0, -\alpha - \beta\}$ and $\{\alpha, \beta, -\alpha\}$ in the cases $\alpha \in \Lambda^0$ and $[L^0_\alpha, T_\beta] \neq 0$; and $\alpha, \beta \in \Lambda^0$ and $[L^0_\alpha, L^0_\beta] \neq 0$ respectively.

(4) Consequence of 1, 2 and 3.

Lemma 3.2. *If $\alpha, \bar{\beta} \in \Lambda^1$ are not connected, then $\bar{\beta}([T_\alpha, T_{-\alpha}]) = 0$.*

Proof. If $[T_\alpha, T_{-\alpha}] = 0$ it is clear. Let us suppose that $[T_\alpha, T_{-\alpha}] \neq 0$ and $\bar{\beta}([T_\alpha, T_{-\alpha}]) \neq 0$. Then $[T_\alpha, T_{-\alpha}, T_{\bar{\beta}}] = \bar{\beta}([T_\alpha, T_{-\alpha}])T_{\bar{\beta}} \neq 0$. So either $[T_{-\alpha}, T_{\bar{\beta}}, T_\alpha] \neq 0$ or $[T_{\bar{\beta}}, T_\alpha, T_{-\alpha}] \neq 0$, contradicting Lemma 3.1(4). Hence, $\bar{\beta}([T_\alpha, T_{-\alpha}]) = 0$.

Lemma 3.3. *Fix $\alpha_0 \in \Lambda^1$ and suppose Λ^0 is symmetric. If $\alpha \in \Lambda^1_{\alpha_0}$ and $\beta, \gamma \in \Lambda^1 \cup \{0\}$, then the following assertions hold*:

(1) *If $[T_\alpha, T_\beta, T_\gamma] \neq 0$, then $\beta, \gamma, \alpha + \beta + \gamma \in \Lambda^1_{\alpha_0} \cup \{0\}$;*
(2) *If $[T_\beta, T_\alpha, T_\gamma] \neq 0$, then $\beta, \gamma, \beta + \alpha + \gamma \in \Lambda^1_{\alpha_0} \cup \{0\}$;*
(3) *If $[T_\beta, T_\gamma, T_\alpha] \neq 0$, then $\beta, \gamma, \beta + \gamma + \alpha \in \Lambda^1_{\alpha_0} \cup \{0\}$.*

Proof.

(1) The fact that $[T_\alpha, T_\beta] \neq 0$ implies by Lemma 3.1 that $\alpha \sim \beta$ in the case $\beta \neq 0$. From here, $\beta \in \Lambda_{\alpha_0} \cup \{0\}$. To show $\gamma, \alpha + \beta + \gamma \in \Lambda_{\alpha_0} \cup \{0\}$ we will distinguish two cases. First, suppose that $\alpha + \beta + \gamma = 0$ and so $\alpha + \beta + \gamma \in \Lambda_{\alpha_0} \cup \{0\}$. If we had $\gamma \neq 0$, then $\alpha + \beta \in \Lambda^0$. As $\alpha + \beta = -\gamma$, then $\{\alpha, \beta, 0\}$ would be a connection from α to γ and we conclude $\gamma \in \Lambda_{\alpha_0} \cup \{0\}$.

Second, suppose $\alpha + \beta + \gamma \neq 0$. If $\alpha + \beta \neq 0$, then $\alpha + \beta \in \Lambda^0$ and so $\{\alpha, \beta, \gamma\}$ is a connection from α to $\alpha + \beta + \gamma$. Hence $\alpha + \beta + \gamma \in \Lambda_{\alpha_0}$. In the case $\gamma \neq 0$, we have $\{\alpha, \beta, -\alpha - \beta - \gamma\}$, is a connection from α to γ. So $\gamma \in \Lambda_{\alpha_0}$. Finally, if $\alpha + \beta = 0$, then necessarily $\gamma \in \Lambda_{\alpha_0} \cup \{0\}$. Indeed, if $\gamma \neq 0$ and α was not connected with γ, by Lemma 3.2, $[T_\alpha, T_\beta, T_\gamma] = [T_\alpha, T_{-\alpha}, T_\gamma] = \gamma([T_\alpha, T_{-\alpha}])T_\gamma = 0$, a contradiction. We also have $\alpha + \beta + \gamma = \gamma \in \Lambda_{\alpha_0} \cup \{0\}$.

Items 2 and 3 are a consequence of the definition of Lie triple system.

Lemma 3.4. *Fix $\alpha_0 \in \Lambda^1$ and suppose Λ^0 is symmetric. If $\alpha, \beta, \gamma \in \Lambda^1_{\alpha_0} \cup \{0\}$ with $\alpha + \beta + \gamma = 0$ and $\delta, \epsilon \in \Lambda^1 \cup \{0\}$. Then the following assertions hold*:

(1) *If $[[T_\alpha, T_\beta, T_\gamma], T_\delta, T_\epsilon] \neq 0$ then $\delta, \epsilon, \delta + \epsilon \in \Lambda^1_{\alpha_0} \cup \{0\}$;*
(2) *If $[T_\delta, [T_\alpha, T_\beta, T_\gamma], T_\epsilon] \neq 0$ then $\delta, \epsilon, \delta + \epsilon \in \Lambda^1_{\alpha_0} \cup \{0\}$;*
(3) *If $[T_\delta, T_\epsilon, [T_\alpha, T_\beta, T_\gamma]] \neq 0$ then $\delta, \epsilon, \delta + \epsilon \in \Lambda^1_{\alpha_0} \cup \{0\}$.*

Proof.

(1) From the fact that $\alpha + \beta + \gamma = 0$ and eq. (2), let us suppose that at least two distinct elements in $\{\alpha, \beta, \gamma\}$ are nonzero and, taking into account $[T_\alpha, T_\beta, T_\gamma] \neq 0$, $\alpha + \beta \neq 0$ and $\gamma \neq 0$.

