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Symmetries, integrals and solutions of ordinary differential equations of maximal symmetry

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Abstract. Second- and third-order scalar ordinary differential equations of maximal symmetry in the traditional sense of point, respectively contact, symmetry are examined for the mappings they produce in solutions and fundamental first integrals. The properties of the 'exceptional symmetries', i.e. those not considered to be generic to scalar equations of maximal symmetry, can be recast into a form which is applicable to all such equations of maximal symmetry. Some properties of these symmetries are demonstrated.

Keywords. Lie symmetry; exceptional symmetries; mapping of integrals and solutions.

1. Introduction

The relationships between the symmetries, integrals and solutions of ordinary differential equations have received some detailed attention in recent years to elaborate the properties already reported by Lie such as the maximal number of point symmetries of a scalar equation and the mapping of solutions into solutions by the transformations generated by the symmetries. The algebras of integrals of scalar ordinary differential equations of low order were considered in [15, 16, 7, 14]. The results of Lie on the maximal number of symmetries for a scalar ordinary differential equation were expanded in [11] and [17].

More recently an alternate representation of the standard categories of the symmetries of scalar second-order ordinary differential equations and third-order ordinary differential equations of maximal symmetry was provided [19] by breaking away from the usual representation in terms of point and contact symmetries. We recall that the standard categories for these equations are homogeneity, solution, $sl(2, R)$ and noncartan [10] for second-order ordinary differential equations and homogeneity, solution, $sl(2, R)$ and intrinsically contact [14] for third-order ordinary differential equations.

The present work is motivated by two observations of Euler and Euler [5] in a work devoted to the use of generalized Sundman symmetries and their associated transformations [26, 4], which may be regarded as an example of the use of nonlocal symmetries. The first observation was of the transformation of one first integral of the nonlinear ordinary differential equation under investigation to another first integral. The second observation was of the generation of the general solution of the source ordinary differential equation under a mapping of a particular solution of the corresponding target equation. Here we explore the mapping of solutions and integrals of elementary scalar ordinary differential equations of maximal symmetry to delineate the properties to be expected without the complexities of the real application considered by Euler and Euler [5].

Since we are dealing with equations of maximal symmetry, we may consider the representative equations, *videlicet* $y'' = 0$ and $y''' = 0$, to make our investigations even more transparent. In §2 we give the results for $y'' = 0$ and in §3 for $y''' = 0$. The properties observed lead to further considerations of the structures of symmetries which are explored in §§4 and 5. A well-known feature of second- and third-order ordinary differential equations of maximal symmetry is the possession of exceptional symmetries, two noncartan symmetries for second-order ordinary differential equations and three intrinsically contact symmetries for third-order ordinary differential equations, by comparison with scalar ordinary differential equations of higher order of maximal symmetry. Examination of the symmetry – integral relations for second- and third-order ordinary differential equations leads to an hypothesis for an additional class of symmetries for nth-order ordinary differential equations of maximal symmetry. The number of symmetries in this class equals the order of equation and so we can associate $2n + 4$ symmetries with a scalar nth-order ordinary differential equation of maximal symmetry thereby removing the exceptional character from second- and third-order ordinary differential equations.

The outstanding question is whether these results extend to linear nth-order ordinary differential equations of less than maximal symmetry. In considering this question we can take consolation in a preliminary result. A linear scalar third-order ordinary differential equation can have 4, 5 or, the maximal, 7 (point; 10 contact) symmetries. In general a linear scalar nth-order ordinary differential equation has $n + 1$, $n + 2$ or $n + 4$ Lie point symmetries [17]. Govinder and Leach [9] have shown the equivalence of all three classes of linear nth-order ordinary differential equations under nonlocal transformations. Consequently all linear (*æq* linearizable by any form of transformation) scalar nth-order ordinary differential equations are equally categorized. Thus the properties which we discuss below are applicable to a very wide class of ordinary differential equations with many subclasses which are generally treated as separate, but which in fact are revealed by our analysis in terms of more general symmetries and transformations to be identical.

In the Conclusion we summarize the salient results and point to further possible avenues of research. In Appendix A we present a point critical for the unification of the discussion of essential oneness of scalar linear nth-order ordinary differential equations which was not considered in [9].

2. The equation $y'' = 0$

The representative second-order ordinary differential equation of maximal point symmetry, *videlicet*

$$y'' = 0, \tag{2.1}$$

has the solution

$$y = A_0 + A_1 x, \tag{2.2}$$

the first integrals, termed fundamental [14],

$$I_1 = y - xy' \quad \text{and} \quad I_2 = y' \tag{2.3}$$

and eight Lie point symmetries, being a representation of the algebra $sl(3, R)$, *videlicet*

$$
\begin{array}{lll}
\Gamma_h = y\partial_y & \Big] & \text{homogeneity; } 1A_1 \\
\Gamma_{s1} = 1\partial_y & \Big] & \text{solution; } 2A_1 \\
\Gamma_{s2} = x\partial_y & & \\
\Gamma_{sl1} = \partial_x & \Big] & \text{special linear; } A_{3,8}\ (sl(2, R)) \\
\Gamma_{sl2} = x\partial_x + \frac{1}{2}y\partial_y & & \\
\Gamma_{sl3} = x^2\partial_x + xy\partial_y & & \\
\Gamma_{nc1} = y\partial_x & \Big] & \text{noncartan; } 2A_1, \\
\Gamma_{nc2} = xy\partial_x + y^2\partial_y & &
\end{array}
\qquad (2.4)
$$

in which the subalgebras of the algebra of the Lie point symmetries, $sl(3, R)$, are listed with their usual descriptors. They are the subalgebras of the homogeneity symmetry, Γ_h, the solution symmetries[1], Γ_{sj}, $j = 1, 2$, the $sl(2, R)$ subalgebra central to all scalar ordinary differential equations of maximal Lie point symmetry and Γ_{ncj}, $j = i, 2$, the so-called noncartan symmetries which are the generators of transformations which are not fibre-preserving [10]. The notation for the subalgebras listed in (2.4) follows the Mubarakzyanov classification scheme [18, 21–23] with $sl(2, R)$ being the common name for $A_{3,8}$.

