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Segal–Bargmann transform and Paley–Wiener theorems on *M*(2)

E K NARAYANAN and SUPARNA SEN

Department of Mathematics, Indian Institute of Science, Bangalore 560 012, India
E-mail: naru@math.iisc.ernet.in; suparna@math.iisc.ernet.in

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Abstract. We study the Segal–Bargmann transform on $M(2)$. The range of this transform is characterized as a weighted Bergman space. In a similar fashion Poisson integrals are investigated. Using a Gutzmer's type formula we characterize the range as a class of functions extending holomorphically to an appropriate domain in the complexification of $M(2)$. We also prove a Paley–Wiener theorem for the inverse Fourier transform.

Keywords. Segal–Bargmann transform; Poisson integrals; Paley–Wiener theorem.

1. Introduction

Consider the following results from Euclidean Fourier analysis:

(I) A function $f \in L^2(\mathbb{R}^n)$ admits a factorization $f(x) = g * p_t(x)$ where $g \in L^2(\mathbb{R}^n)$ and $p_t(x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \mathrm{e}^{\frac{-|x|^2}{4t}}$ (the heat kernel on $\mathbb{R}^n$) if and only if f extends as an entire function to $\mathbb{C}^n$ and we have $\frac{1}{(2\pi t)^{n/2}} \int_{\mathbb{C}^n} |f(z)|^2 \mathrm{e}^{-\frac{|y|^2}{2t}} \mathrm{d}x\mathrm{d}y < \infty (z = x + iy)$. In this case we also have

$$\|g\|_2^2 = \frac{1}{(2\pi t)^{n/2}} \int_{\mathbb{C}^n} |f(z)|^2 \mathrm{e}^{-\frac{|y|^2}{2t}} \mathrm{d}x\mathrm{d}y.$$

The mapping $g \to g * p_t$ is called the Segal–Bargmann transform and the above says that the Segal–Bargmann transform is an unitary map from $L^2(\mathbb{R}^n)$ onto $\mathcal{O}(\mathbb{C}^n) \bigcap L^2(\mathbb{C}^n, \mu)$, where $\mathrm{d}\mu(z) = \frac{1}{(2\pi t)^{n/2}} \mathrm{e}^{-\frac{|y|^2}{2t}} \mathrm{d}x\mathrm{d}y$ and $\mathcal{O}(\mathbb{C}^n)$ denotes the space of entire functions on $\mathbb{C}^n$.

(II) A function $f \in L^2(\mathbb{R})$ admits a holomorphic extension to the strip $\{x + iy \colon |y| < t\}$ such that

$$\sup_{|y| \le s} \int_{\mathbb{R}} |f(x + iy)|^2 \mathrm{d}x < \infty \ \forall s < t$$

if and only if

$$\mathrm{e}^{s|\xi|} \tilde{f}(\xi) \in L^2(\mathbb{R}) \quad \forall s < t$$

where $\tilde{f}$ denotes the Fourier transform of f.

(III) An $f \in L^2(\mathbb{R}^n)$ admits an entire extension to $\mathbb{C}^n$ such that

$$|f(z)| \leq C_N(1+|z|)^{-N} \mathrm{e}^{R|\mathrm{Im}\, z|} \ \forall\, z \in \mathbb{C}^n$$

if and only if $\tilde{f} \in C_c^\infty(\mathbb{R}^n)$ and supp $\tilde{f} \subseteq \mathbb{B}(0, R)$, where $\mathbb{B}(0, R)$ is the ball of radius R centered around 0 in $\mathbb{R}^n$.

In this paper we aim to prove similar results for the non-commutative group $M(2) = \mathbb{R}^2 \ltimes SO(2)$. Some remarks are in order.

As noted above the map $g \to g * p_t$ in (I) is called the Segal–Bargmann transform. This transform has attracted a lot of attention in recent years mainly due to the work of Hall [3] where a similar result was established for an arbitrary compact Lie group K. Let q_t be the heat kernel on K and let $K_{\mathbb{C}}$ be the complexification of K. Then Hall's result (Theorem 2 in [3]) states that the map $f \to f * q_t$ is an unitary map from $L^2(K)$ onto the Hilbert space of ν-square integrable holomorphic functions on $K_{\mathbb{C}}$ for an appropriate positive K-invariant measure ν on $K_{\mathbb{C}}$. Soon after Hall's paper a similar result was proved for compact symmetric spaces by Stenzel in [9]. We also refer to [4–7] for similar results for other groups and spaces.

The second result (II) is originally due to Paley and Wiener. Let

$$\mathcal{H}_t = \{f \in L^2(\mathbb{R}),\ f \text{ has a holomorphic extension to } |\mathrm{Im}\, z| < t \text{ and}$$

$$\sup_{|y|\leq s} \int_{\mathbb{R}} |f(x+iy)|^2 \mathrm{d}x < \infty \quad \forall s < t\}.$$

Then $\bigcup_{t>0} \mathcal{H}_t$ may be viewed as the space of all analytic vectors for the regular representation of $\mathbb{R}$ on $L^2(\mathbb{R})$. This point of view was further developed by Goodman (see [1] and [2]) who studied analytic vectors for representations of Lie groups. The theorem of Paley–Wiener (II) characterizes analytic vectors for the regular representation of $\mathbb{R}$ via a condition on the Fourier transform.

The third result (III) is the classical Paley–Wiener theorem. For a long time the Paley–Wiener theorem has been looked at as a characterization of the image (under Fourier transform) of C_c^∞ functions on the space you are interested in. Recently, using Gutzmer's formula, Thangavelu [11] has proved a Paley–Wiener type result for the inverse Fourier transform (see [8] also for a similar result).

The plan of this paper is as follows: In the remaining of this section we recall the representation theory and Plancherel theorem of $M(2)$ and we prove the unitarity of the Segal–Bargmann transform. In the next section, we study generalized Segal–Bargmann transform and prove an analogue of Theorems 8 and 10 in [3]. Section 3 is devoted to a study of Poisson integrals on $M(2)$. This section is modeled after the work of Goodman [1] and [2]. In the final section, we establish a Paley–Wiener type result for the inverse Fourier transform on $M(2)$.