Since

$$\begin{aligned} 0 &\neq [[T_\alpha, T_\beta, T_\gamma], T_\delta, T_\epsilon] \\ &\subset [T_\alpha, T_\beta, [T_\gamma, T_\delta, T_\epsilon]] + [T_\gamma, [T_\alpha, T_\beta, T_\delta], T_\epsilon] + [T_\gamma, T_\delta, [T_\alpha, T_\beta, T_\epsilon]], \end{aligned}$$

any of the above three summands is nonzero. Suppose

$$[T_\alpha, T_\beta, [T_\gamma, T_\delta, T_\epsilon]] \neq 0.$$

As $\gamma \neq 0$ and $[T_\gamma, T_\delta, T_\epsilon] \neq 0$, Lemma 3.3(1) shows that δ, ϵ and $\gamma + \delta + \epsilon$ are connected with γ in the case of being nonzero roots and so $\delta, \epsilon, \gamma + \delta + \epsilon \in \Lambda^1_{\alpha_0} \cup \{0\}$. If $\gamma + \delta + \epsilon = 0$, then $\delta + \epsilon = -\gamma \in \Lambda^1_{\alpha_0}$. If $\gamma + \delta + \epsilon \neq 0$, taking into account $0 \neq [T_\alpha, T_\beta, [T_\gamma, T_\delta, T_\epsilon]] \subset [T_\alpha, T_\beta, T_{\gamma+\delta+\epsilon}]$, Lemma 3.3(3) and the fact that either $\alpha \in \Lambda^1_{\alpha_0}$ or $\beta \in \Lambda^1_{\alpha_0}$, by the observation at the beginning of the proof, give us that $\alpha + \beta + \gamma + \delta + \epsilon = \delta + \epsilon \in \Lambda^1_{\alpha_0} \cup \{0\}$. Therefore $\delta, \epsilon, \delta + \epsilon \in \Lambda^1_{\alpha_0} \cup \{0\}$.

We argue similarly if either $[T_\gamma, [T_\alpha, T_\beta, T_\delta], T_\epsilon] \neq 0$ or $[T_\gamma, T_\delta, [T_\alpha, T_\beta, T_\epsilon]] \neq 0$ to get $\delta, \epsilon, \delta + \epsilon \in \Lambda^1_{\alpha_0} \cup \{0\}$.

Items (2) and (3) are direct consequences of the definition of Lie triple system and item (1).

Lemma 3.5. *Fix* $\alpha_0 \in \Lambda^1$ *and suppose* Λ^0 *is symmetric. If* $\alpha_1, \alpha_2, \alpha_3 \in \Lambda^1_{\alpha_0} \cup \{0\}$ *with* $\alpha_1 + \alpha_2 + \alpha_3 = 0$ *and* $\bar{\epsilon} \in \Lambda^1 \backslash \Lambda^1_{\alpha_0}$, *then the following assertions hold*:

(1) $[[T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}], T_{\bar{\epsilon}}] = 0$;
(2) *In case* $\bar{\epsilon} \in \Lambda^0$, *then* $[[T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}], L^0_{\bar{\epsilon}}] = 0$;
(3) $\bar{\epsilon}([[T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}], T_0]) = 0$.

Proof.
(1) From the fact $\alpha_1 + \alpha_2 + \alpha_3 = 0$ and eq. (2), let us suppose $\alpha_3 \neq 0$ and $\alpha_i \neq 0$ with $i \in \{1, 2\}$. We have

$$\begin{aligned}[][[T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}], T_{\bar{\epsilon}}] &= [[[T_{\alpha_1}, T_{\alpha_2}], T_{\alpha_3}], T_{\bar{\epsilon}}] \\ &\subset [[T_{\alpha_3}, T_{\bar{\epsilon}}], [T_{\alpha_1}, T_{\alpha_2}]] + [[T_{\bar{\epsilon}}, [T_{\alpha_1}, T_{\alpha_2}]], T_{\alpha_3}]. \qquad (3)\end{aligned}$$

Let us center on the first summand in (3), as $\alpha_3 \neq 0$. Then $[[T_{\alpha_3}, T_{\bar{\epsilon}}], [T_{\alpha_1}, T_{\alpha_2}]] = 0$ by Lemma 3.1(4). Let us now consider the second summand in (3). As either $\alpha_1 \neq 0$ or $\alpha_2 \neq 0$, Jacobi identity, the fact $[T_{\bar{\epsilon}}, T_0] \subset L^0_{\bar{\epsilon}}$ and Lemma 3.1(4) show that $[T_{\bar{\epsilon}}, [T_{\alpha_1}, T_{\alpha_2}]] = 0$. So, the second summand in (3) is also zero and then $[[T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}], T_{\bar{\epsilon}}] = 0$.

(2) A similar argument gives us $[[T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}], L^0_{\bar{\epsilon}}] = 0$ in case $\bar{\epsilon} \in \Lambda^0$.

(3) Suppose $\bar{\epsilon}([[T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}], T_0]) \neq 0$. Then

$$[[[T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}], T_0], T_{\bar{\epsilon}}] = \bar{\epsilon}([[T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}], T_0])T_{\bar{\epsilon}} \neq 0.$$

But

$$\begin{aligned}[][[[T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}], T_0], T_{\bar{\epsilon}}] \subset &[[T_0 T_{\bar{\epsilon}}], [T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}]] \\ &+ [[T_{\bar{\epsilon}}, [T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}]], T_0] = 0\end{aligned}$$

by (1) and (2), a contradiction. Hence $\bar{\epsilon}([[T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}], T_0]) = 0$.

Theorem 3.1. *Suppose* Λ^0 *is symmetric, then the following assertions hold.*

(1) *For any* $\alpha_0 \in \Lambda^1$, *the Lie triple subsystem*

$$T_{\Lambda^1_{\alpha_0}} = T_{0,\Lambda^1_{\alpha_0}} \oplus V_{\Lambda^1_{\alpha_0}}$$

of T *associated to the root subsystem* $\Lambda^1_{\alpha_0}$ *is an ideal of* T.

(2) *If* T *is simple, then there exists a connection from* α *to* β *for any* $\alpha, \beta \in \Lambda^1$.