The actions of the eight Lie point symmetries, once extended, of (2.1) on the first integrals given in (2.3) are

$$
\begin{array}{ll}
\Gamma_h^{[1]} I_1 = I_1 & \Gamma_h^{[1]} I_2 = I_2 \\
\hline
\Gamma_{s1}^{[1]} I_1 = 1 & \Gamma_{s1}^{[1]} I_2 = 0 \\
\Gamma_{s2}^{[1]} I_1 = 0 & \Gamma_{s2}^{[1]} I_2 = 1 \\
\hline
\Gamma_{sl1}^{[1]} I_1 = -I_2 & \Gamma_{sl1}^{[1]} I_2 = 0 \\
\Gamma_{sl2}^{[1]} I_1 = \frac{1}{2} I_1 & \Gamma_{sl2}^{[1]} I_2 = -\frac{1}{2} I_2 \\
\Gamma_{sl3}^{[1]} I_1 = 0 & \Gamma_{sl3}^{[1]} I_2 = I_1 \\
\hline
\Gamma_{nc1}^{[1]} I_1 = -I_1 I_2 & \Gamma_{nc1}^{[1]} I_2 = -I_2^2 \\
\Gamma_{nc2}^{[1]} I_1 = I_1^2 & \Gamma_{nc2}^{[1]} I_2 = I_1 I_2
\end{array}
\qquad (2.5)
$$

in which we see that the actions of the different classes of symmetry have noticeably different results[2].

[1]Note that we have written the solution, 1, explicitly in the expression for Γ_{s1} to emphasise its presence. The same usage is followed in (3.4).

[2]There is always some ambiguity with the homogeneity symmetry and Γ_{sl2}. Here we adopt the standard form for Γ_{sl2} as presented by Mahomed and Leach [17]; see also Moyo and Leach [20]. The ambiguity of the form of Γ_{sl2} is highlighted by the differing forms found for nth-order ordinary differential equations of different orders [17] and their fundamental first integrals [6].

The finite transformations corresponding to each symmetry are given by

$$\begin{array}{llll}\Gamma_h & : & \bar{x}=x & \bar{y}=y\mathrm{e}^a\\ \Gamma_{s1} & : & \bar{x}=x & \bar{y}=y+a\\ \Gamma_{s2} & : & \bar{x}=x & \bar{y}=y+ax\\ \Gamma_{sl1} & : & \bar{x}=x+a & \bar{y}=y\\ \Gamma_{sl2} & : & \bar{x}=x\mathrm{e}^a & \bar{y}=y\mathrm{e}^{a/2}\\ \Gamma_{sl3} & : & \bar{x}=\dfrac{x}{1-ax} & \bar{y}=\dfrac{y}{1-ax}\\ \Gamma_{nc1} & : & \bar{x}=x+ay & \bar{y}=y\\ \Gamma_{nc2} & : & \bar{x}=\dfrac{x}{1-ay} & \bar{y}=\dfrac{y}{1-ay},\end{array} \tag{2.6}$$

where a is the parameter of the finite transformation so chosen that $a=0$ corresponds to the identity. Under these finite transformation the solution (2.2) becomes

$$\begin{array}{lll}\Gamma_h & : & \bar{y}=\mathrm{e}^a\left(A_0+A_1\bar{x}\right)\\ \Gamma_{s1} & : & \bar{y}=A_0+a+A_1\bar{x}\\ \Gamma_{s2} & : & \bar{y}=A_0+\left(A_1+a\right)\bar{x}\\ \Gamma_{sl1} & : & \bar{y}=A_0-aA_1+A_1\bar{x}\\ \Gamma_{sl2} & : & \bar{y}=A_0\mathrm{e}^{a/2}+A_1\bar{x}\mathrm{e}^{-a/2}\\ \Gamma_{sl3} & : & \bar{y}=A_0+\left(A_0a+A_1\right)\bar{x}\\ \Gamma_{nc1} & : & \bar{y}=\dfrac{1}{1+aA_1}\left[A_0(1-aA_1)+A_1\bar{x}\right]\\ \Gamma_{nc2} & : & \bar{y}=\dfrac{1}{1+aA_1}\left(A_0+A_1\bar{x}\right)\end{array} \tag{2.7}$$

in which we note that in the cases of Γ_{s1}, Γ_{s2}, Γ_{sl1} and Γ_{sl3} a one-parameter solution is mapped into a two-parameter solution. In the instances of Γ_{s1} and Γ_{sl1} the constant A_0 may be taken as a specific number including zero, i.e., a particular solution be used, and the transformed solution is a general solution. For Γ_{s2} and Γ_{sl3} the constant A_1 may be taken as a specific number including zero and again the transformed solution is a general solution. Geometrically one pictures the transformation as being out of the surface specified due to one of the parameters being fixed. We see that the observation of this property in the case of a particular ordinary differential equation by Euler and Euler [5] is not peculiar but something to be expected in the case of second-order ordinary differential equations of maximal symmetry. This enhances the possibility of useful application to nonlinear equations arising in applications.

The fundamental first integrals transform as

$$\begin{array}{llll}\Gamma_h & : & \bar{I}_1=I_1\mathrm{e}^a & \bar{I}_2=I_2\mathrm{e}^a\\ \Gamma_{s1} & : & \bar{I}_1=I_1+a & \bar{I}_2=I_2\\ \Gamma_{s2} & : & \bar{I}_1=I_1 & \bar{I}_2=I_2+a\\ \Gamma_{sl1} & : & \bar{I}_1=I_1-aI_2 & \bar{I}_2=I_2\\ \Gamma_{sl2} & : & \bar{I}_1=I_1\mathrm{e}^{a/2} & \bar{I}_2=I_2\mathrm{e}^{-a/2}\\ \Gamma_{sl3} & : & \bar{I}_1=I_1 & \bar{I}_2=I_2+aI_1\\ \Gamma_{nc1} & : & \bar{I}_1=\dfrac{I_1}{1+aI_2} & \bar{I}_2=\dfrac{I_2}{1+aI_2}\\ \Gamma_{nc2} & : & \bar{I}_1=\dfrac{I_1}{1-aI_1} & \bar{I}_2=\dfrac{I_2}{1-aI_1},\end{array} \tag{2.8}$$

where once again a is the parameter of the finite transformation with $a = 0$ being the identity, and all other integrals transform in terms of their decompositions as functions of the fundamental first integrals [6].

