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Homomorphisms between C^*-algebra extensions

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Abstract. In this paper we consider the question when a homomorphism between two extension algebras preserves the essential ideal in the corresponding extension. Some conditions of two essential extensions being isomorphic are given. We also describe the relationship between the induced extensions and the Kasparov products and give the completely positive liftings of the induced extensions.

Keywords. C^*-algebra; extension; homomorphism.

1. Introduction

Brown, Douglas and Fillmore gave the famous BDF theory [3, 4] to study essentially normal operators on an infinite dimensional Hilbert space and extensions of C^*-algebra $C(X)$ by $\mathcal{K}$ in the 1970's, where X is a compact metric space and $\mathcal{K}$ is the C^*-algebra of compact operators on an infinite dimensional separable Hilbert space H. The theory of extensions of C^*-algebras has developed rapidly since then, and has become an important tool for classifications of C^*-algebras together with K-theory and index theory (see [2]).

On the other hand, the classification of amenable C^*-algebras originated from Elliott's work on AF-algebras. Since then a number of classification results appeared ([6, 7, 11, 14], etc.). Extension algebras form an important class of C^*-algebras. There are many classification results of such C^*-algebras ([9, 10, 12], etc.). Unlike classifications of C^*-algebra extensions, Ext groups do not classify extension algebras. So one has to study the isomorphism equivalence of extensions.

In fact, a homomorphism between two extension algebras may not map the essential ideal into the other in general, so we have to consider properties of extension homomorphisms before studying the isomorphism equivalence of extensions. In this paper, we consider when a homomorphism between two extension algebras preserves the essential ideal, i.e., when a homomorphism between two extension algebras can induce an extension homomorphism. Some conditions of two extensions being isomorphic are given. At last, we describe the relationship between the induced extensions and the Kasparov products and give the completely positive liftings of the induced extensions.

2. Preliminaries

Let A and B be C^*-algebras. Recall that an extension of A by B is a short exact sequence $0 \to B \stackrel{\alpha}{\to} E \stackrel{\beta}{\to} A \to 0$. Denote this extension by e or (E, α, β) and the set of all such extensions by $E(A, B)$.

The extension (E, α, β) is called trivial, if the above sequence splits, i.e., if there is a homomorphism $\gamma \colon A \to E$ such that $\beta \circ \gamma = \mathrm{id}_A$. We call (E, α, β) essential, if $\alpha(A)$ is an essential ideal in E.

Let $0 \to B \overset{\alpha}{\to} E \overset{\beta}{\to} A \to 0$ be an extension of A by B. Then there is a unique homomorphism $\sigma \colon E \to M(B)$ such that $\sigma \circ \alpha = \iota$, where $M(B)$ is the multiplier algebra of B, and ι is the inclusion map from B to $M(B)$. It is known that σ is injective if and only if the extension is essential.

The Busby invariant of (E, α, β) is a homomorphism τ from A to the corona algebra $\mathcal{Q}(B) = M(B)/B$ defined by $\tau(a) = \pi(\sigma(b))$ for all $a \in A$, where $\pi \colon M(B) \to \mathcal{Q}(B)$ is the quotient map, and $b \in E$ such that $\beta(b) = a$. Therefore the Busby invariant of (E, α, β) is the unique homomorphism making the following diagram commute:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \xrightarrow{\alpha} & E & \xrightarrow{\beta} & A & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\tau} & & \\ 0 & \longrightarrow & B & \longrightarrow & M(B) & \xrightarrow{\pi} & \mathcal{Q}(B) & \longrightarrow & 0. \end{array}$$

Hence there is a one to one correspondence between extensions of A by B and homomorphisms from A to $\mathcal{Q}(B)$. Denote by τ_e the Busby invariant of an extension e.

If A is unital and the Busby invariant is unital, then (E, α, β) is called unital.