The rigid motion group $M(2)$ is the semi-direct product of $\mathbb{R}^2$ with $SO(2)$ (which will be identified with the circle group S^1) with the group law

$$(x, \mathrm{e}^{i\alpha}) \cdot (y, \mathrm{e}^{i\beta}) = (x + \mathrm{e}^{i\alpha} y, \mathrm{e}^{i(\alpha+\beta)}) \quad \text{where } x, y \in \mathbb{R}^2; \mathrm{e}^{i\alpha}, \mathrm{e}^{i\beta} \in S^1.$$

This group may be identified with a matrix subgroup of $GL(2, \mathbb{C})$ via the map

$$(x, \mathrm{e}^{i\alpha}) \to \begin{pmatrix} \mathrm{e}^{i\alpha} & x \\ 0 & 1 \end{pmatrix}.$$

Unitary irreducible representations of $M(2)$ are completely described by Mackey's theory of induced representations. For any $\xi \in \mathbb{R}^2$ and $g \in M(2)$, we define U_g^ξ as follows:

$$U_g^\xi F(s) = \mathrm{e}^{i\langle x, s\xi\rangle} F(r^{-1}s)$$

for $g = (x, r)$ and $F \in L^2(S^1)$.

It is known that U_ξ is equivalent to $U_{\xi'}$ iff $|\xi| = |\xi'|$. The above collection gives all the unitary irreducible representations of $M(2)$ sufficient for the Plancherel theorem to be true. The Plancherel theorem (see Theorem 4.2 of [10]) reads

$$\int_{M(2)} |f(g)|^2 \mathrm{d}g = \int_{\mathbb{R}^2} \|\hat{f}(\xi)\|_{\mathrm{HS}}^2 \mathrm{d}\xi$$

where $\hat{f}(\xi)$ is the 'group Fourier transform' defined as an operator from $L^2(S^1)$ to $L^2(S^1)$ by

$$\hat{f}(\xi) = \int_{M(2)} f(g) U_g^\xi \mathrm{d}g.$$

Moreover, the group Fourier transform $\hat{f}(\xi)$, for $\xi \in \mathbb{R}^2$ of $f \in L^1(M(2))$ is an integral operator with the kernel $k_f(\xi, \mathrm{e}^{i\alpha}, \mathrm{e}^{i\beta})$ where

$$k_f(\xi, \mathrm{e}^{i\alpha}, \mathrm{e}^{i\beta}) = \tilde{f}(\mathrm{e}^{i\beta}\xi, \mathrm{e}^{i(\beta-\alpha)}),$$

and $\tilde{f}$ is the Euclidean Fourier transform of f in the $\mathbb{R}^2$-variable.

The Lie algebra of $M(2)$ is given by $\left\{\left(\begin{smallmatrix} i\alpha & x \\ 0 & 0\end{smallmatrix}\right) : (x, \mathrm{e}^{i\alpha}) \in M(2)\right\}$. Let

$$X_1 = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}.$$

Then it is easy to see that $\{X_1, X_2, X_3\}$ forms a basis for the Lie algebra of $M(2)$. The 'Laplacian' $\Delta_{M(2)} = \Delta$ is defined by

$$\Delta = -(X_1^2 + X_2^2 + X_3^2).$$

A simple computation shows that $\Delta = -\Delta_{\mathbb{R}^2} - \frac{\partial^2}{\partial\alpha^2}$ where $\Delta_{\mathbb{R}^2}$ is the Laplacian on $\mathbb{R}^2$ given by $\Delta_{\mathbb{R}^2} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. Since $\Delta_{\mathbb{R}^2}$ and $\frac{\partial^2}{\partial\alpha^2}$ commute, it follows that the heat kernel ψ_t associated to $\Delta_{M(2)}$ is given by the product of the heat kernels p_t on $\mathbb{R}^2$ and q_t on $SO(2)$. In other words,

$$\psi_t(x, \mathrm{e}^{i\alpha}) = p_t(x) q_t(\mathrm{e}^{i\alpha}) = \frac{1}{4\pi t} \mathrm{e}^{\frac{-|x|^2}{4t}} \sum_{n\in\mathbb{Z}} \mathrm{e}^{-n^2 t} \mathrm{e}^{in\alpha}.$$

Let $f \in L^2(M(2))$. Expanding f in the $SO(2)$ variable we obtain

$$f(x, \mathrm{e}^{i\alpha}) = \sum_{m\in\mathbb{Z}} f_m(x) \mathrm{e}^{im\alpha},$$

where $f_m(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x, \mathrm{e}^{i\alpha})\mathrm{e}^{-im\alpha}\mathrm{d}\alpha$ and the convergence is understood in the L^2-sense. Since p_t is radial (as a function on $\mathbb{R}^2$) a simple computation shows that

$$f * \psi_t(x, \mathrm{e}^{i\alpha}) = \sum_{m\in\mathbb{Z}} f_m * p_t(x)\mathrm{e}^{-m^2 t}\mathrm{e}^{im\alpha}.$$

Let $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ and $\mathcal{H}(\mathbb{C}^2 \times \mathbb{C}^*)$ be the Hilbert space of holomorphic functions on $\mathbb{C}^2 \times \mathbb{C}^*$ which are square integrable with respect to $\mu \bigotimes \nu(z, w)$ where

$$\mathrm{d}\mu(z) = \frac{1}{2\pi t}\mathrm{e}^{-\frac{|y|^2}{2t}}\mathrm{d}x\mathrm{d}y \text{ on } \mathbb{C}^2$$

and

$$\mathrm{d}\nu(w) = \frac{1}{2\pi}\frac{1}{\sqrt{2\pi t}}\frac{\mathrm{e}^{-\frac{(\ln|w|)^2}{2t}}}{|w|^2}\mathrm{d}w \text{ on } \mathbb{C}^*.$$

Using the Segal–Bargmann result for $\mathbb{R}^2$ and S^1 we can easily prove the following theorem:

Theorem 1.1. *If $f \in L^2(M(2))$, then $f * \psi_t$ extends holomorphically to $\mathbb{C}^2 \times \mathbb{C}^*$. Moreover, the map $f \to f * \psi_t$ is a unitary map from $L^2(M(2))$ onto $\mathcal{H}(\mathbb{C}^2 \times \mathbb{C}^*)$.*

2. Generalizations of Segal–Bargmann transform

In [3], Hall had proved the following generalizations of the Segal–Bargmann transform for $\mathbb{R}$ and compact Lie group:

Theorem 2.1.