We note that, unlike the example reported by Euler and Euler [5] for a nonlinear equation, not one of the fundamental first integrals is transformed into a function which does not contain the original integral. However, we do see that one integral can be transformed into a combination of the two original fundamental first integrals. We can make use of this feature to see a transformation similar to that reported by Euler and Euler [5][3]. If we take Γ_{sl1} then Γ_{sl3} in turn with $a = -1$, for instance, in both cases, $\bar{\bar{I}}_2 = I_1$, and we have an instance of the result observed by Euler and Euler.

3. The equation $y''' = 0$

We list in the same order as in §2 for (2.1) the equivalent features of the equation

$$y'' = 0, \tag{3.1}$$

which are

$$y = A_0 + A_1 x + \frac{1}{2} A_2 x^2, \tag{3.2}$$

$$\begin{aligned} I_1 &= y - xy' + \frac{1}{2} x^2 y'', \\ I_2 &= y' - xy'', \\ I_3 &= y'', \end{aligned} \tag{3.3}$$

(the justification for the ordering is the same as for (2.1))

$$\begin{array}{lll} \Gamma_h = y\partial_y & \} & \text{homogeneity; } 1A_1 \\ \Gamma_{s1} = 1\partial_y & & \\ \Gamma_{s2} = x\partial_y & \} & \text{solution; } 3A_1 \\ \Gamma_{s3} = \frac{1}{2}x^2\partial_y & & \\ \Gamma_{sl1} = \partial_x & & \\ \Gamma_{sl2} = x\partial_x + y\partial_y & \} & \text{special linear; } A_{3,8}\ (sl(2,R)) \\ \Gamma_{sl3} = x^2\partial_x + 2xy\partial_y & & \\ \Gamma_{ic1} = y'\partial_x + \frac{1}{2}y'^2\partial_y & & \\ \Gamma_{ic2} = (xy' - y)\partial_x + \frac{1}{2}xy'^2\partial_y & \} & \text{intrinsically contact; } 3A_1, \\ \Gamma_{ic3} = \left(\frac{1}{2}xy' - y\right)\left[x\partial_x + \left(\frac{1}{2}xy' + y\right)\partial_y\right] & & \end{array} \tag{3.4}$$

[3]That we take two transformations to obtain a result similar to the one obtained by Euler and Euler in one transformation has no serious meaning. The symmetry of (2.1) is an eight-parameter symmetry and we choose a particular basis for its representation in terms of eight one-parameter symmetries. The infinitesimal representation of the transformation of Euler and Euler comes from a representation which is not diagonal in the representation presented here.

with the intrinsically contact symmetries, $\Gamma_{icj},\ j = 1, 3$, replacing the noncartan point symmetries of (2.1) and the subalgebras of the representation of the ten-element of the algebra $sp(4)$ [1] indicated in the same fashion as for (2.4),

$$
\begin{array}{lll}
\Gamma_h^{[1]} I_1 = I_1 & \Gamma_h^{[1]} I_2 = I_2 & \Gamma_h^{[1]} I_3 = I_3 \\
\hline
\Gamma_{s1}^{[1]} I_1 = 1 & \Gamma_{s1}^{[1]} I_2 = 0 & \Gamma_{s1}^{[1]} I_3 = 0 \\
\Gamma_{s2}^{[1]} I_1 = 0 & \Gamma_{s2}^{[1]} I_2 = 1 & \Gamma_{s2}^{[1]} I_3 = 0 \\
\Gamma_{s3}^{[1]} I_1 = 0 & \Gamma_{s3}^{[1]} I_2 = 0 & \Gamma_{s3}^{[1]} I_3 = 1 \\
\hline
\Gamma_{sl1}^{[1]} I_1 = -I_2 & \Gamma_{sl1}^{[1]} I_2 = -I_3 & \Gamma_{sl1}^{[1]} I_3 = 0 \\
\Gamma_{sl2}^{[1]} I_1 = I_1 & \Gamma_{sl2}^{[1]} I_2 = 0 & \Gamma_{sl2}^{[1]} I_3 = -I_3 \\
\Gamma_{sl3}^{[1]} I_1 = 0 & \Gamma_{sl3}^{[1]} I_2 = 2I_1 & \Gamma_{sl3}^{[1]} I_3 = 2I_2 \\
\hline
\Gamma_{ic1}^{[1]} I_1 = -\frac{1}{2} I_2^2 & \Gamma_{ic1}^{[1]} I_2 = -I_2 I_3 & \Gamma_{ic1}^{[1]} I_3 = -I_3^2 \\
\Gamma_{ic2}^{[1]} I_1 = I_1 I_2 & \Gamma_{ic2}^{[1]} I_2 = \frac{1}{2} I_2^2 + I_1 I_3 & \Gamma_{ic2}^{[1]} I_3 = I_2 I_3 \\
\Gamma_{ic3}^{[1]} I_1 = -I_1^2 & \Gamma_{ic3}^{[1]} I_2 = -I_1 I_2 & \Gamma_{ic3}^{[1]} I_3 = -\frac{1}{2} I_2^2
\end{array}
\tag{3.5}
$$

and we see that the pattern with the exceptional symmetries, $\Gamma_{icj},\ j = 1, 3$, does not have the simplicity of that shown by the $\Gamma_{ncj},\ j = 1, 2$, of (2.1) in (2.5). The finite transformations corresponding to each symmetry are given in (3.6).

$$
\begin{array}{lllll}
\Gamma_h & : & \bar{x} = x & \bar{y} = y\mathrm{e}^a & \\
\Gamma_{s1} & : & \bar{x} = x & \bar{y} = y + a & \\
\Gamma_{s2} & : & \bar{x} = x & \bar{y} = y + ax & \\
\Gamma_{s3} & : & \bar{x} = x & \bar{y} = y + \frac{1}{2}ax^2 & \\
\Gamma_{sl1} & : & \bar{x} = x + a & \bar{y} = y & \\
\Gamma_{sl2} & : & \bar{x} = x\mathrm{e}^a & \bar{y} = y\mathrm{e}^a & \\
\Gamma_{sl3} & : & \bar{x} = \dfrac{x}{1 - ax} & \bar{y} = \dfrac{y}{(1 - ax)^2} & \\
\Gamma_{ic1} & : & \bar{x} = x + ay' & \bar{y} = y + \frac{1}{2}ay'^2 & \bar{y}' = y' \\
\Gamma_{ic2} & : & \bar{x} = \dfrac{x - a(y - \frac{1}{2}xy')}{1 - \frac{1}{2}ay'} & \bar{y} = \dfrac{y - ay'\left(y - \frac{1}{2}xy'\right)}{\left(1 - \frac{1}{2}ay'\right)^2} & \bar{y}' = \dfrac{y'}{1 - \frac{1}{2}ay'} \\
\Gamma_{ic3} & : & \bar{x} = \dfrac{x}{1 + a\left(y - \frac{1}{2}xy'\right)} & \bar{y} = \dfrac{y + a\left(y - \frac{1}{2}xy'\right)^2}{\left[1 + a\left(y - \frac{1}{2}xy'\right)\right]^2} & \bar{y}' = \dfrac{y'}{1 + a\left(y - \frac{1}{2}xy'\right)},
\end{array}
\tag{3.6}
$$

