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Nuclearity for dual operator spaces

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Abstract. In this short paper, we study the nuclearity for the dual operator space V^* of an operator space V. We show that V^* is nuclear if and only if V^{***} is injective, where V^{***} is the third dual of V. This is in striking contrast to the situation for general operator spaces. This result is used to prove that V^{**} is nuclear if and only if V is nuclear and V^{**} is exact.

Keywords. Operator space; nuclear; injective.

1. Introduction

The theory of operator spaces is a recently arising area in modern analysis, which is a natural non-commutative quantization of Banach space theory. Many problems in operator spaces are naturally motivated from both Banach space theory and operator algebra theory. Recently, there has been a very important development in the local theory of operator spaces. Some local properties such as local reflexivity, exactness, nuclearity and injectivity have been intensively studied in [7, 8, 12]. In particular, we have for any operator space V,

$$V \text{ is nuclear } \Rightarrow V \text{ is exact } \Rightarrow V \text{ is locally reflexive.}$$

The first implication was proved in [12] and the second in [8]. In [8], Effros, Ozawa and Ruan showed that an operator space V is nuclear if and only if V is locally reflexive and V^{**} is injective. As pointed out in [8], local reflexivity is an essential condition in this result since Kirchberg [10] has constructed a separable non-nuclear operator space V for which $V^{**} = \prod_{n=1}^{+\infty} M_n$. Turning to C^*-algebra theory, using Conne's deep work in [3], Choi and Effros proved the following result in [1, 2]:

$$\text{A } C^*\text{-algebra } A \text{ is nuclear} \Leftrightarrow \text{its second dual } A^{**} \text{ is injective.}$$

For ternary rings of operators (TROs), Kaur and Ruan (Theorem 6.5 of [9]) showed that if V is a TRO, then

$$V \text{ is nuclear} \Leftrightarrow V^{**} \text{ is injective.}$$

The key point of the proof in Theorem 6.5 of [9] is the close relationship between local properties of the TRO V and its linking C^*-algebra $A(V)$. As pointed out in the above, for general operator space V, the nuclearity of V is not equivalent to the injectivity of V^{**}. But how about the situation for the dual operator space V^*? In this short note, we prove that

$$V^* \text{ is nuclear} \Leftrightarrow V^{***} \text{ is injective.}$$

This is in striking contrast to the situation for general operator spaces. In [5], Dong and Ruan have succeeded in showing that a dual operator space V^* is nuclear if and only if it is the direct sum of a row sub-homogeneous operator space and a column sub-homogeneous operator space. As a corollary of this characterization, we give the structure of V, V^{**} and V^{***} when V^* is nuclear.

For the convenience of the readers, we recall some of the basic notations and terminologies in operator spaces, the details of which can be found in [6, 13]. Given a Hilbert space $\mathcal{H}$, we let $\mathcal{B}(H)$ denote the space of all bounded linear operators on $\mathcal{H}$. For each natural number $n \in \mathbf{N}$, there is a canonical norm $\| \cdot \|_n$ on the $n \times n$ matrix space $M_n(\mathcal{B}(H))$ given by identifying $M_n(\mathcal{B}(H))$ with $\mathcal{B}(\mathcal{H}^n)$. We call this family of norms $\{\| \cdot \|_n\}$ an operator space matrix norm on $\mathcal{B}(H)$. An operator space V is a norm closed subspace of some $\mathcal{B}(H)$ equipped with the distinguished operator space matrix norm inherited from $\mathcal{B}(H)$. An abstract matrix norm characterization of operator space was given in [14]. The morphisms in the category of operator spaces are the completely bounded linear maps. Given the operator spaces V and W, a linear map φ: $V \to W$ is completely bounded if the corresponding linear mappings φ_n: $M_n(V) \to M_n(W)$ defined by $\varphi_n([x_{ij}]) = [\varphi(x_{ij})]$ are uniformly bounded, i.e.

$$\| \varphi \|_{cb} = \sup\{\| \varphi_n \| : n \in \mathbf{N}\} < \infty.$$

A map φ is completely contractive (respectively, completely isometric, a complete quotient mapping) if $\| \varphi \|_{cb} \leq 1$ (respectively, for each $n \in \mathbf{N}, \varphi_n$ is an isometry, a quotient mapping). We denote by $CB(V, W)$ the space of all completely bounded maps from V into W. It is known that $CB(V, W)$ is an operator space with the operator space matrix norm given by identifying $M_n(CB(V, W)) = CB(V, M_n(W))$. In particular, if V is an operator space, then its dual space V^* is an operator space with operator space matrix norm given by the identification $M_n(V^*) = CB(V, M_n)$. In the following, we give some definitions of the local properties of operator spaces, given an operator space V.

