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Deficiently extremal Cohen–Macaulay algebras

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Abstract. The aim of this paper is to study homological properties of deficiently extremal Cohen–Macaulay algebras. Eagon–Reiner showed that the Stanley–Reisner ring of a simplicial complex has a linear resolution if and only if the Alexander dual of the simplicial complex is Cohen–Macaulay. An extension of a special case of Eagon–Reiner theorem is obtained for deficiently extremal Cohen–Macaulay Stanley–Reisner rings.

Keywords. Extremal algebras; Stanley–Reisner rings; Betti numbers.

1. Introduction

Let $R = k[x_1, x_2, \ldots, x_n]$ be a standard polynomial ring over a field k. Let I be a graded ideal in R of height g and initial degree p. Suppose the Hilbert series $\mathbf{F}(R/I, t)$ of the algebra R/I is of the form

$$\mathbf{F}(R/I, t) = \frac{\sum_{i=0}^{s} h_i t^i}{(1-t)^d},$$

where $d = n - g$ is the (Krull) dimension of R/I. The vector $(h_0, h_1, \ldots, h_s)$ is called the *h-vector* of R/I and s is its *length*. Suppose R/I is a *Cohen–Macaulay algebra* (or *CM algebra*), i.e. $\dim(R/I) = \operatorname{depth}(R/I)$. Then $s \geq p - 1$. The k-algebra R/I is said to be *an extremal* (or *a p-extremal*) CM algebra if $s = p - 1$. A p-extremal CM algebra is interesting because it has a *p-linear resolution*. Thus using formulae of Herzog and Kühl [5], their Betti numbers are obtained.

Extremal CM algebras with p-linear resolutions have been widely studied. The CM algebra R/I has p-linear resolution if and only if the minimal number of generators in degree p is given by $\nu(I_p) = \binom{p+g-1}{p}$. This characterization of CM algebra having p-linear resolutions is given by Cavaliere, Rossi and Valla [2]. The CM rings with linear resolutions have been studied by Sally [7] for the case $p = 2$ and by Schenzel [8] for the general case. A complete characterization of CM algebras with 2-linear resolutions is given by Fröberg [4]. These results have been extended to CM algebras with 2-pure minimal resolutions by Bruns and Hibi [1].

Let $[n] = \{1, 2, \ldots, n\}$. By an *abstract simplicial complex* Δ on the vertex set $V = [n]$, we mean a collection Δ of subsets of $V = [n]$ called *faces* such that each singleton $\{i\} \in \Delta$ and Δ is closed under taking subsets, i.e. if $\sigma \in \Delta$ and $\tau \subseteq \sigma$, then $\tau \in \Delta$. The *dimension* $\dim(\sigma)$ of a face σ is $\#\sigma - 1$, and *dimension* of Δ is $\dim(\Delta) = \max\{\dim(\sigma) \colon \sigma \in \Delta\}$.

Also, the maximal faces of Δ are called its *facets*. The *f-vector* $f(\Delta)$ of a $d-1$ dimensional simplicial complex Δ is given by

$$f(\Delta) = (f_0, f_1, \ldots, f_{d-1}),$$

where f_i is the number of i-dimensional faces of Δ. To each simplicial complex Δ on $V = [n]$, one associates a (square-free) monomial ideal I_Δ in the standard polynomial ring $R = k[x_1, x_2, \ldots, x_n]$ given by

$$I_\Delta = \langle x^\sigma \colon \sigma \notin \Delta \rangle,$$

where $x^\sigma = \prod_{i \in \sigma} x_i$. Note that I_Δ is minimally generated by x^σ, where σ runs over minimal non-faces of Δ. The monomial ideal I_Δ is called the *Stanley–Reisner ideal* associated to Δ and, the quotient ring $R/I_\Delta = k[\Delta]$ is called the *Stanley–Reisner ring* of Δ. The combinatorial properties of the simplicial complex Δ are intimately related to the algebraic properties of its Stanley–Reisner ring $k[\Delta]$. A CM Stanley–Reisner ring $k[\Delta]$ is p-extremal if and only if the f-vector $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$ is given by