Let $e_i \colon 0 \to B \overset{\alpha_i}{\to} E_i \overset{\beta_i}{\to} A \to 0$ be two extensions of A by B with Busby invariants τ_i for $i = 1, 2$. Then (E_1, α_1, β_1) and (E_2, α_2, β_2) are called congruent (called 'strongly isomorphic' in [2]), denoted by $e_1 \equiv e_2$, if there exists an isomorphism η such that the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \xrightarrow{\alpha_1} & E_1 & \xrightarrow{\beta_1} & A & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\eta} & & \| & & \\ 0 & \longrightarrow & B & \xrightarrow{\alpha_2} & E_2 & \xrightarrow{\beta_2} & A & \longrightarrow & 0. \end{array}$$

Denote by $\mathcal{E}\mathrm{xt}(A, B)$ the set of congruent equivalence classes of essential extensions of A by B.

(E_1, α_1, β_1) and (E_2, α_2, β_2) are called isomorphic (called 'weakly isomorphic' in [2]), denoted by $e_1 \cong e_2$, if there exist isomorphisms β, η, α such that the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \xrightarrow{\alpha_1} & E_1 & \xrightarrow{\beta_1} & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\alpha} & & \\ 0 & \longrightarrow & B & \xrightarrow{\alpha_2} & E_2 & \xrightarrow{\beta_2} & A & \longrightarrow & 0. \end{array}$$

(E_1, α_1, β_1) and (E_2, α_2, β_2) are called unitarily equivalent, denoted by $e_1 \sim e_2$, if there exists a unitary $u \in M(B)$ such that $\tau_2(a) = \pi(u)\tau_1(a)\pi(u)^*$ for all $a \in A$.

Let H be a separable infinite dimensional Hilbert space, and let $\mathcal{K}$ be the ideal of compact operators in $B(H)$. If B is a stable C^*-algebra (i.e., $B \otimes \mathcal{K} \cong B$), then the sum of two extensions τ_1 and τ_2 is defined to be a homomorphism $\tau_1 \oplus \tau_2$, where $\tau_1 \oplus \tau_2 \colon A \to \mathcal{Q}(B) \oplus \mathcal{Q}(B) \subseteq M_2(\mathcal{Q}(B)) \cong \mathcal{Q}(B)$.

Let $\mathrm{Ext}(A, B)$ denote stable strong equivalence classes (see [2] for the definition). If A is a separable nuclear C^*-algebra, then $\mathrm{Ext}(A, B)$ is an abelian group by a theorem of Arveson in [1].

Let $e \in E(A, B)$, $\alpha \in \mathrm{Aut}(A)$ and $\beta \in \mathrm{Aut}(B)$. Then there are two induced extensions βe and $e\alpha$. See [13] for details.

Let A and B be C^*-algebras, and A_1 and A_2 C^*-subalgebras of A. If there is a unitary $u \in M(A)$ such that $\mathrm{Ad}\, u(A_1) = A_2$, then we say that A_2 is unitarily equivalent to A_1 by $\mathrm{Ad}\, u$, and denote it by $A_2 \overset{u}{\sim} A_1$. Let ϕ_1 and ϕ_2 be homomorphisms from A to B. If there is a unitary $u \in M(B)$ such that $\phi_2 = \mathrm{Ad}\, u \circ \phi_1$, then we say that ϕ_2 is unitarily equivalent to ϕ_1 by $\mathrm{Ad}\, u$, and denote it by $\phi_2 \overset{u}{\sim} \phi_1$.

3. Main results

Suppose that ϕ is a homomorphism between extension algebras in essential extensions of A by B. Then ϕ may not map the ideal B into itself in general, even if ϕ is an isomorphism.

Example 1. Let $B = C_0(0, 1]$ and $A = \mathbb{C}$. Then there is an essential unital extension $0 \to C_0(0, 1] \overset{i}{\to} C[0, 1] \overset{\pi}{\to} \mathbb{C} \to 0$, where i is the inclusion map and $\pi(f) = f(0)$ for every $f \in C[0, 1]$. Define a map ϕ from $C[0, 1]$ to itself by assigning f to a function $\phi(f)$, where $\phi(f)(t) = f(1 - t)$. Then ϕ is an automorphism on $C[0, 1]$. But $\phi(C_0(0, 1]) \nsubseteq C_0(0, 1]$, so ϕ does not map the ideal B into itself.