Proof.
(1) Taking into account

$$T_{0,\Lambda^1_{\alpha_0}} := \mathrm{span}_{\mathbb{K}}[[T_\alpha, T_\beta, T_\gamma] \colon \alpha+\beta+\gamma=0;\ \alpha,\beta,\gamma \in \Lambda^1_{\alpha_0}\cup\{0\}\} \subset T_0, \tag{4}$$

we have by eq. (2) that $[T_{0,\Lambda^1_{\alpha_0}}, T_0, T_0] \subset [T_0, T_0, T_0] = 0$. Equation (4) together with Lemma 3.4 imply that $[T_{0,\Lambda^1_{\alpha_0}}, T_0, T_\alpha] + [T_{0,\Lambda^1_{\alpha_0}}, T_\alpha, T_0] + [T_{0,\Lambda^1_{\alpha_0}}, T_\alpha, T_\beta] \subset T_{\Lambda^1_{\alpha_0}}$ for any $\alpha, \beta \in \Lambda^1$. From here,

$$[T_{0,\Lambda^1_{\alpha_0}}, T, T] = \left[T_{0,\Lambda^1_{\alpha_0}}, T_0 \oplus \left(\bigoplus_{\alpha\in\Lambda^1} T_\alpha\right), T_0 \oplus \left(\bigoplus_{\alpha\in\Lambda^1} T_\alpha\right)\right] \subset T_{\Lambda^1_{\alpha_0}}.$$

As $V_{\Lambda^1_{\alpha_0}} := \bigoplus_{\gamma\in\Lambda^1_{\alpha_0}} T_\gamma$, we have by Lemma 3.3 and eq. (4) that

$$\left[\bigoplus_{\gamma\in\Lambda^1_{\alpha_0}} T_\gamma, T_0, T_0\right] + \left[\bigoplus_{\gamma\in\Lambda^1_{\alpha_0}} T_\gamma, T_0, T_\alpha\right] + \left[\bigoplus_{\gamma\in\Lambda^1_{\alpha_0}} T_\gamma, T_\alpha, T_0\right] + \left[\bigoplus_{\gamma\in\Lambda^1_{\alpha_0}} T_\gamma, T_\alpha, T_\beta\right] \subset T_{\Lambda^1_{\alpha_0}}$$

for any $\alpha, \beta \in \Lambda^1$. So

$$[V_{\Lambda^1_{\alpha_0}}, T, T] = \left[\bigoplus_{\gamma\in\Lambda^1_{\alpha_0}} T_\gamma, T_0 \oplus \left(\bigoplus_{\alpha\in\Lambda^1} T_\alpha\right), T_0 \oplus \left(\bigoplus_{\alpha\in\Lambda^1} T_\alpha\right)\right] \subset T_{\Lambda^1_{\alpha_0}}.$$

Then

$$[T_{\Lambda^1_{\alpha_0}}, T, T] = [T_{0,\Lambda^1_{\alpha_0}} + V_{\Lambda^1_{\alpha_0}}, T, T] \subset T_{\Lambda^1_{\alpha_0}}$$

and therefore $T_{\Lambda^1_{\alpha_0}}$ is an ideal of T.

(2) The simplicity of T implies $T_{\Lambda^1_{\alpha_0}} = T$. Hence $\Lambda^1_{\alpha_0} = \Lambda^1$.

Theorem 3.2. *Suppose Λ^0 is symmetric. Then for a vector space complement $\mathcal{U}$ of*

$$\mathrm{span}_{\mathbb{K}}\{[T_\alpha, T_\beta, T_\gamma] \colon \alpha+\beta+\gamma=0,\ \textit{where}\ \alpha,\beta,\gamma\in\Lambda^1\cup\{0\}\}$$

in T_0, we have

$$T = \mathcal{U} + \sum_{[\alpha]\in\Lambda^1/\sim} I_{[\alpha]},$$

where any $I_{[\alpha]}$ is one of the ideals $T_{\Lambda^1_\alpha}$ of T described in Theorem 3.1. Moreover $[I_{[\alpha]}, T, I_{[\beta]}] = 0$ if $[\alpha] \neq [\beta]$.

Proof. Let us denote $\mathfrak{T}_0 := \mathrm{span}_{\mathbb{K}}\{[T_\alpha, T_\beta, T_\gamma]: \alpha+\beta+\gamma = 0, \text{where } \alpha, \beta, \gamma \in \Lambda^1 \cup \{0\}\}$. Then we can write

$$T = T_0 \oplus \left(\bigoplus_{\alpha\in\Lambda^1} T_\alpha\right) = (\mathcal{U} \oplus \mathfrak{T}_0) \oplus \left(\bigoplus_{\alpha\in\Lambda^1} T_\alpha\right).$$

Proposition 2.1(1) lets us consider the quotient set $\Lambda^1/\sim := \{[\alpha]: \alpha \in \Lambda^1\}$. By denoting $I_{[\alpha]} = T_{0,[\alpha]} \oplus V_{[\alpha]}$ where $I_{[\alpha]} := T_{\Lambda^1_\alpha}$, $T_{0,[\alpha]} := T_{0,\Lambda^1_\alpha}$ and $V_{[\alpha]} := V_{\Lambda^1_\alpha}$, we can assert $\bigoplus_{\alpha\in\Lambda^1} T_\alpha = \bigoplus_{[\alpha]\in\Lambda^1/\sim} V_{[\alpha]}$ and $\mathfrak{T}_0 = \sum\limits_{[\alpha]\in\Lambda^1/\sim} T_{0,[\alpha]}$. From here

$$T = (\mathcal{U} \oplus \mathfrak{T}_0) \oplus \left(\bigoplus_{\alpha\in\Lambda^1} T_\alpha\right) = \mathcal{U} + \sum_{[\alpha]\in\Lambda^1/\sim} I_{[\alpha]},$$

each $I_{[\alpha]}$ being an ideal of T by Theorem 3.1. The assertion $[I_{[\alpha]}, T, I_{[\beta]}] = 0$ if $[\alpha] \neq [\beta]$ is a consequence of writing $[I_{[\alpha]}, T, I_{[\beta]}] = [T_{0,[\alpha]} + V_{[\alpha]}, T_0 + \sum_{\gamma\in\Lambda^1} T_\gamma, T_{0,[\beta]} + V_{[\beta]}]$ and applying eq. (2), Lemmas 3.3 and 3.4 taking into account $\alpha \nsim \beta$.