(I) *Let μ be any measurable function on $\mathbb{R}^n$ such that*

- *μ is strictly positive and locally bounded away from zero,*
- *$\forall\, x \in \mathbb{R}^n,\ \sigma(x) = \int_{\mathbb{R}^n} \mathrm{e}^{2x\cdot y}\mu(y)\mathrm{d}y < \infty$.*

Define, for $z \in \mathbb{C}^n$,

$$\psi(z) = \int_{\mathbb{R}^n} \frac{\mathrm{e}^{ia(y)}}{\sqrt{\sigma(y)}}\mathrm{e}^{-iy\cdot z}\mathrm{d}y,$$

where a is a real-valued measurable function on $\mathbb{R}^n$. Then the mapping $C_\psi\colon L^2(\mathbb{R}^n) \to \mathcal{O}(\mathbb{C}^n)$ defined by

$$C_\psi(z) = \int_{\mathbb{R}^n} f(x)\psi(z - x)\mathrm{d}x$$

is an isometric isomorphism of $L^2(\mathbb{R}^n)$ onto $\mathcal{O}(\mathbb{C}^n) \bigcap L^2(\mathbb{C}^n, \mathrm{d}x\mu(y)\mathrm{d}y)$.
(II) *Let K be a compact Lie group and G be its complexification. Let ν be a measure on G such that*

- *ν is bi-K-invariant,*
- *ν is given by a positive density which is locally bounded away from zero,*

- *For each irreducible representation π of K, analytically continued to G,*

$$\delta(\pi) = \frac{1}{\dim V_\pi} \int_G \|\pi(g^{-1})\|^2 \mathrm{d}\nu(g) < \infty.$$

Define $\chi(g) = \sum_{\pi \in \hat{K}} \frac{\dim V_\pi}{\sqrt{\delta(\pi)}} \mathrm{Tr}(\pi(g^{-1})U_\pi)$ *where* $g \in G$ *and* U_π*'s are arbitrary unitary matrices.*

Then the mapping

$$C_\chi f(g) = \int_K f(k)\chi(k^{-1}g)\mathrm{d}k$$

is an isometric isomorphism of $L^2(K)$ *onto*

$$\mathcal{O}(G) \bigcap L^2(G, \mathrm{d}\nu(w)).$$

In this section, we shall prove an analogue of the above theorem for $M(2)$.

Let μ be any radial real-valued function on $\mathbb{R}^2$ such that it satisfies the conditions of Theorem 2.1(I). Define, for $z \in \mathbb{C}^2$

$$\psi(z) = \int_{\mathbb{R}^2} \frac{\mathrm{e}^{ia(y)}}{\sqrt{\sigma(y)}} \mathrm{e}^{-iy.z} \mathrm{d}y,$$

where a is a real-valued measurable function on $\mathbb{R}^2$. Next, let ν be a measure on $\mathbb{C}^*$ such that

- ν is S^1-invariant,
- ν is given by a positive density which is locally bounded away from zero,
- $\forall\, n \in \mathbb{Z}$, $\delta(n) = \int_{\mathbb{C}^*} |w|^{2n} \mathrm{d}\nu(w) < \infty$.

Define $\chi(w) = \sum_{n \in \mathbb{Z}} \frac{c_n}{\sqrt{\delta(n)}} w^n$ for $w \in \mathbb{C}^*$ and $c_n \in \mathbb{C}$ such that $|c_n| = 1$. Also define $\phi(z, w) = \psi(z)\chi(w)$ for $z \in \mathbb{C}^2,\ w \in \mathbb{C}^*$. It is easy to see that $\phi(z, w)$ is a holomorphic function on $\mathbb{C}^2 \times \mathbb{C}^*$. We have the following Paley–Wiener type theorem.

Theorem 2.2. *The mapping*

$$C_\phi f(z, w) = \int_{M(2)} f(\xi, \mathrm{e}^{i\alpha})\phi((\xi, \mathrm{e}^{i\alpha})^{-1}(z, w))\mathrm{d}\xi \mathrm{d}\alpha$$

is an isometric isomorphism of $L^2(M(2))$ *onto*

$$\mathcal{O}(\mathbb{C}^2 \times \mathbb{C}^*) \bigcap L^2(\mathbb{C}^2 \times \mathbb{C}^*, \mu(y)\mathrm{d}x\mathrm{d}y\mathrm{d}\nu(w)).$$

Proof. Let $f \in L^2(M(2))$ and

$$f(x, \mathrm{e}^{i\alpha}) = \sum_{m \in \mathbb{Z}} f_m(x)\mathrm{e}^{im\alpha} \tag{2.1}$$

where $f_m(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x, \mathrm{e}^{i\alpha})\mathrm{e}^{-im\alpha}\mathrm{d}\alpha$. Since the function $\phi(x, \mathrm{e}^{i\alpha})$, for $(x, \mathrm{e}^{i\alpha}) \in M(2)$ is radial in the $\mathbb{R}^2$ variable x, a simple computation shows that the Fourier series of $f * \phi(x, \mathrm{e}^{i\alpha})$ is given by

$$f * \phi(x, \mathrm{e}^{i\alpha}) = \sum_{m \in \mathbb{Z}} f_m * \psi(x) \frac{c_m}{\sqrt{\delta(m)}} \mathrm{e}^{im\alpha}.$$

Now, notice that $f_m \in L^2(\mathbb{R}^2),\ \forall m \in \mathbb{Z}$ and $f_m * \psi$ is a holomorphic function on $\mathbb{C}^2$. Moreover, by Theorem 8 of [3] we have

$$\int_{\mathbb{C}^2} |f_m * \psi(z)|^2 \mu(y) \mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^2} |f_m(x)|^2 \mathrm{d}x. \tag{2.2}$$

Naturally, the analytic continuation of $f * \phi(x, \mathrm{e}^{i\alpha})$ to $\mathbb{C}^2 \times \mathbb{C}^*$ is given by

$$f * \phi(z, w) = \sum_{m\in\mathbb{Z}} f_m * \psi(z) \frac{c_m}{\sqrt{\delta(m)}} w^m. \tag{2.3}$$

We show that the series in (2.3) converges uniformly on compact sets in $\mathbb{C}^2 \times \mathbb{C}^*$ proving the holomorphicity. Let K be a compact set in $\mathbb{C}^2 \times \mathbb{C}^*$. For $(z, w) \in K$, we have

$$\left|\sum_{m\in\mathbb{Z}} f_m * \psi(z) \frac{c_m}{\sqrt{\delta(m)}} w^m\right| \leq \sum_{m\in\mathbb{Z}} |f_m * \psi(z)| \frac{|w|^m}{\sqrt{\delta(m)}}. \tag{2.4}$$

By Fourier inversion (see also Theorem 8 in [3])

$$f_m * \psi(z) = \int_{\mathbb{R}^2} \widetilde{f_m}(\xi) \frac{\mathrm{e}^{ia(\xi)}}{\sqrt{\sigma(\xi)}} \mathrm{e}^{-i\xi(x+iy)} \mathrm{d}\xi,$$

where $z = x + iy \in \mathbb{C}^2$ and $\widetilde{f_m}$ is the Fourier transform of f_m. Hence, if z varies in a compact subset of $\mathbb{C}^2$, we have

$$\begin{aligned} |f_m * \psi(z)| &\leq \|f_m\|_2 \left(\int_{\mathbb{R}^2} \frac{\mathrm{e}^{2\xi\cdot y}}{\sigma(\xi)} \mathrm{d}\xi\right)^{\frac{1}{2}} \\ &\leq C\|f_m\|_2. \end{aligned}$$