$$f_{j-1} = \sum_{i=0}^{\min(j,p-1)} \binom{d-i}{j-i} \binom{n-d+i-1}{i},$$

for $1 \leq j \leq d$ (see [1]). The Alexander dual Δ^* of Δ is a simplicial complex given by

$$\Delta^* = \{\sigma \subseteq [n] \colon [n] - \sigma \notin \Delta\}.$$

Using a dual version of Hochster's formula, Eagon and Reiner proved that the Stanley–Reisner ring $k[\Delta]$ has a p-linear resolution if and only if $k[\Delta^*]$ is Cohen–Macaulay (cf. Theorem 2.2).

We again consider the CM algebra R/I, where I is a graded ideal in the polynomial ring $R = k[x_1, x_2, \ldots, x_n]$ of height g and initial degree p. We say that the k-algebra R/I is *c-deficiently extremal Cohen–Macaulay algebra* (or *c-deficient CM algebra*) if $s = p - 1 + c$, where s is the length of the h-vector of R/I. In other words, a c-deficiently extremal CM algebra is c-steps away from being an extremal CM algebra. Thus 0-deficient CM algebra is nothing but an extremal CM algebra whereas a 1-deficient CM algebra is a nearly extremal CM algebra introduced in Kumar, Singh and Kumar [6]. We shall investigate the properties of c-deficiently extremal CM algebras vis-á-vis extremal or nearly extremal CM algebras.

2. Deficiently extremal CM algebras

For a c-deficiently extremal Cohen–Macaulay algebra ($c \geq 1$), we have the following characterization theorem.

Theorem 2.1. *Let $R = k[x_1, x_2, \ldots, x_n]$ be a standard polynomial ring over a field k and I be a Cohen–Macaulay graded ideal in R of height g and initial degree p. Then the following conditions are equivalent.*

1) *R/I is a c-deficiently extremal Cohen–Macaulay algebra.*

2) *The minimal resolution of R/I is of the form*

$$0 \to F_g \to F_{g-1} \to \cdots \to F_2 \to F_1 \to R \to R/I \to 0,$$

where $F_i = \oplus_{j=0}^{c} R[-(p+i+j-1)]^{b_{ij}}$ *and,* $\beta_{i,p+i+j-1} = b_{ij}$ *are graded Betti numbers with* $\beta_{g,p+g+c-1} \neq 0$.

3) *The Hilbert series of R/I is of the form*

$$\mathbf{F}(R/I, t) = \frac{\sum_{i=0}^{p-1} \binom{i+g-1}{g-1} t^i + \sum_{j=0}^{c-1} h_{p+j} t^{p+j}}{(1-t)^d},$$

where $d = \dim(R/I)$, $0 < h_{p+j} < \binom{g+p+j-1}{p+j}$; *for* $j = 0, 1, \ldots, c-1$ *and* $h_p = \binom{g+p-1}{p} - \nu(I_p)$, *where* $\nu(I_p)$ *is the minimal number of generators of I in degree p.*

Proof. It can be assumed that k is an infinite field. Since R/I is Cohen–Macaulay, there is a regular system of parameters $\mathbf{y} = \{y_1, \ldots, y_d\}$ of R/I such that each y_i is of degree 1 in R. Then $\bar{R} = R/\mathbf{y}R$ is a polynomial ring in g variables and $\bar{R}/\bar{I} = R/(\mathbf{y}, I)$ is Artinian. The h-vector of R/I satisfies $h_i = H(\bar{R}/\bar{I}, i) = \dim_k(\bar{R}_i/\bar{I}_i)$. Clearly, $h_i = \binom{g+i-1}{i}$ for $0 \leq i < p$, and $h_{p+j} = \binom{g+p+j-1}{p+j} - \dim_k(\bar{I}_{p+j})$ for $j = 0, 1, \ldots, p-1+c$. Also, $\dim_k(\bar{I}_p) = \nu(I_p)$. On substituting the value of h-vector in the Hilbert series $\mathbf{F}(R/I, t) = (h_0 + h_1 t + \cdots + h_s t^s)/(1-t)^d$, we see that (1) and (3) are equivalent. Since R/I is Cohen–Macaulay, *projective dimension* $\mathrm{pd}(R/I) = ht(I) = g$. Thus R/I has a minimal resolution of the form