Observe that the fact $[I_{[\alpha]}, T, I_{[\beta]}] = 0$ if $[\alpha] \neq [\beta]$ implies $[I_{[\alpha]}, I_{[\beta]}, T] = [[I_{[\alpha]}, I_{[\beta]}], T] = 0$. As any element in $[I_{[\alpha]}, I_{[\beta]}] \subset L^0$ is a linear mapping from T onto itself we conclude

$$[I_{[\alpha]}, I_{[\beta]}] = 0 \tag{5}$$

if $[\alpha] \neq [\beta]$.

COROLLARY 3.1

Suppose Λ^0 *is symmetric. If* $\mathrm{Ann}(T) = 0$ *and* $[T, T, T] = T$, *then* T *is the direct sum of the ideals given in Theorem* 3.2,

$$T = \bigoplus_{[\alpha]\in\Lambda^1/\sim} I_{[\alpha]}.$$

Proof. The fact $[T, T, T] = T$ and Theorem 3.2 give

$$\left[\mathcal{U} + \sum_{[\alpha]\in\Lambda^1/\sim} I_{[\alpha]}, \mathcal{U} + \sum_{[\alpha]\in\Lambda^1/\sim} I_{[\alpha]}, \mathcal{U} + \sum_{[\alpha]\in\Lambda^1/\sim} I_{[\alpha]}\right] = \mathcal{U} + \sum_{[\alpha]\in\Lambda^1/\sim} I_{[\alpha]}.$$

Taking into account $\mathcal{U} \subset T_0$, eq. (2), Lemma 3.3 and the fact $[I_{[\alpha]}, T, I_{[\beta]}] = 0$ if $[\alpha] \neq [\beta]$ (see Theorem 3.2), let us conclude $\mathcal{U} = 0$, that is $T = \sum_{[\alpha]\in\Lambda^1/\sim} I_{[\alpha]}$. The direct character of the sum can be obtained as follows. If $x \in I_{[\alpha]} \cap \sum\limits_{\substack{[\beta]\in\Lambda^1/\sim \\ \beta\nsim\alpha}} I_{[\beta]}$ then $[x, T, I_{[\alpha]}] = [x, T, \sum\limits_{\substack{[\beta]\in\Lambda^1/\sim \\ \beta\nsim\alpha}} I_{[\beta]}] = 0$ as consequence of $[I_{[\alpha]}, T, I_{[\beta]}] = 0$ when $[\alpha] \neq [\beta]$. From here $[x, T, T] = 0$ and as $\mathrm{Ann}(T) = 0$, $x = 0$.

4. The simple components

In this section we study if any of the component in the decomposition given in Corollary 3.1 is simple. Under certain conditions we give an affirmative answer. From now on $\mathrm{char}(\mathbb{K}) = 0$.

Lemma 4.1. *Let $T = T_0 \oplus (\bigoplus_{\alpha\in\Lambda^1} T_\alpha)$ be a split Lie triple system. If I is an ideal of T then $I = (I \cap T_0) \oplus (\bigoplus_{\alpha\in\Lambda^1}(I \cap T_\alpha))$.*

Proof. We can see that $T = T_0 \oplus (\bigoplus_{\alpha\in\Lambda^1} T_\alpha)$ as a weight module with respect to the split Lie algebra L^0 with MASA H^0. The character of ideal of I and the fact $L^0 = [T, T]$ give us that I is a submodule of T. It is well-known that a submodule of a weight module is again a weight module. From here, I is a weight module with respect to L^0 (and H^0), and so $I = (I \cap T_0) \oplus (\bigoplus_{\alpha\in\Lambda^1}(I \cap T_\alpha))$.

DEFINITION 4.1

We say that a split Lie triple system T is root-multiplicative if $\alpha, \beta, \gamma \in \Lambda^1 \cup \{0\}$ are such that $\alpha + \beta \in \Lambda^0$ and $\alpha + \beta + \gamma \in \Lambda^1$ then $[T_\alpha, T_\beta, T_\gamma] \neq 0$.

Let us see some examples of root-multiplicative split Lie triple systems: Consider $L = H \oplus \bigoplus_{\gamma\in\Lambda} L_\gamma$ a simple split Lie algebra, and the Lie triple system $\mathcal{T}(L)$, where $\mathcal{T}(L)$ agrees with L as vector spaces, and the triple product is defined by $[x, y, z] := [[x, y], z]$. The standard embedding of $\mathcal{T}(L)$ is $L \oplus L$ with the product $[(x, y), (z, t)] = ([x, z] + [y, t], [x, t] + [y, z])$. It is straightforward to show that $\mathcal{T}(L) = (0, L)$ is split with respect to the MASA $(H, 0)$ of $L^0 := (L, 0)$, with $\Lambda^1 = \Lambda$, that its root spaces are $\mathcal{T}(L)_\gamma = (0, L_\gamma)$ for any $\gamma \in \Lambda$, and $\mathcal{T}(L)_0 = (0, H)$. Observe that the root systems of the split Lie algebra $L^0 = (L, 0)$ and of the split Lie triple system $\mathcal{T}(L) = (0, L)$ agree. Now, if we take L as a simple separable L^*-algebra or a simple locally finite split Lie algebra over a field of characteristic zero [11, 1], it is well-known that any such algebras satisfy and that if $\alpha, \beta, \alpha + \beta \in \Lambda$ then $[L_\alpha, L_\beta] = L_{\alpha+\beta}$ (see Proposition I.7 (v) and Theorem III.19 of [12]). From here, we obtain that the Lie triple system $\mathcal{T}(L)$ is root-multiplicative.