where in the case of the intrinsically contact symmetries we present the transformations of the first derivatives to emphasize that they follow from the transformations engendered in the basic variables, x and y,

$$
\begin{aligned}
\Gamma_h &: \quad \bar{y} = \mathrm{e}^a \left(A_0 + A_1\bar{x} + \tfrac{1}{2}A_2\bar{x}^2\right)\\
\Gamma_{s1} &: \quad \bar{y} = A_0 + a + A_1\bar{x} + \tfrac{1}{2}A_2\bar{x}^2\\
\Gamma_{s2} &: \quad \bar{y} = A_0 + (A_1 + a)\,\bar{x} + \tfrac{1}{2}A_2\bar{x}^2\\
\Gamma_{s3} &: \quad \bar{y} = A_0 + A_1\bar{x} + \tfrac{1}{2}\,(A_2 + a)\,\bar{x}^2\\
\Gamma_{sl1} &: \quad \bar{y} = A_0 - aA_1 + \tfrac{1}{2}A_2a^2 + (A_1 - aA_2)\,\bar{x} + \tfrac{1}{2}A_2\bar{x}^2\\
\Gamma_{sl2} &: \quad \bar{y} = A_0\mathrm{e}^a + A_1\bar{x} + \tfrac{1}{2}A_2\bar{x}^2\mathrm{e}^{-a}\\
\Gamma_{sl3} &: \quad \bar{y} = A_0 + (2A_0a + A_1)\,\bar{x} + \left(a^2A_0 + aA_1 + \tfrac{1}{2}A_2\right)\bar{x}^2\\
\Gamma_{ic1} &: \quad \bar{y} = \frac{1}{1 + aA_2}\left[A_0(1 + aA_2) - \tfrac{1}{2}aA_1^2 + A_1\bar{x} + \tfrac{1}{2}A_2\bar{x}^2\right] \qquad (3.7)\\
\Gamma_{ic2} &: \quad \bar{y} = \frac{A_0\left(1 - \tfrac{1}{2}aA_1\right)\left(1 - \tfrac{1}{2}aA_1 - \tfrac{1}{2}aA_0\right)}{\left[\left(1 - \tfrac{1}{2}aA_1\right)^2 - \tfrac{1}{2}a^2A_0A_2\right]^2}\\
&\qquad\quad + \frac{\bar{x}\left[A_1(1 - \tfrac{1}{2}aA_1) + \tfrac{1}{2}aA_0A_2\right] + \tfrac{1}{2}A_2\bar{x}^2}{\left(1 - \tfrac{1}{2}aA_1\right)^2 - \tfrac{1}{2}a^2A_0A_2}\\
\Gamma_{ic3} &: \quad \bar{y} = \frac{A_0 + A_1\bar{x}}{1 + aA_0} + \tfrac{1}{2}\bar{x}^2\left[A_2 - \frac{a^2A_1^2}{1 + aA_0}\right]
\end{aligned}
$$

and finally the integrals transform as

$$
\begin{array}{lcll}
 & & \bar{I}_1 & \bar{I}_2 & \bar{I}_3\\
\Gamma_h & : & I_1\mathrm{e}^a & I_2\mathrm{e}^a & I_3\mathrm{e}^a\\
\Gamma_{s1} & : & I_1 + a & I_2 & I_3\\
\Gamma_{s2} & : & I_1 & I_2 + a & I_3\\
\Gamma_{s3} & : & I_1 & I_2 & I_3 + a\\
\Gamma_{sl1} & : & = I_1 - aI_2 + \frac{1}{2}a^2I_3 & I_2 - aI_3 & I_3\\
\Gamma_{sl2} & : & I_1\mathrm{e}^a & I_2 & I_3\mathrm{e}^{-a}\\
\Gamma_{sl3} & : & I_1 & \bar{I}_2 = I_2 + 2aI_1 & I_3 + 2aI_2 + 2a^2I_1\\
\Gamma_{ic1} & : & I_1 - \frac{1}{2}\dfrac{a}{1 + aI_3}I_2^2 & \dfrac{I_2}{1 + aI_3} & \dfrac{I_3}{1 + aI_3}\\
\Gamma_{ic2} & : & \dfrac{I_1}{\left(1 - \frac{1}{2}aI_2\right)^2 - \frac{1}{2}I_1I_3a^2} & \dfrac{I_2 - \frac{1}{2}a\left(I_2^2 - 2I_1I_3\right)}{\left(1 - \frac{1}{2}aI_2\right)^2 - \frac{1}{2}a^2I_1I_3} & \dfrac{I_3}{\left(1 - \frac{1}{2}aI_2\right)^2 - \frac{1}{2}a^2I_1I_3}\\
\Gamma_{ic3} & : & \dfrac{I_1}{1 + aI_1} & \dfrac{I_2}{1 + aI_1} & \dfrac{I_3 - \frac{1}{2}a\left(I_2^2 - 2I_1I_3\right)}{1 + aI_1}.
\end{array}
\qquad (3.8)
$$

We observe that the possibilities for the transformation of particular solutions to general solutions exist for all of the listed symmetries. In some cases one of the elements of the solution set, $\left\{1, x, \frac{1}{2}x^2\right\}$, can be missing, i.e., its arbitrary multiplying constant is zero, and the transformation, by its introduction of the missing element of the solution set multiplied by the parameter of the transformation, i.e. a third arbitrary constant, leads to the general solution. In other cases all elements of the solution set must be present, but one of what would normally be an arbitrary constant takes a specific nonzero value. Then the transformation leads to the general solution since the parameter of the transformation becomes the necessary arbitrary constant. If one thinks of the solution as an arbitrary vector in the three-dimensional space of the solution set, the former transformation maps a solution from one in the plane of just two of the solutions to the general vector whereas the latter transformation maps a solution from a plane not of two of the solutions to the general solution.