Example 2. Even if B is stable, an automorphism of an extension algebra in an essential unital extension also may not map B into B. By Example 1, there is an essential unital extension $0 \to C_0(0, 1] \otimes \mathcal{K} \overset{i\otimes\mathrm{id}}{\to} C[0, 1] \otimes \mathcal{K} \overset{\pi\otimes\mathrm{id}}{\to} \mathbb{C} \otimes \mathcal{K} \to 0$. Set $\phi(f)(t) = f(1-t)$. Then $\phi \otimes \mathrm{id}\colon C[0, 1] \otimes \mathcal{K} \to C[0, 1] \otimes \mathcal{K}$ is an isomorphism. By Example 1, $\phi \otimes \mathrm{id}$ does not map $C_0(0, 1] \otimes \mathcal{K}$ into itself.

In the case of finite algebras, we have the following results.

PROPOSITION 3.1

Let A_i and B_i be finite algebras, with B_i stable and A_i unital and simple for $i = 1, 2$. Let $e_i = (\varphi_i, E_i, \psi_i) \in \mathcal{E}\mathrm{xt}(A_i, B_i)$ be unital extensions for $i = 1, 2$.

(1) *If ϕ is a homomorphism from E_1 to E_2 and $RR(B_i) = 0$, then $\phi(B_1) \subset B_2$.*
(2) *If $e_1 \cong e_2$, then $E_1 \cong E_2$. Conversely, if $RR(B_i) = 0$ and $E_1 \cong E_2$, then $e_1 \cong e_2$.*

Proof.

(1) Since B_1 is stable, for any projection $p_0 \in B_1 \backslash \{0\}$, there is a sequence $\{p_i\}_{i=1}^{\infty}$ consisting of projections in B_1 such that $p_n \sim p_0$ and $\{p_i\}_{i=0}^{\infty}$ are mutually orthogonal. Then there is a projection $p \in E_1$ such that $p_1 + p_2 + \cdots + p_n \le p$ for any $n \in \mathbb{N}$.

Suppose $\pi\colon E_2 \to A_2$ is the quotient map. Then $\{\pi(\phi(p_n))\}$ are mutually orthogonal equivalent projections in A_2. Moreover, $\pi(\phi(p_1)) + \pi(\phi(p_2)) + \cdots + \pi(\phi(p_n)) \le \pi(\phi(p))$ for any $n \in \mathbb{N}$. Since A_2 is a finite C^*-algebra, $\pi(\phi(p_0)) = 0$. By $RR(B_1) = 0$, we have $\phi(B_1) \subset B_2$.

(2) If e_1 is isomorphic to e_2, then E_1 is isomorphic to E_2, obviously.

Conversely, suppose ϕ is an isomorphism from E_1 to E_2. By (1), we have $\phi(B_1) \subset B_2$. Similarly, ϕ^{-1} is an isomorphism from E_2 to E_1, so we have $\phi^{-1}(B_2) \subset B_1$. Then $\phi(B_1) = B_2$. Therefore, $\varphi = \phi|_{B_1}: B_1 \to B_2$ is an isomorphism. Let ψ denote the map induced by ϕ. Then ψ is an isomorphism from A_1 to A_2. Therefore $(\varphi, \phi, \psi): e_1 \to e_2$ is an extension isomorphism. □

COROLLARY 3.2

(1) *Let A_i and B_i be finite algebras, with B_i stable and A_i unital and simple. If $RR(B_i) = 0$ and $e_i = (\varphi_i, E_i, \psi_i) \in \mathcal{E}\mathrm{xt}(A_i, B_i)$ for $i = 1, 2$, then $E_1 \cong E_2$ if and only if there are isomorphisms $\alpha: A_1 \to A_2$ and $\beta: B_1 \to B_2$ such that $e_1 \cdot \beta = \alpha \cdot e_2$.*