More examples of root-multiplicative split Lie triple systems are the split Lie triple systems considered in §4 of [4], §5 of [5] and §4 of [6].

Lemma 4.2. *Let T be a root-multiplicative split Lie triple system with* $\mathrm{Ann}(T) = 0$. *If for any $\alpha \in \Lambda^1$ we have* $\dim L^0_\alpha \leq 1$. *Then there is not any nonzero ideal of T contained in T_0.*

Proof. Suppose there exists a nonzero ideal I of T such that $I \subset T_0$. Taking into account eq. (2), $[I, T_0, T_0] = 0$. Given $\alpha \in \Lambda^1$, as $[I, T_0, T_\alpha] + [I, T_\alpha, T_0] \subset T_\alpha \cap T_0$, $[I, T_0, T_\alpha] = [I, T_\alpha, T_0] = 0$. Given also $\beta \in \Lambda^1$ with $\alpha + \beta \neq 0$, $[I, T_\alpha, T_\beta] \subset T_{\alpha+\beta} \cap T_0 = 0$. As $\mathrm{Ann}(T) = 0$, $[I, T_\alpha, T_{-\alpha}] \neq 0$ for some $\alpha \in \Lambda^1$. Thus, there exist $t_{\pm\alpha} \in T_{\pm\alpha}$ and $t_0 \in I$ such that $[t_0, t_\alpha, t_{-\alpha}] \neq 0$. Hence $0 \neq [t_0, t_\alpha] \in L^0_\alpha$ and so necessarily $\dim L^0_\alpha = 1$. The root-multiplicativity of T (consider the roots $0, \alpha, 0 \in \Lambda^1 \cup \{0\}$), and the fact that $\dim L^0_\alpha = 1$ give us the existence of $0 \neq t'_0 \in T_0$ such that $0 \neq [t_0, t_\alpha, t'_0] \in T_\alpha$. As $t_0 \in I$, we conclude $0 \neq t'_\alpha := [t_0, t_\alpha, t'_0] \in I \subset T_0$, a contradiction. Hence I is not contained in T_0

Theorem 4.1. *Let T be root-multiplicative*, *with* $\mathrm{Ann}(T) = 0$ *and satisfying* $T = [T, T, T]$. *If Λ^0 is symmetric and for any $\alpha \in \Lambda^1$ we have* $\dim T_\alpha = 1$ *and* $\dim L^0_\alpha \leq 1$, *then T is simple if and only if it has all its nonzero roots connected.*

Proof. If T is simple then it has all its nonzero roots connected as a consequence of Theorem 3.1(1). Let us prove the converse: Consider I a nonzero ideal of T. By Lemmas 4.1 and 4.2, $I = (I \cap T_0) \oplus (\bigoplus_{\alpha \in \Lambda^1} (I \cap T_\alpha))$ with $I \cap T_{\alpha_0} \neq 0$ for some $\alpha_0 \in \Lambda^1$. Taking into account $\dim T_{\alpha_0} = 1$, we have $T_{\alpha_0} \subset I$. Given any $\beta_0 \in \Lambda^1$ with $\beta_0 \notin \{\alpha_0, -\alpha_0\}$, as α_0 and β_0 are connected, the root-multiplicativity of T and the assumption $\dim T_\alpha = 1$ for any $\alpha \in \Lambda^1$, give us a connection $\{\alpha_1, \ldots, \alpha_{2r+1}\}$ from α_0 to β_0 such that $\alpha_1 = \alpha_0, \alpha_1 + \alpha_2 + \alpha_3, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{2r+1} \in \Lambda^1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{2r} \in \Lambda^0$ and $\alpha_1 + \alpha_2 + \cdots + \alpha_{2r+1} \in \{\beta_0, -\beta_0\}$, with $T_{\alpha_1} = T_{\alpha_0}$, $[T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}] = T_{\alpha_1+\alpha_2+\alpha_3}$, $[[T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}], T_{\alpha_4}, T_{\alpha_5}] = T_{\alpha_1+\cdots+\alpha_5}, \cdots,$

$$[[\ldots[[T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}], T_{\alpha_4}, T_{\alpha_5}]]\ldots], T_{\alpha_{2r+1}}] \in \{T_{\beta_0}, T_{-\beta_0}\}.$$

From here, either

$$T_{\beta_0} \subset I \text{ or } T_{-\beta_0} \subset I \tag{6}$$

for any $\beta_0 \in \Lambda^1$, and so $[T_{\beta_0}, T_{-\beta_0}, T] \subset I$.

Observe that as a consequence of $T = [T, T, T]$, we have

$$T_0 = \sum_{\substack{\alpha+\beta+\gamma=0 \\ \alpha,\beta,\gamma \in \Lambda^1 \cup \{0\}}} [T_\alpha, T_\beta, T_\gamma]. \tag{7}$$

Let us study the products $[T_\alpha, T_\beta, T_\gamma]$ of eq. (7) in order to show $T_0 \subset I$. Taking into account $[T_0, T_0, T_0] = 0$ (see eq. (2)), and the fact $\alpha+\beta+\gamma = 0$ with $\alpha, \beta, \gamma \in \Lambda^1 \cup \{0\}$, we can suppose $\gamma \neq 0$ and either $\alpha \neq 0$ or $\beta \neq 0$. Suppose $\alpha \neq 0$ and $\beta = 0$ (resp. $\alpha = 0$ and $\beta \neq 0$), then $\alpha = -\gamma$ (resp. $\beta = -\gamma$), and by eq. (6), $[T_\alpha, T_\beta, T_\gamma] = [T_{-\gamma}, T_0, T_\gamma] \subset I$, (resp. $[T_\alpha, T_\beta, T_\gamma] = [T_0, T_{-\gamma}, T_\gamma] \subset I$). If the three elements in $\{\alpha, \beta, \gamma\}$ are nonzero, in case some $T_\epsilon \subset I$, $\epsilon \in \{\alpha, \beta, \gamma\}$, then clearly $[T_\alpha, T_\beta, T_\gamma] \subset I$. Finally, consider the case in which any of the T_ϵ does not belong to I. If $[T_\alpha, T_\beta, T_\gamma] = 0$ then $[T_\alpha, T_\beta, T_\gamma] \subset I$. If $[T_\alpha, T_\beta, T_\gamma] \neq 0$, necessarily $\alpha + \beta \neq 0$ and so $\alpha + \beta \in \Lambda^0$. From here, we have by root-multiplicativity $[T_\alpha, T_\beta, T_{-\beta}] = T_\alpha$. Equation (6) gives us $T_{-\beta} \subset I$, then $T_\alpha \subset I$ and so $[T_\alpha, T_\beta, T_\gamma] \subset I$. Therefore eq. (7) implies