Using the above in (2.4) and assuming $|w| \leq R$ (as w varies in a compact set in $\mathbb{C}^*$) we have

$$\left|\sum_{m\in\mathbb{Z}} f_m * \psi(z) \frac{c_m}{\sqrt{\delta(m)}} w^m\right| \leq C \sum_{m\in\mathbb{Z}} \|f_m\|_2 \frac{R^m}{\sqrt{\delta(m)}}.$$

Applying Cauchy–Schwarz inequality to the above, noting that

$$\sum_{m\in\mathbb{Z}} \|f_m\|_2^2 = \|f\|_2^2 \quad \text{and} \quad \sum_{m\in\mathbb{Z}} \frac{R^{2m}}{\delta(m)} < \infty$$

we prove the above claim. Applying Theorem 10 in [3] for S^1, we obtain

$$\int_{\mathbb{C}^*} |f * \phi(z, w)|^2 \mathrm{d}\nu(w) = \sum_{m\in\mathbb{Z}} |f_m * \psi(z)|^2.$$

Integrating the above against $\mu(y)\mathrm{d}x\mathrm{d}y$ on $\mathbb{C}^2$ and using (2.2) we obtain that

$$\int_{\mathbb{C}^2} \int_{\mathbb{C}^*} |f * \phi(z, w)|^2 \mu(y) \mathrm{d}x\mathrm{d}y\mathrm{d}\nu(w) = \|f\|_2^2.$$

To prove that the map C_ϕ is surjective it suffices to prove that the range of C_ϕ is dense in $\mathcal{O}(\mathbb{C}^2 \times \mathbb{C}^*) \cap L^2(\mathbb{C}^2 \times \mathbb{C}^*, \mu(y)\mathrm{d}x\mathrm{d}y\mathrm{d}\nu(w))$. For this, consider functions of the form $f(x, \mathrm{e}^{i\alpha}) = g(x)\mathrm{e}^{im\alpha} \in L^2(M(2))$ where $g \in L^2(\mathbb{R}^2)$. Then a simple computation shows that

$$f * \phi(z, w) = g * \psi(z) w^m \text{ for } (z, w) \in \mathbb{C}^2 \times \mathbb{C}^*.$$

Suppose $F \in \mathcal{O}(\mathbb{C}^2 \times \mathbb{C}^*) \cap L^2(\mathbb{C}^2 \times \mathbb{C}^*, \mu(y)\mathrm{d}x\mathrm{d}y\mathrm{d}\nu(w))$ be such that

$$\int_{\mathbb{C}^2 \times \mathbb{C}^*} F(z, w)\overline{g * \psi(z)}\bar{w}^m \mu(y)\mathrm{d}x\mathrm{d}y\mathrm{d}\nu(w) = 0, \tag{2.5}$$

for all $g \in L^2(\mathbb{R}^2)$ and for all $m \in \mathbb{Z}$. From (2.5) we have

$$\int_{\mathbb{C}^*} \left(\int_{\mathbb{C}^2} F(z, w)\overline{g * \psi(z)}\mathrm{d}\mu(z) \right) \bar{w}^m \mathrm{d}\nu(w) = 0,$$

which by Theorem 10 of [3] implies that

$$\int_{\mathbb{C}^2} F(z, w)\overline{g * \psi(z)}\mathrm{d}\mu(z) = 0.$$

Finally, an application of Theorem 8 of [3] shows that $F \equiv 0$. Hence the proof. □

3. Poisson integrals and Paley–Wiener type theorems

In this section we study the Poisson integrals on $M(2)$. We also find conditions on the 'group Fourier transform' of a function so that it extends holomorphically to an appropriate domain in the complexification of the group. We start with the following Gutzmer-type lemma:

Lemma 3.1. *Let $f \in L^2(M(2))$ extend holomorphically to the domain*

$$\left\{ (z, w) \in \mathbb{C}^2 \times \mathbb{C}^* \colon |\mathrm{Im}\, z| < t, \frac{1}{R} < |w| < R \right\}$$

and

$$\sup_{\left\{|y|<s, \frac{1}{N}<|w|<N\right\}} \int_{M(2)} |f(x + iy, |w|\mathrm{e}^{i\theta})|^2 \mathrm{d}x\mathrm{d}\theta < \infty$$

for all $s < t$ and $N < R$. Then

$$\int_{M(2)} |f(x + iy, |w|\mathrm{e}^{i\theta})|^2 \mathrm{d}x\mathrm{d}\theta = \sum_{n \in \mathbb{Z}} \left(\int_{\mathbb{R}^2} |\widetilde{f}_n(\xi)|^2 \mathrm{e}^{-2\xi \cdot y} \mathrm{d}\xi \right) |w|^{2n}$$

provided $|y| < t$ and $\frac{1}{R} < |w| < R$. Conversely, if

$$\sup_{\left\{|y|<s, \frac{1}{N}<|w|<N\right\}} \sum_{n \in \mathbb{Z}} \left(\int_{\mathbb{R}^2} |\widetilde{f}_n(\xi)|^2 \mathrm{e}^{-2\xi \cdot y} \mathrm{d}\xi \right) |w|^{2n} < \infty \, \forall s < t \text{ and } N < R$$

then f extends holomorphically to the domain

$$\left\{(z, w) \in \mathbb{C}^2 \times \mathbb{C}^*: |\mathrm{Im}\, z| < t, \frac{1}{R} < |w| < R\right\}$$

and

$$\sup_{\left\{|y|<s, \frac{1}{N}<|w|<N\right\}} \int_{M(2)} |f(x+iy, |w|\mathrm{e}^{i\theta})|^2 \mathrm{d}x\mathrm{d}\theta < \infty\ \forall\, s < t \text{ and } N < R.$$

Proof. Notice that $f_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x, \mathrm{e}^{i\alpha})\mathrm{e}^{-in\alpha}\mathrm{d}\alpha$. It follows that $f_n(x)$ has a holomorphic extension to $\{z \in \mathbb{C}^2: |\mathrm{Im}\, z| < t\}$ and

$$\sup_{|y|<s} \int_{\mathbb{R}^2} |f_n(x+iy)|^2 \mathrm{d}x < \infty\ \forall\, s < t.$$

Consequently,

$$\int_{\mathbb{R}^2} |f_n(x+iy)|^2 \mathrm{d}x = \int_{\mathbb{R}^2} |\widetilde{f_n}(\xi)|^2 \mathrm{e}^{-2\xi\cdot y}\mathrm{d}\xi \text{ for } |y| < s\ \forall\, s < t.$$

Now, for each fixed $z \in \mathbb{C}^2$ with $|\mathrm{Im}\, z| < s$ the function $w \to f(z, w)$ is holomorphic in the annulus $\{w \in \mathbb{C}^*: \frac{1}{N} < |w| < N\}$ for every $s < t$ and $N < R$ and so admits a Laurent series expansion

$$f(z, w) = \sum_{m\in\mathbb{Z}} a_m(z) w^m.$$

It follows that $a_m(z) = f_m(z)\ \forall m \in \mathbb{Z}$. The first part of the lemma is proved now by appealing to the Plancherel theorem on S^1 and $\mathbb{R}^2$. Converse can also be proved similarly.