In terms of a single mapping we have the following results for the mapping of a particular solution to the general solution. We list the parameter of interest. The others must be arbitrary.

$$
\begin{aligned}
\Gamma_h &: \text{any one of } A_0,\ A_1 \text{ or } A_3 \text{ fixed nonzero}\\
\Gamma_{s1} &: A_0 \text{ fixed}\\
\Gamma_{s2} &: A_1 \text{ fixed}\\
\Gamma_{s3} &: A_1 \text{ fixed}\\
\Gamma_{sl1} &: A_0 \text{ or } A_1 \text{ fixed}\\
\Gamma_{sl2} &: A_0 \text{ or } A_2 \text{ fixed}\\
\Gamma_{sl1} &: A_1 \text{ or } A_2 \text{ fixed}\\
\Gamma_{is1} &: A_0 \text{ fixed; } A_1 \text{ or } A_2 \text{ fixed nonzero}\\
\Gamma_{is2} &: A_0,\ A_1 \text{ or } A_2 \text{ fixed nonzero}\\
\Gamma_{is3} &: A_0 \text{ or } A_1 \text{ fixed nonzero; } A_2 \text{ fixed.}
\end{aligned}
$$

Since all integrals can be expressed in terms of the fundamental integrals, the transformation rules for the integrals can be computed from the rules given in (3.8). For example the autonomous integral J, $(= yy'' - \frac{1}{2}y'^2)$ is $I_1 I_3 - \frac{1}{2}I_2^2$ and under the finite transformation corresponding to Γ_{ic1},

$$
\begin{aligned}
\bar{J} &= \bar{I}_1\bar{I}_3 - \tfrac{1}{2}\bar{I}_2^2\\
&= \frac{J}{1 + aI_3}.
\end{aligned}
$$

Naturally a succession of transformations can map a solution from the origin to the general solution, for example the application of the three solution symmetries – obviously with different transformation parameters! – in turn.

4. A standardized form for the exceptional symmetries

The exceptional symmetries, the noncartan, Γ_{ncj}, of (2.1) and the intrinsically contact, Γ_{icj}, of (3.1) have a generally quadratic effect on the fundamental first integrals, but there is hot insufficient evidence to support a postulate of a general formula. For the cartan symmetries the situation is clear-cut for the homogeneity and solution symmetries and suggestive for the $sl(2, R)$ symmetries. We propose a set of symmetries to replace the exceptional symmetry for (2.1) and (3.1) to produce simpler actions than those listed in (2.5) and (3.5). We propose that the symmetries have the property

$$\Gamma_{ei} I_j = I_i I_j, \tag{4.1}$$

where Γ_{ei} is the ith exceptional symmetry and I_j is the jth fundamental first integral.

We take these exceptional symmetries to have the form

$$\Gamma_{ei} = \xi_i \partial_x + \eta_i \partial_y, \quad i = 1, n, \tag{4.2}$$

in which the variable dependence in the coefficient functions, ξ_i and η_i, is not yet specified. The required dependence is established in the subsequent treatment.

We commence with the archetypal second-order ordinary differential equation (2.1). The symmetry (4.2) is required to have the properties[4]

$$\begin{aligned} &\eta_1'' - y'\xi_1'' = 0 \\ &\eta_1 - \xi_1 y' - (\eta_1' - y'\xi_1')x = (y - xy')^2 \\ &\eta_1' - y'\xi_1' = y'(y - xy') \end{aligned} \tag{4.3}$$

in the case of Γ_{e1} and

$$\begin{aligned} &\eta_2'' - y'\xi_2'' = 0 \\ &\eta_2 - \xi_2 y' - (\eta_2' - y'\xi_2')x = y'(y - xy') \\ &\eta_2' - y'\xi_2' = y'^2 \end{aligned} \tag{4.4}$$

in the case of Γ_{e2}, where (4.3a) and (4.4a) are the requirements that the Γ_{ei}, $i = 1, 2$, be symmetries of (2.1). We note that both are differential consequences of (4.3c) and (4.4c), equally (4.3b) and (4.4b). If we substitute (4.3c) and (4.4c) into (4.3b) and (4.4b) respectively, we obtain

$$\eta_1 - y'\xi_1 = y(y - xy') = yI_1 \tag{4.5}$$

$$\eta_2 - y'\xi_2 = yy' = yI_2 \tag{4.6}$$

which in each case gives an explicit relationship between the coefficient functions of the symmetries and the first integrals. The explicitness of the relationship is reminiscent of that of Noether's theorem [24] even though the noncartan symmetries of (2.1) are not normally counted amongst the Noether's symmetries of the action integral usually associated with (2.1)[5].

In (4.5) and (4.6) we note that the noncartan symmetries are recovered by the choices $\eta_1 = y^2$, $\xi_1 = xy$ (Γ_{nc2}) and $\eta_2 = 0$, $\xi_2 = -y$ (Γ_{nc1}) respectively. The generality of the requirement (4.1) does not impinge upon the functional forms of the coefficient functions of the symmetries in their standard representation as point symmetries.

[4] A prime indicates total differentiation with respect to the independent variable, x.

[5] Some association has been made, see for example Moyo *et al* [19], but the resulting integral presents problems of interpretation as a meaningful construct (see also Govinder *et al* [8]).

In the case of (3.1) we obtain three sets of equations which are similar in structure to those in (4.3) and (4.4). By the same process, this time a twofold substitution, we find that

$$\begin{aligned} \eta_1 - y'\xi_1 &= y\left(y - xy' + \tfrac{1}{2}x^2y''\right) = yI_1 \\ \eta_2 - y'\xi_2 &= y(y' - xy'') = yI_2 \\ \eta_3 - y'\xi_3 &= yy'' = yI_3 \end{aligned} \tag{4.7}$$

in which we find a repetition of the structures given in (4.5) and (4.6). The detail of the reduction is given in Appendix B.