$$0 \to F_g \to F_{g-1} \to \cdots \to F_2 \to F_1 \to R \to R/I \to 0,$$

where $F_i = \oplus_{j=1}^{b_i} R[-d_{ij}]$ with $d_{i1} \leq d_{i2} \leq \cdots \leq d_{ib_i}$ for all i. By the minimality of resolution, we have $d_{11} < d_{21} < \cdots < d_{g1}$ and $p + i - 1 \leq d_{ij}$ for all i, j. On tensoring the minimal resolution of R/I by $R/(\mathbf{y})$, we get a minimal resolution of the Artinian ring $\bar{R}/\bar{I} = R/(\mathbf{y}, I)$. In this case, the *Socle* of $\bar{R}/\bar{I}$ is isomorphic to $\oplus_{j=1}^{b_g} k[-(d_{gj} - g)]$. If R/I is c-deficiently extremal CM algebra, then the Socle of $\bar{R}/\bar{I}$ can live in degrees $\leq p - 1 + c$. Thus $d_{gj} - g \leq p - 1 + c$. Combining the two inequalities, we get $p + (g-1) \leq d_{gj} \leq p - 1 + c + g$. As R/I is Cohen–Macaulay, we have $p \leq d_{1b_1} < d_{2b_2} < \cdots < d_{gb_g} \leq p - 1 + c + g$ (see Proposition 4.2.3 of [9]). Therefore, d_{ij} can take the values $p + i - 1, \ldots, p + i + c - 1$. Since R/I is c-deficiently extremal CM algebra, $d_{gb_g} = p + g + c - 1$ and $\beta_{g,p+g+c-1} \neq 0$. This proves that (1) implies (2). Finally, we shall show that (2) implies (1). If the minimal resolution of R/I is given as in (2), then its Hilbert series $\mathbf{F}(R/I, t)$ is of the form

$$\mathbf{F}(R/I, t) = \frac{1 + \sum_{i=1}^{g} (-1)^i \left(\sum_{j=0}^{c} b_{ij} t^{p+i+j-1} \right)}{(1-t)^n}.$$

Since $d = \dim(R/I) = n - g$, the Hilbert series of R/I is of the form

$$\mathbf{F}(R/I, t) = \frac{\sum_{i=0}^{p-1+c} h_i t^i}{(1-t)^d}.$$

Thus the h-vector of R/I has length exactly $p - 1 + c$. ■

We see that if $h_p = 0$, then R/I becomes an extremal CM algebra. In this case R/I has a p-linear resolution and its Hilbert series is given by

$$\mathbf{F}(R/I, t) = \frac{\sum_{i=0}^{p-1} \binom{i+g-1}{g-1} t^i}{(1-t)^d}, \quad d = \dim(R/I).$$

On the other hand, if $h_p > 0$ and $h_{p+1} = 0$, then R/I becomes a nearly extremal CM algebra [6]. The Hilbert series of a nearly extremal CM algebra is given by

$$\mathbf{F}(R/I, t) = \frac{\sum_{i=0}^{p-1} \binom{i+g-1}{g-1} t^i + h_p t^p}{(1-t)^d},$$

where $d = \dim(R/I)$ and $h_p = \binom{g+p-1}{p} - \nu(I_p)$. Since a p-extremal CM algebra R/I has a p-linear resolution, its Betti numbers can be specified by Herzog and Kühl formulae. By manipulating the techniques of Herzog and Kühl, one can obtain some specific bounds for graded Betti numbers of nearly extremal CM algebras. We would like to remark that the techniques of Herzog–Kühl cannot be extended beyond nearly extremal (or 1-deficient) CM algebras. Nevertheless, proceeding exactly on the lines of Theorem 2.2 in [6], the initial Betti numbers $\beta_{i,p+i-1}$ can be shown to satisfy the inequality

$$0 \le \beta_{i,p+i-1} \le \binom{p+g-1}{g-i} \binom{p+i-2}{i-1}$$

for $i = 1, 2, \ldots, g$.