Recall from the Introduction that the Laplacian Δ on $M(2)$ is given by $\Delta = -\Delta_{\mathbb{R}^2} - \frac{\partial^2}{\partial\alpha^2}$. If $f \in L^2(M(2))$ it is easy to see that

$$\mathrm{e}^{-t\Delta^{\frac{1}{2}}} f(x, \mathrm{e}^{i\alpha}) = \sum_{m\in\mathbb{Z}} \left(\int_{\mathbb{R}^2} \widetilde{f_m}(\xi)\mathrm{e}^{-t(|\xi|^2+m^2)^{\frac{1}{2}}} \mathrm{e}^{ix\cdot\xi}\mathrm{d}\xi\right) \mathrm{e}^{im\alpha}.$$

We have the following (almost) characterization of the Poisson integrals. Let Ω_s denote the domain in $\mathbb{C}^2 \times \mathbb{C}^*$ defined by

$$\Omega_s = \{(z, w): |\mathrm{Im}\, z| < s, \mathrm{e}^{-s} < |w| < \mathrm{e}^s\}.$$

Theorem 3.2. *Let $f \in L^2(M(2))$. Then $g = \mathrm{e}^{-t\Delta^{\frac{1}{2}}} f$ extends to a holomorphic function on the domain $\Omega_{\frac{t}{\sqrt{2}}}$ and*

$$\sup_{\left\{|y|<\frac{t}{\sqrt{2}}, \mathrm{e}^{-\frac{t}{\sqrt{2}}}<|w|<\mathrm{e}^{\frac{t}{\sqrt{2}}}\right\}} \int_{M(2)} |g(x+iy, |w|\mathrm{e}^{i\alpha})|^2 \mathrm{d}x\mathrm{d}\alpha < \infty.$$

Conversely, let g be a holomorphic function on Ω_t and

$$\sup_{\{|y|<s, \mathrm{e}^{-s}<|w|<\mathrm{e}^s\}} \int_{M(2)} |g(x+iy, |w|\mathrm{e}^{i\alpha})|^2 \mathrm{d}x\mathrm{d}\alpha < \infty \text{ for } s < t.$$

Then $\forall s < t$, $\exists f \in L^2(M(2))$ such that $\mathrm{e}^{-s\Delta^{\frac{1}{2}}} f = g$.

Proof. We know that, if $f \in L^2(M(2))$ then

$$g(x, \mathrm{e}^{i\alpha}) = \mathrm{e}^{-t\Delta^{\frac{1}{2}}} f(x, \mathrm{e}^{i\alpha}) = \sum_{m\in\mathbb{Z}} \left(\int_{\mathbb{R}^2} \widetilde{f_m}(\xi) \mathrm{e}^{-t(|\xi|^2+m^2)^{\frac{1}{2}}} \mathrm{e}^{ix\cdot\xi} \mathrm{d}\xi \right) \mathrm{e}^{im\alpha}.$$

Also, $g(x, \mathrm{e}^{i\alpha}) = \sum_{m\in\mathbb{Z}} g_m(x)\mathrm{e}^{im\alpha}$ with $\widetilde{g_m}(\xi) = \widetilde{f_m}(\xi)\mathrm{e}^{-t(|\xi|^2+m^2)^{\frac{1}{2}}}$. If $s \leq \frac{t}{\sqrt{2}}$ it is easy to show that

$$\sup_{\{\xi\in\mathbb{R}^2, m\in\mathbb{Z}\}} \mathrm{e}^{-2t(|\xi|^2+m^2)^{\frac{1}{2}}} \mathrm{e}^{2|\xi||y|} \mathrm{e}^{2|m|s} \leq C < \infty \text{ for } |y| \leq \frac{t}{\sqrt{2}}.$$

It follows that

$$\sup_{\left\{|y|<\frac{t}{\sqrt{2}}, \mathrm{e}^{-\frac{t}{\sqrt{2}}}<|w|<\mathrm{e}^{\frac{t}{\sqrt{2}}}\right\}} \sum_{m\in\mathbb{Z}} \left(\int_{\mathbb{R}^2} |\widetilde{g_m}(\xi)|^2 \mathrm{e}^{-2\xi\cdot y} \mathrm{d}\xi \right) |w|^{2m} < \infty.$$

By the previous lemma we prove the first part of the theorem.

Conversely, let g be a holomorphic function on Ω_t and

$$\sup_{\{|y|<s, \mathrm{e}^{-s}<|w|<\mathrm{e}^s\}} \int_{M(2)} |g(x+iy, |w|\mathrm{e}^{i\alpha})|^2 \mathrm{d}x\mathrm{d}\alpha < \infty \text{ for } s < t.$$

By Lemma 3.1 we have

$$\sup_{\{|y|<s, \mathrm{e}^{-s}<|w|<\mathrm{e}^s\}} \sum_{n\in\mathbb{Z}} \left(\int_{\mathbb{R}^2} |\widetilde{g_n}(\xi)|^2 \mathrm{e}^{-2\xi\cdot y} \mathrm{d}\xi \right) |w|^{2n} < \infty \text{ for } s < t.$$

Integrating the above over $|y| = s < t$, we obtain

$$\sum_{n\in\mathbb{Z}} \left(\int_{\mathbb{R}^2} |\widetilde{g_n}(\xi)|^2 J_0(i2s|\xi|) \mathrm{d}\xi \right) |w|^{2n} < \infty,$$

where J_0 is the Bessel function of first kind. Noting that $J_0(i2s|\xi|) \sim \mathrm{e}^{2s|\xi|}$ for large $|\xi|$ we obtain

$$\sum_{n\in\mathbb{Z}} \int_{\mathbb{R}^2} |\widetilde{g_n}(\xi)|^2 \mathrm{e}^{2s|\xi|} \mathrm{e}^{2|n|s} \mathrm{d}\xi < \infty \text{ for } s < t.$$

This surely implies that

$$\sum_{n\in\mathbb{Z}} \int_{\mathbb{R}^2} |\widetilde{g_n}(\xi)|^2 \mathrm{e}^{2s(|\xi|^2+m^2)^{\frac{1}{2}}} \mathrm{d}\xi < \infty \text{ for } s < t.$$

Defining $\widetilde{f_m}(\xi)$ by $\widetilde{f_m}(\xi) = \widetilde{g_m}(\xi)\mathrm{e}^{s(|\xi|^2+m^2)^{\frac{1}{2}}}$ we obtain

$$f(x, \mathrm{e}^{i\alpha}) = \sum_{m\in\mathbb{Z}} f_m(x)\mathrm{e}^{im\alpha} \in L^2(M(2))$$

and $g = \mathrm{e}^{-s\Delta^{\frac{1}{2}}} f$.