In the case of the third-order ordinary differential equation, (3.1), the connection with Noether's theorem becomes less than somewhat tenuous since the relationship between the symmetry and integral requires that the Lagrangian be regular [25] and (3.1) cannot be a consequence of the variational principle applied to a regular Lagrangian.

The symmetries in (4.7) must necessarily be generalized symmetries rather than contact symmetries due to the presence of the second derivatives in $\eta_i - y'\xi_i$. The actions of the intrinsically contact symmetries of (3.1) on the fundamental first integrals do not fit the simplicity of the posited (4.1). One pays a price for simplicity!

In the case of the third-order ordinary differential equation, (3.1), a set of symmetries equivalent to the intrinsically contact symmetries but expressed in terms of a generalized Wronskian structure has been reported by Moyo *et al* [19]. Specifically they are

$$\Gamma_{icej} = [(w_j'y - w_jy')'y - (w_j'y - w_jy')y']\partial_y \tag{4.8}$$

in which for (3.1) $\{w_j,\ j = 1, 3\} \Leftrightarrow \left\{1, x, \frac{1}{2}x^2\right\}$, i.e. the solution set of (3.1) used above. With this specific solution set from (4.8) we obtain the symmetries

$$\begin{aligned} \Gamma_{ice1} &= (y'^2 - yy'')\partial_y \\ \Gamma_{ice2} &= (x(y'^2 - yy'') - yy')\partial_y \\ \Gamma_{ice3} &= \left[\tfrac{1}{2}x^2(y'^2 - yy'') + y(y - xy')\right]\partial_y. \end{aligned} \tag{4.9}$$

The symmetries are neither the Γ_{icj} of (3.4) nor the symmetries of (4.7) which give the property (4.1).

5. Higher-order equations

In §§2 and 3 we treated the properties of the symmetries, in particular the 'exceptional symmetries', in considerable detail. We now enunciate some propositions for the exceptional symmetries, which have now become nonexceptional, for higher-order equations of maximal symmetry.

PROPOSITION I

The differential operator

$$\Gamma_{ej} = \xi_j\partial_x + \eta_j\partial_y \Leftrightarrow (\eta_j - y'\xi_j)\partial_y = yI_j\partial_y, \quad j = 1, n, \tag{5.1}$$

where I_j, $j = 1, n$, is a fundamental first integral of the nth-order ordinary differential equation of maximal order, i.e. belonging to the equivalence class of $y^{(n)} = 0$ under transformation, is a Lie symmetry of that equation.

Proof. The nth extension of Γ_{ej} is

$$\Gamma_{ej}^{[n]} = I_j \sum_{i=0}^{n} y^{(i)} \partial_{y^{(i)}}, \tag{5.2}$$

whence the result follows by the application of (5.2) to the differential equation.

Remark. We observe that

$$\Gamma_{ej}^{[k]} = I_i \Gamma_h^{[k]}, \tag{5.3}$$

where Γ_h is the homogeneity symmetry.

PROPOSITION II

The set of n symmetries of the form (5.1) *constitutes an abelian algebra.*

Proof. Since the symmetries are generalized, the $(n-1)$th extension is required for the calculation of the Lie Bracket. The Lie Bracket of Γ_{ei} and Γ_{ej} is found as follows:

$$\begin{aligned}
[\Gamma_{ei}, \Gamma_{ej}]_{\mathrm{LB}} &= [\Gamma_{ei}^{[n-1]}, \Gamma_{ej}^{[n-1]}]_{\mathrm{LB}} \\
&= [I_i \Gamma_h^{[n-1]},\ I_j \Gamma_h^{[n-1]}]_{\mathrm{LB}} \\
&= I_i (\Gamma_h^{[n-1]} I_j) \Gamma_h^{[n-1]} - I_j (\Gamma_h^{[n-1]} I_i) \Gamma_h^{[n-1]} \\
&= I_i I_j \Gamma_h^{[n-1]} - I_j I_i \Gamma_h^{[n-1]} \\
&= 0.
\end{aligned} \tag{5.4}$$

PROPOSITION III

The action of Γ_{ei} *on* I_j *is*

$$\Gamma_{ei}^{[n-1]} I_j = I_i I_j. \tag{5.5}$$

Proof. We have

$$\begin{aligned}
\Gamma_{ei}^{[n-1]} I_j &= I_i \Gamma_h^{[n-1]} I_j \\
&= I_i I_j.
\end{aligned} \tag{5.6}$$

We note that there are general results for the standard homogeneity, solution and special linear symmetries. In the case of the first the results are somewhat obvious, but we state them for the purposes of completion.

PROPOSITION IV

The representative nth-order ordinary differential equation, $y^{(n)} = 0$, *possesses the homogeneity symmetry*, $\Gamma_h = y\partial_y$, *with the property that*

$$\Gamma_h^{[n-1]} I_j = I_j, \tag{5.7}$$

where I_j *is a fundamental first integral.*

Proof. The proof is evident.

PROPOSITION V

The representative nth-order ordinary differential equation, $y^{(n)} = 0$, possesses n solution symmetries of the form $\Gamma_{si} = s_i\partial_y$, $i = 1, n$, where $s_i^{(n)} = 0$, with the property that

$$\Gamma_{si}^{[n-1]} I_j = \delta_{ij}, \;\; i, j = 1, n. \tag{5.8}$$

Proof. Again the proof is trivial.

PROPOSITION VI

The representative nth-order ordinary differential equation, $y^{(n)} = 0$, possesses the three-element algebra $sl(2, R)$ in the representation

$$\begin{aligned} \Gamma_{sl1} &= \partial_x \\ \Gamma_{sl2} &= x\partial_x + \tfrac{1}{2}(n-1)y\partial_y \\ \Gamma_{sl3} &= x^2\partial_x + (n-1)xy\partial_y \end{aligned} \tag{5.9}$$

with actions on the fundamental first integrals of

$$\begin{aligned} \Gamma_{sl1}^{[n-1]} I_j &= -I_{j+1}, \quad I_{n+1} = 0 \\ \Gamma_{sl2}^{[n-1]} I_j &= \tfrac{1}{2}(n+1-2j)I_j \\ \Gamma_{sl3}^{[n-1]} I_j &= (j-1)(n+1-j)I_{j-1}, \quad I_0 = 0. \end{aligned} \tag{5.10}$$

Proof. The proof follows from direct computation.