(2) *Let A and B be finite algebras, with B stable and A unital and simple. If $RR(B) = 0$ and $e_1, e_2 \in \mathcal{E}\mathrm{xt}(A, B)$, then $E_1 \cong E_2$ if and only if there are isomorphisms $\alpha \in \mathrm{Aut}(A)$ and $\beta \in \mathrm{Aut}(B)$ such that $e_1 \cdot \beta = \alpha \cdot e_2$.*

Example 3. When $B = \mathcal{K}$, any isomorphism between extension algebras in essential unital extensions of A by B must map $\mathcal{K}$ onto itself, but a homomorphism between such extension algebras may not have this property. Let $A = \mathcal{O}_2$. Suppose that $0 \to \mathcal{K} \to E \to \mathcal{O}_2 \to 0$ is an essential unital extension of A by $\mathcal{K}$, and $0 \to \mathcal{K} \to E' \xrightarrow{\pi} \mathcal{O}_2 \to 0$ is an essential unital trivial extension with a homomorphism λ from $\mathcal{O}_2$ to E' such that $\pi \circ \lambda = \mathrm{id}_{\mathcal{O}_2}$. Since E is exact, by Kirchberg's exact embedding theorem, there is an injective unital homomorphism η from E to $\mathcal{O}_2$. Set $\phi = \lambda \circ \eta$. Then ϕ is an injective homomorphism from E to E' such that $\phi(\mathcal{K}) \cap \mathcal{K} = \{0\}$.

When B is simple and e_i are essential extensions for $i = 1, 2$, then the isomorphisms between such extension algebras map the essential ideal B into itself.

Theorem 3.3. *Let A and B be C^*-algebras, with B simple. If $e_i: 0 \to B \to E_i \xrightarrow{\psi_i} A \to 0$ are essential unital extensions of A by B with the Busby invariants $\tau_i: A \to \mathcal{Q}(B)$ for $i = 1, 2$, then*

(1) *$E_1 \cong E_2 \Longleftrightarrow e_1 \cong e_2$; and*

(2) *if there is a unitary $u \in M(B)$ such that $\tau_1(A)$ is unitarily equivalent to $\tau_2(A)$ by $\mathrm{Ad}\,\pi(u)$, i.e., $\tau_2(A) \overset{\pi(u)}{\sim} \tau_1(A)$, then $e_1 \cong e_2$.*

Proof.

(1) We only need to prove the necessary. If $\phi: E_1 \to E_2$ is an isomorphism, then $\phi(B)$ is a nonzero ideal in E_2. Since B is essential, $\phi(B) \cap B \neq \{0\}$. Note that B is simple. Then we have $\phi(B) \cap B = B$, so $B \subset \phi(B)$. Similarly, since $\phi^{-1}: E_1 \to E_2$ an isomorphism, $B \subset \phi^{-1}(B)$. Hence $B = \phi(B)$. Therefore, there is an isomorphism $\psi: A \to A$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & E_1 & \xrightarrow{\psi_1} & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi|_B} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi} & & \\ 0 & \longrightarrow & B & \longrightarrow & E_2 & \xrightarrow{\psi_2} & A & \longrightarrow & 0, \end{array}$$

i.e., $e_1 \cong e_2$.

(2) Suppose that there is a unitary $u \in B(H)$ such that $\tau_2(A) \overset{\pi(u)}{\sim} \tau_1(A)$. Since

$$\pi(\operatorname{Ad} u(E_1)) = \operatorname{Ad} \pi(u)\pi(E_1) = \operatorname{Ad} \pi(u)(\tau_1(A)) = \tau_2(A),$$

it follows that $\operatorname{Ad} u(E_1) \subset E_2$.

Similarly, we have $\operatorname{Ad} u^*(E_2) \subset E_1$. Then $\phi = \operatorname{Ad} u\colon E_1 \to E_2$ is an isomorphism. Obviously, $\phi(B) = B$. Hence $\phi|_B$ is an isomorphism. Set $\psi = \tau_2^{-1} \circ \operatorname{Ad} \pi(u) \circ \tau_1$. Then $\psi \in \operatorname{Aut}(A)$. Note that $\psi_i = \tau_i^{-1} \circ \pi$. We have a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B & \longrightarrow & E_1 & \xrightarrow{\psi_1} & A & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \phi|_B} & & \downarrow{\scriptstyle \operatorname{Ad} u} & & \downarrow{\scriptstyle \psi} & & \\
0 & \longrightarrow & B & \longrightarrow & E_2 & \xrightarrow{\psi_2} & A & \longrightarrow & 0.
\end{array}$$

Therefore $e_1 \cong e_2$.