Remark 3.3. *A similar result may be proved for the operator* $\mathrm{e}^{-t\Delta_{\mathbb{R}^2}^{\frac{1}{2}}}\mathrm{e}^{-t\Delta_{S^1}^{\frac{1}{2}}}$.

It is known that a function $f \in L^2(\mathbb{R})$ extends holomorphically to the complex plane $\mathbb{C}$ with

$$\int_{\mathbb{R}} |f(x+iy)|^2 \mathrm{d}x < \infty, \quad \forall y \in \mathbb{R}$$

if and only if

$$\int_{\mathbb{R}} \mathrm{e}^{s|\xi|}|\tilde{f}(\xi)|^2 \mathrm{d}\xi < \infty, \quad \forall s \in \mathbb{R}$$

where $\tilde{f}$ denotes the Fourier transform of f. The second condition is the same as

$$\int_{\mathbb{R}} |\mathrm{e}^{i(x+iy)\xi}|^2|\tilde{f}(\xi)|^2 \mathrm{d}\xi < \infty, \quad \forall y \in \mathbb{R}.$$

Here $\xi \mapsto \mathrm{e}^{i(x+iy)\xi}$ may be seen as the complexification of the parameters of the unitary irreducible representations $\xi \mapsto \mathrm{e}^{ix\xi}$ of $\mathbb{R}$. This point of view was further developed by Roe Goodman (see Theorem 3.1 of [2]). We shall prove an analogue of the above theorem in the case of $M(2)$.

Analytic vectors

Let π be an unitary representation of a Lie group G on a Hilbert space H. A vector $v \in H$ is called an analytic vector for π if the function $g \to \pi(g)v$ is analytic.

Recall the representations U_g^{ξ} from the Introduction. Denote by U_g^a the representations $U_g^{(a,0)}$ for $a > 0$. If $e_n(\theta) = \mathrm{e}^{in\theta} \in L^2(S^1)$, it is easy to see that e_n's are analytic vectors for these representations. For $g = (x, \mathrm{e}^{i\alpha}) \in M(2)$, we have

$$(U_g^a e_n)(\theta) = \mathrm{e}^{i\langle x, a\mathrm{e}^{i\theta}\rangle}\mathrm{e}^{in(\theta-\alpha)}.$$

This action of U_g^a on e_n can clearly be analytically continued to $\mathbb{C}^2 \times \mathbb{C}^*$ and we obtain

$$(U_{(z,w)}^a e_n)(\theta) = \mathrm{e}^{i\langle x, a\mathrm{e}^{i\theta}\rangle}\mathrm{e}^{-\langle y, a\mathrm{e}^{i\theta}\rangle} w^{-n}\mathrm{e}^{in\theta}$$

where $(z, w) \in \mathbb{C}^2 \times \mathbb{C}^*$ and $z = x + iy \in \mathbb{C}^2$.

We also note that the action of S^1 on $\mathbb{R}^2$ naturally extends to an action of $\mathbb{C}^*$ on $\mathbb{C}^2$ given by

$$w(z_1, z_2) = (z_1 \cos\zeta - z_2 \sin\zeta, z_1 \sin\zeta + z_2 \cos\zeta),$$

where $w = \mathrm{e}^{i\zeta} \in \mathbb{C}^*$ and $(z_1, z_2) \in \mathbb{C}^2$. Then we have the following theorem:

Theorem 3.4. *Let $f \in L^2(M(2))$. Then f extends holomorphically to $\mathbb{C}^2 \times \mathbb{C}^*$ with*

$$\int_{|y|=r} \int_{M(2)} |f(w^{-1}(x+iy), |w|\mathrm{e}^{i\alpha})|^2 \mathrm{d}x\mathrm{d}\alpha\mathrm{d}\sigma_r(y) < \infty$$

(*where σ_r is the normalized surface area measure on the sphere* $\{|y| = r\} \subseteq \mathbb{R}^2$) *iff*

$$\int_0^\infty \int_{|y|=r} \|U^a_{(z,w)} \hat{f}(a)\|^2_{\mathrm{HS}} \mathrm{d}\sigma_r(y) a \mathrm{d}a < \infty$$

where $z = x + iy \in \mathbb{C}^2$ and $w \in \mathbb{C}^*$. In this case we also have

$$\int_0^\infty \int_{|y|=r} \|U^a_{(z,w)} \hat{f}(a)\|^2_{\mathrm{HS}} \mathrm{d}\sigma_r(y) a \mathrm{d}a$$
$$= \int_{|y|=r} \int_{M(2)} |f(w^{-1}(x+iy), |w|\mathrm{e}^{i\alpha})|^2 \mathrm{d}x \mathrm{d}\alpha \mathrm{d}\sigma_r(y).$$

Proof. First assume that $f \in L^2(M(2))$ satisfies the transformation property

$$f(\mathrm{e}^{i\theta}x, \mathrm{e}^{i\alpha}) = \mathrm{e}^{im\theta} f(x, \mathrm{e}^{i\alpha}) \tag{3.1}$$

for some fixed $m \in \mathbb{Z}$ and $\forall\ (x, \mathrm{e}^{i\alpha}) \in M(2)$. As earlier we have

$$(\hat{f}(a)e_n)(\theta) = \widetilde{f}_n(a\mathrm{e}^{i\theta})\mathrm{e}^{in\theta}.$$

By the Hecke–Bochner identity, we have

$$\widetilde{f}_n(a\mathrm{e}^{i\theta}) = i^{-|m|}a^{|m|}(\mathcal{F}_{2+2|m|}g)(a)\mathrm{e}^{im\theta}$$

where $\mathcal{F}_{2+2|m|}(g)$ is the $2+2|m|$-dimensional Fourier transform of $g(x) = \frac{f_n(|x|)}{|x|^{|m|}}$, considered as a radial function on $\mathbb{R}^{2+2|m|}$.