Finally we present the general structures based upon a symmetry of the form $\Gamma = \xi\partial_x + \eta\partial_y$ which follows from the requirements (5.4), (5.6), (5.8) and (5.10).

PROPOSITION VII

The general forms of a symmetry $\Gamma = \xi\partial_x + \eta\partial_y$ possessing the characteristic properties of the homogeneity symmetry, the solution symmetries, the special linear symmetries and the exceptional symmetries are

$$\begin{aligned} \Gamma_h &= \xi_h\partial_x + \eta_h\partial_y &&: \eta_h - y'\xi_h = y \\ \Gamma_{si} &= \xi_{si}\partial_x + \eta_{si}\partial_y &&: \eta_{si} - y'\xi_{si} = s_i, \;\; i = 1, n, \\ \Gamma_{sli} &= \xi_{sli}\partial_x + \eta_{sli}\partial_y &&: \eta_{sli} - y'\xi_{sli} = \begin{cases} -y', & i = 1 \\ \tfrac{1}{2}(n-1)y - xy', & i = 2 \\ (n-1)xy - x^2yi, & i = 3 \end{cases} \\ \Gamma_{ei} &= \xi_{ei}\partial_x + \eta_{ei}\partial_y &&: \eta_{ei} - y'\xi_{ei} = yI_i, \;\; i = 1, n, \end{aligned}$$

where s_i, $i = 1, n$, and I_i, $i = 1, n$, are the solutions and fundamental first integrals of the differential equation $y^{(n)} = 0$ respectively, determined from the requirements that

$$\Gamma_h^{[n-1]} I_i = I_i$$

$$\Gamma_{si}^{[n-1]} I_j = \delta_{ij}$$

$$\Gamma_{sli}^{[n-1]} I_j = \begin{cases} -I_{j+1}, & i = 1 \\ \frac{1}{2}(n+1-2j) I_j, & i = 2 \\ (j-1)(n+1-j) I_{j-1}, & i = 3 \end{cases} \tag{5.11}$$

$$\Gamma_{ei}^{[n-1]} I_j = I_i I_j, \tag{5.12}$$

for which in the cases of Γ_{si} and Γ_{ei} the index i runs from 1 to n and $j = 1, n$ and we set $I_{n+1} = 0$.

Proof. Trivial.

6. Conclusion

In this paper we have reviewed the properties of the standard symmetries of the representative equations of the second- and third-order of maximal symmetry of point (second-order) and contact (third-order) symmetries in terms of the corresponding finite transformations and actions on solutions and fundamental first integrals. In particular we noted that in the general property of a transformation obtained from a symmetry mapping a solution into a solution one regularly obtains the general solution from a particular solution.

In the case of the so-called exceptional symmetries we showed how to construct an equivalence class with a property as coherent as found for the other classes of symmetry. This class of symmetries is applicable to all linear nth-order ordinary differential equations and equations linearizable by means of transformation. The actions of the four classes of symmetry are given in (5.12). In the case of second- and third-order equations the exceptional symmetries have an equivalent definition which enables them to be classed within the limits of point and contact symmetries for second- and third-order ordinary differential equations respectively although their generic structures are generalized of first- and second-order in the derivative respectively. This reclassification is not possible for equations of higher order.

The idea of a generalization of the symmetries Γ_{ei} defined in (5.1) is easy to develop. For example, we may define

$$\Gamma_{eij} = y I_i I_j \partial_y \tag{6.1}$$

which clearly has the properties

$$\Gamma_{eij}^{[n-1]} I_k = I_i I_j I_k, \quad \left[\Gamma_{eij}, \, \Gamma_{elk}\right]_{LB} = 0. \tag{6.2}$$

Further generalizations are evident.

Finally we observe that the subalgebras of the different classes of symmetry are abelian with the exception of the three-dimensional subalgebra, $sl(2, R)$. This subalgebra, as represented by symmetries, is essential for all equations of maximal symmetry and yet is not sufficient for linearity which is the characteristic of all equations of maximal symmetry.

Appendix

Appendix A: Complete symmetry group of a scalar linear nth-order ordinary differential equation

The equivalence of all linear nth-order ordinary differential equations under nonlocal transformation, be the number of Lie point symmetries $n+4$, $n+2$ or $n+1$, which extends the results obtained here for linear equations of maximal point symmetry to all linear equations relies upon any linear equation with the number of symmetries being representative of the class of linear equations with that number of symmetries. This point is not made explicit in [9]. The result relies on the representation of complete symmetry group of the class of differential equations being automatically included in the given seven symmetries for that class. We recall that the complete symmetry group of a differential equation is represented by the algebra of minimal dimension which specifies completely differential equation [12, 13, 2, 3]. In the case of nth-order linear differential equations[6] the Lie point symmetries are

number of symmetries	$n+4$	$n+2$	$n+1$
homogeneity symmetry	$y\partial_y$	$y\partial_y$	$y\partial_y$
solution	$s_i(x)\partial_y$	$s_i(x)\partial_y$	$s_i(x)\partial_y$
special linear/ autonomy	$w_i(x)\partial_x$ $+\frac{1}{2}(n-1)w_I'(x)y\partial_y$	∂_x	
algebra	$\{sl(2,R)\oplus_s A_1\}\oplus_s nA_1$	$A_1\oplus_s\{A_1\oplus_s nA_1\}$	$A_1\oplus_s nA_1$,

(A.1)

where the n linearly independent functions, $s_i(x)$, are solutions of the nth-order linear differential equation and the three functions, $w_i(x)$, are the three linearly independent solutions of the third-order equation [17]

$$\frac{(n+1)!}{(n-2)!4!}w''' + B_{n-2}w' + \tfrac{1}{2}B_{n-2}'w = 0 \tag{A.2}$$

for an equation written in normal form and with the coefficient of $y^{(n-2)}$ being B_{n-2}. The equation with $n+2$ Lie point symmetries is assumed to be written in autonomous form. The equivalence is a consequence of the following proposition.

PROPOSITION

The complete symmetry group of a scalar nth-order linear ordinary differential equation is represented by the algebra $A_1\oplus_s nA_1$, *where the one-element subalgebra*, A_1, *is represented by* Γ_h *and the n-element subalgebra*, nA_1, *is represented by* Γ_{si}, $i=1,n$, *of solution synergies.*

[6]As always the equation is taken to be homogeneous. The dependent variable of a nonhomogeneous linear equation is related to the dependent variable of the corresponding homogeneous linear equation by means of a translation involving a function of the independent variable.