(1) *Nuclearity*. If there exists a diagram of complete contractions

$$\begin{array}{ccccc} & & M_{n(\alpha)} & & \\ & \overset{r_\alpha}{\nearrow} & & \overset{s_\alpha}{\searrow} & \\ V & & \overset{\mathrm{id}_V}{\longrightarrow} & & V \end{array}$$

which approximately commute in the point-norm topology, we say V is nuclear.

(2) *Exactness*. If for any finite dimensional subspace L of V and every $\epsilon > 0$, there exist an integer n and a subspace $S \subseteq M_n$ such that $d_{cb}(L, S) < 1 + \epsilon$.

(3) *Local lifting property* (*LLP*). If given any operator spaces $W \subseteq Y$ and a complete contraction φ: $V \to Y/W$, for every finite dimensional subspace E of V and $\epsilon > 0$, there exists a completely bounded linear map $\tilde{\varphi}$: $E \to Y$ such that $\| \tilde{\varphi} \|_{cb} < 1 + \epsilon$ and $q \circ \tilde{\varphi} = \varphi|_E$, where q: $Y \to Y/W$ is the canonical quotient mapping.

(4) *Injectivity*. If for any operator spaces $X \subseteq Y$ and a complete contraction φ: $X \to V$ there is a completely contractive extension $\tilde{\varphi}$: $Y \to V$.

2. Nuclearity of dual operator spaces

We say that a diagram of operator spaces and complete contractions

$$0 \to X \overset{\varphi}{\hookrightarrow} Y \overset{\psi}{\to} Z \to 0$$

is 1-exact if φ is a complete isometry, ψ is a complete quotient mapping, and $\ker \psi = \operatorname{Im} \varphi$.

The following result was discussed in Pisier [12], who attributed the result to Kirchberg and Valliant.

Lemma 2.1. *An operator space V is nuclear if and only if it has the following property. For any* 1*-exact sequence of operator spaces*

$$0 \to X \overset{\varphi}{\hookrightarrow} Y \overset{\psi}{\to} Z \to 0,$$

it follows that

$$0 \to X \check{\otimes} V \overset{\varphi \otimes \mathrm{id}}{\longrightarrow} Y \check{\otimes} V \overset{\psi \otimes \mathrm{id}}{\longrightarrow} Z \check{\otimes} V \to 0$$

is 1*-exact.*

Lemma 2.2. *For any* 1*-exact sequence of operator spaces*

$$0 \to X \overset{\varphi}{\hookrightarrow} Y \overset{\psi}{\to} Z \to 0, \tag{1}$$

then

$$0 \to Z^* \overset{\psi^*}{\hookrightarrow} Y^* \overset{\varphi^*}{\to} X^* \to 0 \tag{2}$$

is also 1*-exact.*

Proof. It is obvious that φ^* is a complete quotient mapping, ψ^* is a complete isometry and $\varphi^* \circ \psi^* = 0$. It suffices to show that $\ker \varphi^* = \operatorname{Im} \psi^*$. Since $\varphi^* \circ \psi^* = 0$, $\operatorname{Im} \psi^* \subseteq \ker\ \varphi^*$. Conversely, suppose that $y^* \in \ker\ \varphi^* \subseteq Y^*$. Thus for any $x \in X$,

$$0 = \varphi^*(y^*)(x) = y^*(\varphi(x)).$$

This implies that $y^* \in \varphi(X)^{\perp} \subseteq Y^*$. From Theorem 10.2 of [4], the mapping defined by