Hence,

$$(\hat{f}(a)e_n)(\theta) = i^{-|m|}a^{|m|}(\mathcal{F}_{2+2|m|}g)(a)\mathrm{e}^{i(m+n)\theta}.$$

It follows that $\hat{f}(a)e_n$ is an analytic vector and we can apply $U^a_{(z,w)}$ to the above. We obtain

$$(U^a_{(z,w)}\hat{f}(a)e_n)(\theta)$$
$$= \mathrm{e}^{i\langle x, a\mathrm{e}^{i\theta}\rangle}\mathrm{e}^{-\langle y, a\mathrm{e}^{i\theta}\rangle} i^{-|m|}a^{|m|}(\mathcal{F}_{2+2|m|}g)(a)w^{-(m+n)}\mathrm{e}^{i(m+n)\theta}.$$

Thus,

$$\int_{S^1} |[U^a_{(z,w)}\hat{f}(a)e_n](\theta)|^2 \mathrm{d}\theta$$
$$= |w|^{-2(m+n)} \int_{S^1} a^{2m}|(\mathcal{F}_{2+2m}g)(a)|^2 \mathrm{e}^{-2\langle y, a\mathrm{e}^{i\theta}\rangle}\mathrm{d}\theta$$
$$= |w|^{-2(m+n)} \int_{S^1} \mathrm{e}^{-2\langle y, a\mathrm{e}^{i\theta}\rangle}|\widetilde{f}_n(a\mathrm{e}^{i\theta})|^2\mathrm{d}\theta.$$

Hence,

$$\int_0^\infty \|U^a_{(z,w)}\hat{f}(a)\|^2_{\mathrm{HS}} a \mathrm{d}a$$
$$= |w|^{-2m} \sum_{n\in\mathbb{Z}} |w|^{-2n} \left(\int_{\mathbb{R}^2} \mathrm{e}^{-2\langle y, \xi\rangle}|\widetilde{f}_n(\xi)|^2 \mathrm{d}\xi\right). \tag{3.2}$$

Notice that, if f extends holomorphically to $\mathbb{C}^2 \times \mathbb{C}^*$ we must have $f(w^{-1}z, w) = w^{-m} f(z, w)\ \forall\ (z, w) \in \mathbb{C}^2 \times \mathbb{C}^*$ because of (3.1).

In view of Lemma 3.1, the above remark and the identity (3.2), the theorem is established for functions with transformation property (3.1) and we obtain

$$\int_0^\infty \int_{|y|=r} \|U^a_{(z,w)} \hat{f}(a)\|^2_{\mathrm{HS}} \mathrm{d}\sigma_r(y) a \mathrm{d}a$$
$$= \int_{|y|=r} \int_{M(2)} |f(w^{-1}(x+iy), |w|\mathrm{e}^{i\alpha})|^2 \mathrm{d}x \mathrm{d}\alpha \mathrm{d}\sigma_r(y). \tag{3.3}$$

Next, we deal with the general case. For $f \in L^2(M(2))$ define

$$f^m(x, \mathrm{e}^{i\alpha}) = \int_{S^1} f(\mathrm{e}^{i\theta} x, \mathrm{e}^{i\alpha}) \mathrm{e}^{-im\theta} \mathrm{d}\theta.$$

Then $f^m(\mathrm{e}^{i\theta} x, \mathrm{e}^{i\alpha}) = \mathrm{e}^{im\theta} f^m(x, \mathrm{e}^{i\alpha})$ and f_m's are orthogonal on $M(2)$. Assume that f extends holomorphically to $\mathbb{C}^2 \times \mathbb{C}^*$. Then, so does f^m for all $m \in \mathbb{Z}$ and we have

$$\int_{|y|=r} \int_{M(2)} |f(w^{-1}(x+iy), |w|\mathrm{e}^{i\alpha})|^2 \mathrm{d}x \mathrm{d}\alpha \mathrm{d}\sigma_r(y)$$
$$= \sum_{m\in\mathbb{Z}} \int_{|y|=r} \int_{M(2)} |f^m(w^{-1}(x+iy), |w|\mathrm{e}^{i\alpha})|^2 \mathrm{d}x \mathrm{d}\alpha \mathrm{d}\sigma_r(y). \tag{3.4}$$

This follows from the fact that

$$\int_{|y|=r} \int_{\mathbb{R}^2} f^m(w^{-1}(x+iy), w) \overline{f^l(w^{-1}(x+iy), w)} \mathrm{d}x \mathrm{d}\sigma_r(y) = 0 \text{ if } m \neq l.$$

Applying identity (3.3) we get from (3.4)

$$\sum_{m\in\mathbb{Z}} \int_0^\infty \int_{|y|=r} \|U^a_{(z,w)} \widehat{f^m}(a)\|^2_{\mathrm{HS}} a \mathrm{d}a \mathrm{d}\sigma_r(y) < \infty. \tag{3.5}$$

Now, let $\langle T, S\rangle_{\mathrm{HS}} = \sum_{n\in\mathbb{Z}} \langle Te_n, Se_n\rangle$ denote the inner product in the space of Hilbert–Schmidt operators on $L^2(S^1)$. Then we notice that

$$\int_{|y|=r} \langle U^a_{(z,w)} \widehat{f^m}(a), U^a_{(z,w)} \widehat{f^l}(a)\rangle_{\mathrm{HS}} \mathrm{d}\sigma_r(y)$$
$$= \delta_{ml} \sum_{n\in\mathbb{Z}} (-1)^{-l} \left(\frac{a}{i}\right)^{m+l} (\mathcal{F}_{2+2|m|} g^m)(a)$$
$$\times \overline{(\mathcal{F}_{2+2|l|} g^l)(a)} w^{-(m+n)} (\overline{w})^{-(l+n)} J_0(2ira). \tag{3.6}$$

Hence (3.5) implies that

$$\int_0^\infty \int_{|y|=r} \|U^a_{(z,w)} \hat{f}(a)\|^2_{\mathrm{HS}} a \mathrm{d}a \mathrm{d}\sigma_r(y) < \infty.$$

To prove the converse, we first show that f has a holomorphic extension to whole of $\mathbb{C}^2 \times \mathbb{C}^*$. Recall that we have

$$(\hat{f}(a)e_n)(\theta) = \widetilde{f}_n(a\mathrm{e}^{i\theta})\mathrm{e}^{in\theta}.$$

Expanding $\widetilde{f}_n(a\mathrm{e}^{i\theta})$ into Fourier series we have

$$\widetilde{f}_n(a\mathrm{e}^{i\theta}) = \sum_{k\in\mathbb{Z}} C_{a,n}(k)\mathrm{e}^{ik\theta}.$$

Hence $(U^a_{(z,w)}\hat{f}(a)e_n)(\theta)$ is given by

$$\sum_{k\in\mathbb{Z}} C_{a,n}(k)\mathrm{e}^{i\langle x,a\mathrm{e}^{i\theta}\rangle}\mathrm{e}^{-\langle y,a\mathrm{e}^{i\theta}\rangle}w^{-(k+n)}\mathrm{e}^{i(k+n)\theta}.$$