We now describe a result of Eagon and Reiner [3]. Let Δ be a simplicial complex on the vertex set $V = [n]$ and $k[\Delta] = R/I_\Delta$ be the Stanley–Reisner ring of Δ, as mentioned in the Introduction. Let Δ^* be the Alexander dual of Δ. Then Eagon and Reiner showed that the Stanley–Reisner rings $k[\Delta]$ and $k[\Delta^*]$ are intimately related to each other.

Theorem 2.2 (Eagon and Reiner). *Let Δ be a simplicial complex on the vertex set $V = [n]$. Then the Stanley–Reisner ring $k[\Delta]$ has linear resolution if and only if $k[\Delta^*]$ is Cohen–Macaulay.*

Proof. For a complete proof, see Theorem 3 of [3]. ■

Suppose Δ is Cohen–Macaulay over k and the Stanley–Reisner ideal I_Δ is of initial degree p. Then it is clear that $k[\Delta]$ has a linear (or p-linear) resolution if and only if $k[\Delta]$ is p-extremal. Thus Theorem 2.2 has the following corollary.

COROLLARY 2.3

Let Δ be a Cohen–Macaulay simplicial complex on the vertex set $V = [n]$ such that the Stanley–Reisner ideal I_Δ is of initial degree p. Then $k[\Delta]$ is p-extremal if and only if $k[\Delta^*]$ is Cohen–Macaulay, i.e. $\dim(k[\Delta^*]) = \operatorname{depth}(k[\Delta^*])$.

We now consider a CM Stanley–Reisner ring $k[\Delta]$ which is non p-extremal. Then $\operatorname{depth}(k[\Delta^*]) < \dim(k[\Delta^*])$ and we shall determine the difference.

Example 2.4. Let $R = k[x_1, \dots, x_6]$ and $V = \{1, 2, \dots, 6\}$. Let Δ be a simplicial complex on V generated by its facets as

$$\Delta = \langle \{1, 2, 3\}, \{1, 3, 4\}, \{2, 3, 5\}, \{2, 3, 6\}, \{2, 5, 6\}, \{3, 4, 5\}, \{3, 5, 6\} \rangle.$$

Then the Stanley–Reisner ideal $I_\Delta = \langle x_1x_5, x_1x_6, x_2x_4, x_4x_6, x_2x_3x_5x_6 \rangle$ is of initial degree $p = 2$, and the Stanley–Reisner ring $k[\Delta] = R/I_\Delta$ is of dimension $d = 3$. Here $n = 6$ and $d = n - g = 3$ implies that the height of I_Δ is $g = 3$. The minimal resolution of $k[\Delta]$ is of the form

$$0 \to R[-5] \oplus R[-6] \to R[-3]^3 \oplus R[-4] \oplus R[-5]^2$$
$$\to R[-2]^4 \oplus R[-4] \to R \to k[\Delta] \to 0.$$

Thus projective dimension $\mathrm{pd}(k[\Delta]) = 3$. From the Auslander–Buchsbaum formula $\mathrm{pd}(k[\Delta]) + \mathrm{depth}(k[\Delta]) = \mathrm{depth}(R)$, we deduce that $\mathrm{depth}(k[\Delta]) = 3 = \dim(k[\Delta])$. Thus $k[\Delta]$ is a non extremal CM Stanley–Reisner ring with deficiency $c = 2$. The Alexander dual Δ^* of Δ is given by

$$\Delta^* = \langle \{1, 4\}, \{1, 2, 3, 5\}, \{1, 3, 5, 6\}, \{2, 3, 4, 6\}, \{2, 3, 4, 5\} \rangle,$$

and, its Stanley–Reisner ideal I_{Δ^*} is given by

$$I_{\Delta^*} = \langle x_1x_2x_4, x_1x_2x_6, x_1x_3x_4, x_1x_4x_5, x_1x_4x_6, x_2x_5x_6, x_4x_5x_6 \rangle.$$