$$\rho(f) = f \circ \pi, \qquad \forall f \in (Y/\varphi(X))^*$$

is an isometric linear isomorphism from $(Y/\varphi(X))^*$ onto $\varphi(X)^{\perp}$, where $\pi \colon Y \to Y/\varphi(X)$ is the canonical quotient mapping. It follows from the 1-exactness of sequence (1) that

$$\tilde{\psi} \colon Y/\varphi(X) \to Z, \qquad \tilde{\psi}(\pi(y)) = \psi(y)$$

is a (completely) isometric linear isomorphism. Thus the composition $\rho \circ \tilde{\psi}^*$ is a bounded linear isomorphism from Z^* onto $\varphi(X)^{\perp}$. Hence there exists an element $z^* \in Z^*$ such that $y^* = \rho \circ \tilde{\psi}^*(z^*)$. Now for any $y \in Y$, we have

$$\begin{aligned} \rho \circ \tilde{\psi}^*(z^*)(y) &= \tilde{\psi}^*(z^*)(\pi(y)) \\ &= z^*(\tilde{\psi}(\pi(y))) \\ &= z^*(\psi(y)) \\ &= \psi^*(z^*)(y). \end{aligned}$$

This implies that $y^* = \rho \circ \tilde{\psi}^*(z^*) = \psi^*(z^*)$ and $\ker\ \varphi^* \subseteq \operatorname{Im} \psi^*$. Hence $\ker\ \varphi^* = \operatorname{Im} \psi^*$ and the sequence (2) is also 1-exact. □

Lemma 2.3. *Suppose that V is an operator space, then V^* is nuclear if and only if V^* is injective and exact.*

Proof.
Necessity. The exactness of V^* follows from [12]. For any finite operator spaces $E \subseteq F$, we have the 1-exact sequence

$$0 \to E \hookrightarrow F \to F/E \to 0.$$

From Lemma 2.2, the sequence of operator spaces

$$0 \to (F/E)^* \hookrightarrow F^* \to E^* \to 0$$

is also 1-exact. It follows from Lemma 2.1 that the top row of the commutative diagram

$$\begin{array}{ccc} F^*\check{\otimes}V^* & \longrightarrow & E^*\check{\otimes}V^* \\ \| & & \| \\ CB(F, V^*) & \longrightarrow & CB(E, V^*) \end{array}$$

is a complete quotient mapping, and the same is true for the bottom row. In other words, any mapping $\varphi: E \to V^*$ with $\| \varphi \|_{cb} < 1$ can be extended to a mapping $\psi: F \to V^*$, i.e. V^* is finitely injective. Corollary 4.4 in [11] shows that V^* is injective.

Sufficiency. Given a finite dimensional subspace $L \subseteq V^*$ and $\epsilon > 0$, it follows from the exactness of V^* that we may choose an $n \in \mathbf{N}$, a subspace $G \subseteq M_n$, and a linear isomorphism $r: L \to G$ with $\| r \|_{cb} = 1$ and $\| r^{-1} \|_{cb} < 1 + \epsilon$. Since V^* is injective, we may find a corresponding extension $s: M_n \to V^*$ of r^{-1} with $\| s \|_{cb} < 1 + \epsilon$. We thus obtain a diagram

$$\begin{array}{ccccc} & & M_n & & \\ & \overset{t_0}{\nearrow} & \cup\!\!| & \overset{s}{\searrow} & \\ L & \overset{r}{\to} & G & \overset{r^{-1}}{\to} & V^* \end{array}$$

in which $t_0: L \to M_n$ is just the inclusion mapping composed with r. We may extend t_0 to a complete contraction $t: V^* \to M_n$. From this construction it is evident that V^* is nuclear. □

Now we are in the position to prove the main result of this paper.

Theorem 2.4. *Suppose that V is an operator space, then V^* is nuclear if and only if V^{***} is injective.*

Proof.
Necessity. The injectivity of V^{***} follows from Theorem 4.5 in [8].