Thus

$$\int_{|y|=r}\int_{S^1} |[U^a_{(z,w)}\hat{f}(a)e_n](\theta)|^2\mathrm{d}\theta\mathrm{d}\sigma_r(y) = J_0(2ira)\sum_{k\in\mathbb{Z}} |C_{a,n}(k)|^2|w|^{-(k+n)}.$$

Notice that $J_0(2ira) \sim \mathrm{e}^{2ra}$ for large a and $\sum_{k\in\mathbb{Z}} |C_{a,n}(k)|^2 = \int_{S^1} |\widetilde{f}_n(a\mathrm{e}^{i\theta})|^2\mathrm{d}\theta$. If $\mathrm{e}^{-r} < |w| < \mathrm{e}^r$, we obtain

$$\sum_{n\in\mathbb{Z}}\left(\int_0^\infty\int_{S^1} |\widetilde{f}_n(a\mathrm{e}^{i\theta})|^2\mathrm{e}^{2ra}\mathrm{d}\theta a\mathrm{d}a\right)|w|^{2n} < \infty,$$

which implies

$$\sum_{n\in\mathbb{Z}}\left(\int_{\mathbb{R}^2} |\widetilde{f}_n(\xi)|^2\mathrm{e}^{2r|\xi|}\mathrm{d}\xi\right)|w|^{2n} < \infty.$$

Since this is true for all $r > 0$ and $w \in \mathbb{C}^*$, by Lemma 3.1 f extends holomorphically to $\mathbb{C}^2 \times \mathbb{C}^*$. It follows that $f^m(x, \mathrm{e}^{i\alpha})$ defined by

$$f^m(x, \mathrm{e}^{i\alpha}) = \int_{S^1} f(\mathrm{e}^{i\theta}x, \mathrm{e}^{i\alpha})\mathrm{e}^{-im\theta}\mathrm{d}\theta$$

also extends holomorphically to $\mathbb{C}^2 \times \mathbb{C}^*$ and

$$\int_{M(2)} |f^m((x+iy), |w|\mathrm{e}^{i\alpha})|^2\mathrm{d}x\mathrm{d}\alpha < \infty.$$

Now the proof can be completed using the identity (3.3), orthogonality of $U^a_{(z,w)}\widehat{f^m}(a)$ (see (3.6)) and (3.4).

4. A Paley–Wiener theorem for the inverse Fourier transform

Recall from the Introduction that the 'group Fourier transform' $\hat{f}(\xi)$, for $\xi \in \mathbb{R}^2$ of $f \in L^1(M(2))$ is an integral operator with the kernel $k_f(\xi, \mathrm{e}^{i\alpha}, \mathrm{e}^{i\beta})$ where $k_f(\xi, \mathrm{e}^{i\alpha}, \mathrm{e}^{i\beta}) = \tilde{f}(\mathrm{e}^{i\beta}\xi, \mathrm{e}^{i(\beta-\alpha)})$, $\tilde{f}$ being the Euclidean Fourier transform of f in the $\mathbb{R}^2$-variable. We have the following Paley–Wiener theorem for the inverse Fourier transform:

Theorem 4.1. *Let $f \in L^1(M(2))$ be such that $\hat{f}(\xi) \equiv 0$ for all $|\xi| > R$ and the kernel k_f of $\hat{f}(\xi)$ is smooth on $\mathbb{R}^2 \times S^1 \times S^1$. Then $x \to f(x, \mathrm{e}^{i\alpha})$ extends to an entire function of exponential type R such that*

$$\sup_{\mathrm{e}^{i\alpha} \in S^1} |z^m f(z, \mathrm{e}^{i\alpha})| \leq c_m \mathrm{e}^{R|\mathrm{Im}\, z|}, \quad \forall\, z \in \mathbb{C}^2, \ \forall\, m \in \mathbb{N}^2. \tag{4.1}$$

Conversely, if f extends to an entire function on $\mathbb{C}^2$ in the first variable and satisfies (4.1) *then $\hat{f}(\xi) = 0$ for $|\xi| > R$ and k_f is smooth on $\mathbb{R}^2$.*

Proof. We have

$$(\hat{f}(\xi)F)(\mathrm{e}^{i\alpha}) = \int_{S^1} k_f(\xi, \mathrm{e}^{i\alpha}, \mathrm{e}^{i\beta}) F(\mathrm{e}^{i\beta}) \mathrm{d}\beta, \quad \text{for } F \in L^2(S^1)$$

where $k_f(\xi, \mathrm{e}^{i\alpha}, \mathrm{e}^{i\beta}) = \tilde{f}(\mathrm{e}^{i\beta}\xi, \mathrm{e}^{i(\beta-\alpha)})$.

Assume that $\hat{f}(\xi) \equiv 0$ for all $|\xi| > R$. Since k_f is smooth we have $\tilde{f}(\cdot, \mathrm{e}^{i\alpha}) \in C_c^\infty(\mathbb{R}^2)$. By the Paley–Wiener theorem on $\mathbb{R}^2$ we obtain that $f(\cdot, \mathrm{e}^{i\alpha})$ extends to an entire function on $\mathbb{C}^2$ of exponential type. Moreover, if $m \in \mathbb{N}^2$,

$$z^m f(z, \mathrm{e}^{i\alpha}) = \int_{|\xi| \leq R} \frac{\partial^m \tilde{f}}{\partial \xi^m}(\xi, \mathrm{e}^{i\alpha}) \mathrm{e}^{i\xi \cdot z} \mathrm{d}\xi.$$

It follows that

$$\sup_{\mathrm{e}^{i\alpha} \in S^1} |z^m f(z, \mathrm{e}^{i\alpha})| \leq \sup_{\mathrm{e}^{i\alpha} \in S^1} \left\| \frac{\partial^m \tilde{f}}{\partial \xi^m}(\cdot, \mathrm{e}^{i\alpha}) \right\|_1 \mathrm{e}^{R|\mathrm{Im}\, z|}.$$

Conversely, if f is holomorphic on $\mathbb{C}^2$ and satisfies (4.1), by the Paley–Wiener theorem on $\mathbb{R}^2$ we get $\tilde{f}(\cdot, \mathrm{e}^{i\alpha}) \in C_c^\infty(\mathbb{R}^2)$ and $\tilde{f}(\xi, \mathrm{e}^{i\alpha}) = 0$ for all $|\xi| > R$. Hence $\hat{f}(\xi) \equiv 0$ for all $|\xi| > R$. Moreover, k_f is smooth on $\mathbb{R}^2$ since $\tilde{f}$ is smooth.

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