$$T_0 \subset I. \tag{8}$$

Fix now any $\alpha_0 \in \Lambda^1$. By (6) either $T_{\alpha_0} \subset I$ or $T_{-\alpha_0} \subset I$. Write $T_{\rho\alpha_0} \subset I$ with $\rho \in \pm 1$, then we can show $T_{-\rho\alpha_0} \subset I$. Indeed, since $\alpha_0 \neq 0$, there exists $h_0 \in H^0$ such that $\alpha_0(h_0) \neq 0$ and so we have

$$t_{-\rho\alpha_0} = -\rho\alpha_0(h_0)^{-1}[h_0, t_{-\rho\alpha_0}] \tag{9}$$

for any $t_{-\rho\alpha_0} \in T_{-\rho\alpha_0}$. As

$$H^0 = [T_0, T_0] + \sum_{\alpha \in \Lambda^1} [T_\alpha, T_{-\alpha}],$$

(see eq. (1)), we can suppose that either $h_0 = [t_0, t_0']$ with $t_0, t_0' \in T_0$ or $h_0 = [t_\alpha, t_{-\alpha}]$ with $t_{\pm\alpha} \in T_{\pm\alpha}$ and $\alpha \in \Lambda^1$. In the first case, we have by eqs. (9) and (8) that

$$t_{-\rho\alpha_0} = -\rho\alpha_0(h_0)^{-1}[t_0, t_0', t_{-\rho\alpha_0}] \in I,$$

and in the second case eqs. (9) and (6) give us

$$t_{-\rho\alpha_0} = -\rho\alpha_0(h_0)^{-1}[t_\alpha, t_{-\alpha}, t_{-\rho\alpha_0}] \in I$$

for any $t_{-\rho\alpha_0} \in T_{-\rho\alpha_0}$. Since $\dim T_{-\rho\alpha_0} = 1$ we conclude $T_{-\rho\alpha_0} \subset I$ and so $T_{\pm\alpha_0} \subset I$ for any $\alpha_0 \in \Lambda^1$. From here, and taking into account eq. (8) we conclude $I = T$ and so T is simple.

Theorem 4.2. *Let T be root-multiplicative, with* $\mathrm{Ann}(T) = 0$ *and satisfying* $T = [T, T, T]$. *If Λ^0 is symmetric and for any $\alpha \in \Lambda^1$ we have* $\dim T_\alpha = 1$ *and* $\dim L^0_\alpha \leq 1$, *then T is the direct sum of the family of its minimal ideals, each one being a simple split Lie triple system with a symmetric root system and having all its nonzero roots connected.*

Proof. By Corollary 3.1, $T = \bigoplus_{[\alpha]\in\Lambda^1/\sim} I_{[\alpha]}$ is the direct sum of the ideals

$$I_{[\alpha_0]} = T_{0,[\alpha_0]} \oplus V_{[\alpha_0]} = \sum_{\substack{\alpha+\beta+\gamma=0 \\ \alpha,\beta,\gamma\in[\alpha_0]\cup\{0\}}} [T_\alpha, T_\beta, T_\gamma] \oplus \left(\bigoplus_{\alpha\in[\alpha_0]} T_\alpha \right). \tag{10}$$

Let us show that any of the Lie triple systems $I_{[\alpha_0]}$ is split with respect to the MASA

$$H^0_{[\alpha_0]} := \sum_{\substack{\alpha+\beta+\gamma=0 \\ \alpha'+\beta'+\gamma'=0 \\ \alpha,\beta,\gamma,\alpha',\beta',\gamma'\in[\alpha_0]\cup\{0\}}} [[T_\alpha, T_\beta, T_\gamma], [T_{\alpha'}, T_{\beta'}, T_{\gamma'}]] + \sum_{\delta\in[\alpha_0]} [T_\delta, T_{-\delta}] \tag{11}$$

of

$$L^0_{[\alpha_0]} := [I_{[\alpha_0]}, I_{[\alpha_0]}] = H^0_{[\alpha_0]} + \sum_{\alpha\in[\alpha_0]} [T_{0,[\alpha_0]}, T_\alpha] + \sum_{\alpha,\beta\in[\alpha_0];\alpha+\beta\neq 0} [T_\alpha, T_\beta]. \tag{12}$$

Observing eqs (1) and (7), let us write

$$H^0 = \sum_{\substack{\alpha+\beta+\gamma=0 \\ \alpha'+\beta'+\gamma'=0 \\ \alpha,\beta,\gamma,\alpha',\beta',\gamma'\in\Lambda^1\cup\{0\}}} [[T_\alpha, T_\beta, T_\gamma], [T_{\alpha'}, T_{\beta'}, T_{\gamma'}]] + \sum_{\delta\in\Lambda^1} [T_\delta, T_{-\delta}]. \tag{13}$$