Sufficiency. Suppose that the third dual V^{***} is injective. Let $\iota: V \hookrightarrow V^{**}$ be the canonical inclusion and its adjoint $\iota^*: V^{***} \to V^*$. Now for any $v^* \in V^* \hookrightarrow V^{***}$ and $v \in V$, we have

$$\iota^*(v^*)(v) = v^*(\iota(v)) = v^*(v).$$

Thus $\iota^*(v^*) = v^*$ for any $v^* \in V^*$, this shows that ι^* is a completely contractive projection from V^{***} onto V^*. Since V^{***} is injective by the hypothesis, Proposition 4.16 in [6] implies that V^* is also injective. Thus $V^* = e\mathcal{R}e^{\perp}$ from Theorem 1.3 in [8], where e is a projection in an injective von Neumann algebra $\mathcal{R}$. In particular, V^* is a TRO. Hence from Theorem 6.5 in [9], the injectivity of V^{***} implies the nuclearity of V^*. This completes the proof. □

COROLLARY 2.5

For any operator space V, *we have*

$$V^{(k)} \text{ is nuclear} \Leftrightarrow V^{(k+2)} \text{ is injective, for any integer } k \geq 1,$$

where $V^{(k)}$ *is the* k*-th dual operator space of* V.

Suppose that

$$W = \prod_{k=1}^{+\infty} (M_{n_k,m_k} \overline{\otimes} L_\infty(X_k, \mu_k)),$$

if $\sup\{n_k\} < +\infty$ (or $\sup\{m_k\} < +\infty$), W is said to be row sub-homogeneous (or column sub-homogeneous). In [5], Dong and Ruan proved that a dual operator space V^* is nuclear if and only if

$$V^* = V_{\mathrm{r}} \oplus_\infty V_{\mathrm{c}},$$

where V_{r} is row sub-homogeneous and V_{c} is column sub-homogeneous. Notice that the row subhomogenuous space V_{r} (respectively, the column subhomogenuous space V_{c}) is a dual TRO and thus has a unique predual. From this, we can easily obtain the following corollary.

COROLLARY 2.6

Suppose that the dual operator space V^* *is nuclear and* $V^* = V_{\mathrm{r}} \oplus_\infty V_{\mathrm{c}}$, *then*

(1) $V = (V_{\mathrm{r}})_* \oplus_1 (V_{\mathrm{c}})_*$;
(2) $V^{**} = (V_{\mathrm{r}})^* \oplus_1 (V_{\mathrm{c}})^*$;
(3) $V^{***} = (V_{\mathrm{r}})^{**} \oplus_\infty (V_{\mathrm{c}})^{**}$.

In particular, if

$$V_{\mathrm{r}} = \prod_{k=1}^{+\infty} (M_{n_k,m_k} \overline{\otimes} L_\infty(X_k, \mu_k)),$$

we have

$$(V_{\mathrm{r}})_* = l_1(\{T_{n_k,m_k} \hat{\otimes} L_1(X_k, \mu_k)\}).$$

In the following, we will use Theorem 2.4 to discuss the relationship between the nuclearity of V and V^{**}.

PROPOSITION 2.7

For any operator space V, V^{***} is injective if and only if V^* is injective and exact.

Proof. It follows from Lemma 2.3 and Theorem 2.4 directly. □

Exactness of V^* is an essential condition in this result. For example, let $V = \mathcal{T}(l^2)$, the injectivity of $V^* = \mathcal{B}(l^2)$ does not imply that $V^{***} = \mathcal{B}(l^2)^{**}$ is injective.

COROLLARY 2.8

For any operator space V, $V^{(k+2)}$ is injective if and only if $V^{(k)}$ is injective and exact, for any integer $k \geq 1$.

Theorem 2.9. *Suppose that V is an operator space, then V^{**} is nuclear if and only if V is nuclear and V^{**} is exact.*

Proof.
Necessity. Suppose that V^{**} is nuclear. It follows from Lemma 2.3 that V^{**} is injective and exact. From the definition of exactness, the exactness of V^{**} implies that V is also exact. Thus V is locally reflexive and V^{**} is injective, Theorem 4.5 in [8] shows that V is nuclear.

Sufficiency. Suppose that V is nuclear and V^{**} is exact. From Theorem 4.5 in [8] again, V^{**} is injective. Thus the nuclearity of V^{**} follows from Lemma 2.3. □

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