Proof. The proof is constructive. We apply the n solution symmetries in turn to the equation

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}) \tag{A.3}$$

to obtain a nonhomogeneous equation which is linear in y and its derivatives. The application of the homogeneity symmetry removes the nonhomogeneous term and gives the desired result.

We illustrate the first few steps. The application of $\Gamma_{s1}^{[n]}$ to (A.3) gives

$$s_1^{(n)} = s_1^{(j)} \frac{\partial f}{\partial y^{(j)}} \tag{A.4}$$

in which summation over the repeated index is implied[7]. The associated Lagrange's system is

$$\frac{\mathrm{d}x}{0} = \frac{\mathrm{d}y^{(j)}}{s_1^{(j)}} = \frac{\mathrm{d}f}{s_1^{(n)}}, \tag{A.5}$$

where the middle term implies a succession of equalities, with the characteristics x,

$$u_j = \frac{y^{(j)}}{s_1^{(j)}} = \frac{y}{s_1}, \;\; j = 1, n-1, \tag{A.6}$$

and

$$w_1 = \frac{f}{s_1^{(n)}} - \frac{y}{s_1} \tag{A.7}$$

so that (A.3) may now be written as

$$y^{(n)} = \frac{s_1^{(n)}}{s_1} y + s_1^{(n)} F_1(x, u_j). \tag{A.8}$$

The application of $\Gamma_{s2}^{[n]}$ to (A.8) leads to

$$\frac{s_2^{(n)}}{s_1^{(n)}} - \frac{s_2}{s_1} = \left(\frac{s_2^{(j)}}{s_1^{(j)}} - \frac{s_2}{s_1} \right) \frac{\partial F_1}{\partial u_j} \tag{A.9}$$

and this in turn to the new form of (A.8) as

$$y^{(n)} = \left(\frac{s_2^{(n)}}{s_1^{(n)}} - \frac{s_2}{s_1} \right) \left\{ \frac{u_1}{\dfrac{s_2^{(1)}}{s_1^{(1)}} - \dfrac{s_2}{s_1}} + F_2(x, v_j) \right\}, \tag{A.10}$$

[7]Note that we are assuming that the equation is linear, homogeneous and of the nth-order. Otherwise it can be quite general in form. For a general equation at least one solution must be such that $s^{[n]}$ is not zero. We take that to be s_1.

for which the new characteristic is

$$v_j = \frac{u_j}{\dfrac{s_2^{(j)}}{s_1^{(j)}} - \dfrac{s_2}{s_1}} - \frac{u_1}{\dfrac{s_2^{(1)}}{s_1^{(1)}} - \dfrac{s_2}{s_1}}, \quad j = 2, n-1. \tag{A.11}$$

The process is continued for all of the symmetries, Γ_{si}. On the application of the n-th of the symmetries of the equation corresponding to (A.9) contains only one derivative and so the integration is elementary. The current function $F(x, w)$, in which w is the penultimate characteristic, is an affine function of w which is linear in y and its first $n-1$ derivatives. The application of the homogeneity symmetry removes the arbitrary function of x and we are left with an homogeneous equation of precisely determined coefficients. In fact the equation has the standard form of a linear homogeneous equation of known solution, *videlicet*

$$\begin{vmatrix} s_1 & s_2 & s_3 & \dots & s_n & y \\ s_1^{(1)} & s_2^{(1)} & s_3^{(1)} & \dots & s_n^{(1)} & y^{(1)} \\ s_1^{(2)} & s_2^{(2)} & s_3^{(2)} & \dots & s_n^{(2)} & y^{(2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ s_1^{(n)} & s_2^{(n)} & s_3^{(n)} & \dots & s_n^{(n)} & y^{(n)} \end{vmatrix} = 0. \tag{A.12}$$

With this demonstration of the equivalence of all linear equations within a class of given number of Lie point symmetries the completeness of the results of [9] is established. *A fortiori* the results presented in this paper for linear equations of maximal symmetry are extended to all linear equations and to all equations which can be reduced to linear equations by means of transformation.

Appendix B: *A demonstration of the derivation of one of* (4.7)

We illustrate the manner of derivation of the set of equations (4.7) using Γ_1 and the integral I_1. The symmetry, Γ_1, is a symmetry of (3.1) and is required to satisfy (4.1). Consequently we have the set of four equations

$$\eta_1''' - 3y''\xi_1'' - y'\xi_1''' = 0, \tag{B.1}$$

$$\xi_1(xy'' - y') + \eta_1 - x(\eta_1' - y'\xi_1') + \frac{1}{2}x^2(\eta_1'' - 2y''\xi_1' - y'\xi_1'') = \left(\frac{1}{2}x^2y'' - xy' + y\right)^2, \tag{B.2}$$

$$\xi_1 y'' + x(\eta_1'' - 2y''\xi_1' - y'\xi_1'') - (\eta_1' - y'\xi_1') = \left(\frac{1}{2}x^2y'' - xy' + y\right)(xy'' - y'), \tag{B.3}$$

$$\eta_1'' - 2y''\xi_1' - y'\xi_1'' = \left(\frac{1}{2}x^2y'' - xy' + y\right)y'', \tag{B.4}$$

where once again the prime denotes total differentiation with respect to the independent variable, x.

We use (B.4) to eliminate $\eta_1'' - 2y''\xi_1' - y'\xi_1''$ from (B.2) and (B.3) to obtain

$$\xi_1(xy'' - y') + \eta_1 - x(\eta_1' - y'\xi') + \frac{1}{2}x^2y''\left(\frac{1}{2}x^2y'' - xy' + y\right)$$
$$= \left(\frac{1}{2}x^2y'' - xy' + y\right)^2, \tag{B.5}$$

$$\xi_1 y'' - (\eta_1' - y'\xi_1') = -\left(\frac{1}{2}x^2y'' - xy' + y\right)y'. \tag{B.6}$$

We substitute for $\eta_1' - y'\xi_1'$ from (B.6) into (B.5) and are left with

$$\eta_1 - \xi_1 y' = \left(\frac{1}{2}x^2y'' - xy' + y\right)y \tag{B.7}$$

as required. The satisfaction of (B.1) is automatic.

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