Let us consider eq. (13). If $\gamma = 0$ or $\gamma' = 0$, then $[[T_\alpha, T_\beta, T_\gamma], [T_{\alpha'}, T_{\beta'}, T_{\gamma'}]] = 0$ by eq. (2). If $\gamma \neq 0$ and $\gamma' \neq 0$, Lemma 3.3 implies

$$[[T_\alpha, T_\beta, T_\gamma], [T_{\alpha'}, T_{\beta'}, T_{\gamma'}]] = 0$$

in the cases $\alpha \notin [\gamma]\cup\{0\}$ or $\beta \notin [\gamma]\cup\{0\}$ or $\alpha' \notin [\gamma']\cup\{0\}$ or $\beta' \notin [\gamma']\cup\{0\}$. Finally, in the case $\alpha, \beta \in [\gamma]\cup\{0\}$ and $\alpha', \beta' \in [\gamma']\cup\{0\}$, eq. (5) shows $[[T_\alpha, T_\beta, T_\gamma], [T_{\alpha'}, T_{\beta'}, T_{\gamma'}]] \subset [I_{[\gamma]}, I_{[\gamma']}] = 0$ if $\gamma' \notin [\gamma]$. From here, the only possible nonzero summands in

$$\sum_{\substack{\alpha+\beta+\gamma=0 \\ \alpha'+\beta'+\gamma'=0 \\ \alpha,\beta,\gamma,\alpha',\beta',\gamma'\in\Lambda^1\cup\{0\}}} [[T_\alpha, T_\beta, T_\gamma], [T_{\alpha'}, T_{\beta'}, T_{\gamma'}]]$$

are of the form $[[T_\alpha, T_\beta, T_\gamma], [T_{\alpha'}, T_{\beta'}, T_{\gamma'}]]$ with $\alpha, \beta, \gamma, \alpha', \beta', \gamma' \in [\alpha_0] \cup \{0\}$ for some $\alpha_0 \in \Lambda^1$ and $\alpha+\beta+\gamma = 0, \alpha'+\beta'+\gamma' = 0$. Now, taking into account that the connection relation is of equivalence in Λ^1 and equations (11) and (13), we deduce

$$H^0 = \sum_{[\alpha]\in\Lambda^1/\sim} H^0_{[\alpha]}.$$

Moreover, this is a direct sum. Indeed, if $x \in H^0_{[\alpha]} \cap \sum_{[\beta]\in\Lambda^1/\sim\setminus[\alpha]} H^0_{[\beta]}$ then $[x, T] = 0$ as a consequence of writing $T = T_0 \oplus \left(\bigoplus_{\alpha\in\Lambda^1} T_\alpha\right)$, Lemmas 3.3 and 3.4, and the fact $[T_0, T_0, T_0] \subset [H^0, T_0] = 0$. So $x = 0$ because $x \in L^0$. Then we can write

$$H^0 = \bigoplus_{[\alpha]\in\Lambda^1/\sim} H^0_{[\alpha]}. \tag{14}$$

Also observe that $[I_{[\alpha_0]}, T, I_{[\beta_0]}] = 0$ when $[\alpha_0] \neq [\beta_0]$ (see Theorem 3.2) implies

$$[L^0_{[\alpha_0]}, L^0_{[\beta_0]}] = [[I_{[\alpha_0]}, I_{[\alpha_0]}], [I_{[\beta_0]}, I_{[\beta_0]}]] = 0.$$

From here, the abelian maximal character of $H^0_{[\alpha_0]}$ in $L^0_{[\alpha_0]}$ follows that of H^0, eqs (12), (14) and the fact $[L^0_{[\alpha_0]}, H^0_{[\beta_0]}] \subset [L^0_{[\alpha_0]}, L^0_{[\beta_0]}] = 0$ if $[\alpha_0] \neq [\beta_0]$.

Let us show that the set of nonzero roots of $I_{[\alpha_0]}$ with respect to $H^0_{[\alpha_0]}$ is

$$\Lambda^1_{[\alpha_0]} := \{\alpha|_{H^0_{[\alpha_0]}} : \alpha \in [\alpha_0]\} \subset (H^0_{[\alpha_0]})^*\setminus\{0\}. \tag{15}$$

Indeed, eq. (11) and Lemmas 3.5(3), 3.2 give us

$$\beta(H^0_{[\alpha_0]}) = 0 \tag{16}$$

for any $\beta \notin [\alpha_0]$. By eqs (14) and (16) we deduce

$$\alpha|_{H^0_{[\alpha_0]}} \neq 0 \tag{17}$$

for any $\alpha \in [\alpha_0]$, and taking into account eqs (10), (14), (16) and (17) that the root spaces of $I_{[\alpha_0]}$ are

$$(I_{[\alpha_0]})_0 = \sum_{\substack{\alpha+\beta+\gamma=0 \\ \alpha,\beta,\gamma\in[\alpha_0]\cup\{0\}}} [T_\alpha, T_\beta, T_\gamma]$$

and

$$(I_{[\alpha_0]})_{\alpha|_{H^0_{[\alpha_0]}}} = T_\alpha. \tag{18}$$

We have proved that any $I_{[\alpha_0]}$ is a split Lie triple system with respect to $H^0_{[\alpha_0]}$ with the following decomposition

$$I_{[\alpha_0]} = \left(\sum_{\substack{\alpha+\beta+\gamma=0 \\ \alpha,\beta,\gamma\in[\alpha_0]\cup\{0\}}} [T_\alpha, T_\beta, T_\gamma]\right) \oplus \left(\bigoplus_{\alpha\in[\alpha_0]} T_\alpha\right).$$

We would like to apply Theorem 4.1 to $I_{[\alpha_0]}$. First we need to check that the split Lie triple system $I_{[\alpha_0]}$ satisfies some hypothesis: We begin by observing that the above split decomposition of $I_{[\alpha_0]}$ and the fact $L^0_{[\alpha_0]} := [I_{[\alpha_0]}, I_{[\alpha_0]}]$ give us that the set of nonzero roots of $L^0_{[\alpha_0]}$, with respect to $H^0_{[\alpha_0]}$ is

$$\Lambda^0_{[\alpha_0]} = \mathfrak{A} \cup \mathfrak{B}, \tag{19}$$

where

$$\mathfrak{A} := \{(\alpha_1 + \alpha_2)|_{H^0_{[\alpha_0]}} : \alpha_1, \alpha_2 \in [\alpha_0] \text{ with } \alpha_1 + \alpha_2 \neq 0 \text{ and } [T_{\alpha_1}, T_{\alpha_2}] \neq 0\}$$

and

$$\mathfrak{B} := \{\delta|_{H^0_{[\alpha_0]}} : \delta \in [\alpha_0] \text{ and } [[T_\alpha, T_\beta, T_\gamma], T_\delta] \neq 0 \text{ with}$$
$$\alpha, \beta, \gamma \in \Lambda^1_{\alpha_0} \cup \{0\} \text{ and } \alpha + \beta + \gamma = 0\}.$$