Remark. Suppose that B is a simple and stable C^*-algebra. If e_i are not essential extensions, then an isomorphism between extension algebras may not map the essential ideal B into itself. See Example 4 below.

Example 4. Let $B = A = \mathcal{K}$, $E = \mathcal{K} \oplus \mathcal{K}$ and $0 \to \mathcal{K} \to \mathcal{K} \oplus \mathcal{K} \to \mathcal{K} \to 0$ a trivial extension. Define a map by $\phi(a, b) = (b, a)$ for any $(a, b) \in E$. Then ϕ is an automorphism on E, but $\phi(B) \cap B = \{0\}$.

Suppose that A is a separable C^*-algebra. By Theorem 15.7.1 of [2], an extension $\tau\colon A \to \mathcal{Q}(B)$ is invertible if and only if τ can lift to a completely positive map from A to $M(B)$. This is equivalent to that the exact sequence $0 \to B \to E \to A \to 0$ corresponding to τ is semsplit (see [8]).

Theorem 3.4. *Let A and B are C^*-algebra with A separable, and let τ_1 and τ_2 be invertible essential extensions of A. Then*

(1) $\tau_2 \overset{\pi(u)}{\sim} \tau_1$ *if and only if there is a completely positive map $\sigma_i\colon A \to M(B)$ such that $\tau_i = \pi \circ \sigma_i$ and $\sigma_2 \overset{u}{\sim} \sigma_1$ for $i = 1, 2$, where u is a unitary in $M(B)$;*

(2) $\tau_2(A) \overset{\pi(u)}{\sim} \tau_1(A)$ *if and only if there is a completely positive map $\sigma_i\colon A \to M(B)$ such that $\tau_i = \pi \circ \sigma_i$ and $\sigma_2(A) \overset{u}{\sim} \sigma_1(A)$ for $i = 1, 2$, where u is a unitary in $M(B)$.*

Proof.

(1) Since τ_1 is invertible, there is a completely positive map $\sigma_1\colon A \to M(B)$ such that $\tau_1 = \pi \circ \sigma_1$. Suppose that there is a unitary $u \in M(B)$ such that $\tau_2 = \operatorname{Ad} \pi(u) \circ \tau_1$. Let $\sigma_2 = \operatorname{Ad} u \circ \sigma_1\colon A \to M(B)$. Since $\operatorname{Ad} \pi(u) \circ \pi \circ \sigma_1 = \pi \circ \operatorname{Ad} u \circ \sigma_1$, it follows that $\sigma_2 \overset{u}{\sim} \sigma_1$ and $\tau_2 = \pi \circ \sigma_2$.

Conversely, suppose that there is a unitary $u \in M(B)$ and a completely positive map $\sigma_i\colon A \to M(B)$ such that $\tau_i = \pi \circ \sigma_i$ and $\sigma_2 \overset{u}{\sim} \sigma_1$ for $i = 1, 2$. Then $\tau_2 = \operatorname{Ad} \pi(u) \circ \tau_1$.

(2) If there is a unitary $u \in M(B)$ and a completely positive map $\sigma_i\colon A \to M(B)$ such that $\tau_i = \pi \circ \sigma_i$ and $\sigma_2(A) \overset{u}{\sim} \sigma_1(A)$ for $i = 1, 2$, then $\pi \circ \sigma_2(A) \overset{\pi(u)}{\sim} \pi \circ \sigma_1(A)$. Since $\tau_i = \pi \circ \sigma_i$ for $i = 1, 2$, $\tau_2(A) \overset{\pi(u)}{\sim} \tau_1(A)$.