Therefore, $k[\Delta^*] = R/I_{\Delta^*}$ and $\dim(k[\Delta^*]) = 4$. By Theorem 2.2, $k[\Delta^*]$ has minimal 3-linear resolution of the form

$$0 \to R[-6] \to R[-5]^5 \to R[-4]^{10} \to R[-3]^7 \to R \to k[\Delta^*] \to 0.$$

Thus $\mathrm{pd}(k[\Delta^*]) = 4$, and again by Auslander–Buchsbaum formula, we have $\mathrm{depth}(k[\Delta^*]) = 2$. Now we have $\dim(k[\Delta^*]) - \mathrm{depth}(k[\Delta^*]) = 2 = c$.

The phenomenon shown in Example 2.3 is not a pathalogical case. In fact, the following theorem extends the Corollary 2.3 to any c-deficiently extremal CM Stanley–Reisner ring.

Theorem 2.5. *Let Δ be a Cohen–Macaulay simplicial complex on the vertex set $V = [n]$. Suppose that the Stanley–Reisner ideal I_Δ is of initial degree p and height g. Then $k[\Delta]$ is c-deficiently extremal Cohen–Macaulay algebra if and only if* $\dim(k[\Delta^*]) - \mathrm{depth}(k[\Delta^*]) = c$.

Proof. Since I_Δ has initial degree p, it follows from the definition of Δ^* that $\dim(\Delta^*) = n - p - 1$. Thus $\dim(k[\Delta^*]) = \dim(\Delta^*) + 1 = n - p$. Now assume that $k[\Delta]$ is a c-deficiently extremal CM algebra. Then $s = p - 1 + c$. Since Δ is Cohen–Macaulay, it follows from Theorem 3, 4 of [3], that $k[\Delta^*]$ has a linear resolution and its Betti numbers satisfy the identity

$$\sum_{i \geq 1} \beta_i(k[\Delta^*]) t^{i-1} = \sum_{i=0}^{p-1+c} h_i(\Delta)(t+1)^i.$$

Clearly, $\beta_{p+c}(k[\Delta^*]) = h_{p-1+c}(\Delta)$ and all the higher Betti numbers are zero. Thus projective dimension $\mathrm{pd}(k[\Delta^*]) = p + c$. By Auslander–Buchsbaum formula

$$\mathrm{pd}(k[\Delta^*]) + \mathrm{depth}(k[\Delta^*]) = \mathrm{depth}(R) = n,$$

we have $\mathrm{depth}(k[\Delta^*]) = n - p - c$. This shows that

$$\dim(k[\Delta^*]) - \mathrm{depth}(k[\Delta^*]) = c.$$

Conversely, if the condition holds i.e. $\dim(k[\Delta^*]) - \mathrm{depth}(k[\Delta^*]) = c$, then going backward, we see that $\mathrm{pd}(k[\Delta^*]) = p + c$. Now the above identity becomes

$$\sum_{i=1}^{p+c} \beta_i(k[\Delta^*])t^{i-1} = \sum_{i=0}^{s} h_i(\Delta)(t+1)^i,$$

from which we conclude that $s = p - 1 + c$. Hence $k[\Delta]$ is a c-deficiently extremal CM Stanley–Reisner ring. ■

If $c = 0$, then clearly $\dim(k[\Delta^*]) = \mathrm{depth}(k[\Delta^*])$, i.e. $k[\Delta^*]$ is Cohen–Macaulay. Thus Theorem 2.5 reflects that if a CM simplicial complex Δ is c-steps away from being extremal, then its Alexander dual Δ^* is c-steps away from being Cohen–Macaulay and vice versa.

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