If $\delta|_{H^0_{[\alpha_0]}} \in \mathfrak{B}$, then $\delta \in [\alpha_0]$ and eq. (17) gives $\delta|_{H^0_{[\alpha_0]}} \neq 0$. If $\delta|_{H^0_{[\alpha_0]}} \in \mathfrak{A}$, then $\delta = \alpha_1 + \alpha_2$ with $\alpha_1, \alpha_2 \in [\alpha_0]$ and $\alpha_1 + \alpha_2 \neq 0$. We know as above that $\alpha_1|_{H^0_{[\alpha_0]}} \neq 0$ and $\alpha_2|_{H^0_{[\alpha_0]}} \neq 0$. Moreover, by eq. (16), $\alpha_1(H_{[\beta]}) = \alpha_2(H_{[\beta]}) = 0$ for any $\beta \notin [\alpha_0]$. So, in case $(\alpha_1 + \alpha_2)|_{H^0_{[\alpha_0]}} = 0$ we would have by eq. (14) that $\alpha_1 + \alpha_2 = 0$, a contradiction. From here $\delta|_{H^0_{[\alpha_0]}} = (\alpha_1 + \alpha_2)|_{H^0_{[\alpha_0]}} \neq 0$ and we conclude

$$\delta|_{H^0_{[\alpha_0]}} \neq 0$$

for any $\delta \in \Lambda^0_{[\alpha_0]}$. From here, as $\dim L^0_\alpha = 1$ for any $\alpha \in \Lambda^0_{[\alpha_0]}$, we deduce that the nonzero root spaces of $L^0_{[\alpha_0]}$, with respect to $H^0_{[\alpha_0]}$ are

$$(L^0_{[\alpha_0]})_{\alpha|_{H^0_{[\alpha_0]}}} = L^0_\alpha \tag{20}$$

for any $\alpha \in \Lambda^0_{[\alpha_0]}$.

The character root-multiplicative of $I_{[\alpha_0]}$ now follows that of T and eqs. (18) and (20). The fact $\mathrm{Ann}(I_{[\alpha_0]}) = 0$ is consequence of $\mathrm{Ann}(T) = 0$, $T = \bigoplus_{[\alpha]\in\Lambda^1/\sim} I_{[\alpha]}$ and eq. (5). We obtain $[I_{[\alpha_0]}, I_{[\alpha_0]}, I_{[\alpha_0]}] = I_{[\alpha_0]}$ by taking into account $[T, T, T] = T$, $T = \bigoplus_{[\alpha]\in\Lambda^1/\sim} I_{[\alpha]}$ and eq. (5). It is clear that $I_{[\alpha_0]}$ has its root system $\Lambda^1_{[\alpha_0]}$ symmetric as consequence of eq. (18), $\Lambda^0_{[\alpha_0]}$ is also symmetric by eq. (20). From $\dim T_\alpha = 1$ and $\dim L^0_\alpha \leq 1$ for any $\alpha \in \Lambda^1$ and eqs. (18) and (20) we have $\dim(I_{[\alpha_0]})_{\alpha|_{I_{[\alpha_0]}}} = 1$ and $\dim(L^0_{[\alpha_0]})_{\alpha|_{I_{[\alpha_0]}}} \leq 1$ for any $\alpha|_{I_{[\alpha_0]}} \in \Lambda^1_{[\alpha_0]}$. Finally, $I_{[\alpha_0]}$ has all its nonzero roots connected (through roots in $\Lambda^1_{[\alpha_0]}$ and $\Lambda^0_{[\alpha_0]}$), as consequence of eqs (15) and (19) and the root multiplicativity of T. By applying Theorem 4.1 to $I_{[\alpha_0]}$ we conclude $I_{[\alpha_0]}$ is simple and the proof is complete.

Acknowledgements

The first author was supported by the PCI of the UCA 'Teoría de Lie y Teoría de Espacios de Banach', by the PAI with project numbers FQM-298, FQM-3737, FQM-2467, by the project of the Spanish Ministerio de Educación y Ciencia MTM2004-06580-C02-02 and with fondos FEDER.

References

[1] Balachandran V K, Real L^*-algebras, *Indian J. Pure Appl. Math.* **3(6)** (1972) 1224–1246

[2] Calderón A J and Forero M, On locally finite split Lie triple systems, *Comm. Algebra* **32(6)** (2004) 2443–2455

[3] Calderón A J, Split Lie algebras with symmetric root systems, *Proc. Indian Acad. Sci. (Math. Sci.)* **118(23)** (2008) 351–356

[4] Calderón A J, On split Lie triple systems, *Proc. Indian. Acad. Sci. (Math. Sci.)* **119(2)** (2009) 165–177

[5] Calderón A J, On simple split Lie triple systems, *Algbr. Represent. Theor.* **12** (2009) 401–415

[6] Calderón A J, Integrable roots in split Lie triple systems, *Acta Mathematica Sinica* **25** (2009) 1759–74

[7] Cuenca J A, García A and Martín C, Structure theory for L^*-algebras, *Math. Proc. Cambridge Philos. Soc.* **107** (1990) 361–365

[8] Faulkner J R, Dynkin diagrams for Lie triple systems, *J. Algebra* **62** (1980) 384–392

[9] Lister W G, A structure theory of Lie triple systems, *Trans. Am. Math. Soc.* **72** (1952) 217–242

[10] Neeb K-H, Integrable roots in split graded Lie algebras, *J. Algebra* **225** (2000) 534–580

[11] Schue J R, Hilbert space methods in the theory of Lie algebras, *Trans. Am. Math. Soc.* **95** (1960) 69–80

[12] Stumme N, The structure of locally finite split Lie algebras, *J. Algebra* **220** (1999) 664–693