Conversely, if $\tau_2(A) \overset{\pi(u)}{\sim} \tau_1(A)$, where u is a unitary in $M(B)$, then by Theorem 3.3 there is an isomorphism $(\alpha, \beta, \gamma)\colon E_1 \to E_2$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & E_1 & \xrightarrow{\psi_1} & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & B & \longrightarrow & E_2 & \xrightarrow{\psi_2} & A & \longrightarrow & 0, \end{array}$$

where $\beta = \operatorname{Ad} u$.

Since τ_1 is invertible, there is a completely positive map $\sigma_1\colon A \to E_1 \subset M(B)$ such that $\psi_1 \circ \sigma_1 = \mathrm{id}_A$. Hence $\tau_1 = \pi \circ \sigma_1$. Set $\sigma_2 = \beta \circ \sigma_1 \circ \gamma^{-1}$. Then $\psi_2 \circ \sigma_2 = \mathrm{id}_A$. Note that $\psi_i = \tau_i^{-1} \circ \pi$. Then $\pi \circ \sigma_2 = \tau_2$. Since $\beta \circ \sigma_1 = \sigma_2 \circ \gamma$ and $\beta = \operatorname{Ad} u$, it follows that $\operatorname{Ad} u \circ \sigma_1 = \sigma_2 \circ \gamma$. Therefore,

$$\operatorname{Ad} u(\sigma_1(A)) = \sigma_2(\gamma(A)) = \sigma_2(A). \qquad \square$$

Induced extensions play an important role in the classification of C^*-algebra extensions. Rørdam (Proposition 1.1 of [15]) described the relationship between the Kasparov products and the extensions induced by isomorphisms. In the following theorem we generalize the result to the general case.

Theorem 3.5. *Let A be a separable nuclear C^*-algebra and B a σ-unital C^*-algebra. Let $e \in \mathcal{E}\mathrm{xt}(A, B)$, $\alpha \in \mathrm{Hom}(D, A)$ and $\beta\colon B \to C$ a surjective homomorphism. Then*

$$[e\alpha] = KK(\alpha) \cdot [e] \quad \textit{and} \quad [\beta e] = [e] \cdot KK(\beta).$$

Proof. Let τ be the Busby invariant of e. It follows from Lemma 3.3.13 of [8] that $\tilde{\beta} \circ \tau$ is the Busby invariant of βe, where $\tilde{\beta}$ is the unique homomorphism such that the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & M(B) & \xrightarrow{\pi_B} & Q(B) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\bar{\beta}} & & \downarrow{\scriptstyle\tilde{\beta}} & & \\ 0 & \longrightarrow & C & \longrightarrow & M(C) & \xrightarrow{\pi_C} & Q(C) & \longrightarrow & 0, \end{array}$$

where $\bar{\beta}$ is the unique extension of β.

Identify $\mathrm{Ext}(A, B)$ with $KK^1(A, B)$. By 17.6.4 of [2], we have

$$KK^1(A, B) = \{[\varphi, p] \mid \varphi\colon A \to M(B),\ p^* = p^2 = p \in M(B),\ \text{such that } p\varphi - \varphi p \in B\}.$$

Let e be the extension corresponding to (φ, p). Then $\tau = \pi_B(p\varphi p)$ is the Busby invariant of e. Since

$$[e] \cdot KK(\beta) = [\varphi, p] \cdot KK(\beta) = [\bar{\beta} \circ \varphi, \bar{\beta}(p)],$$

it follows that $\tau' = \pi_C(\bar{\beta}(p)\bar{\beta} \circ \varphi\bar{\beta}(p))$ is the Busby invariant of $[e] \cdot KK(\beta)$.

Note that

$$\pi_C(\bar{\beta}(p)\bar{\beta} \circ \varphi\bar{\beta}(p)) = \pi_C \circ \bar{\beta}(p\varphi p) = \tilde{\beta} \circ \pi_B(p\varphi p).$$

This implies that $\tau' = \tilde{\beta} \circ \tau$, so we have $[\beta e] = [e] \cdot KK(\beta)$.

From Proposition 4.2 of [5], it follows that there is a natural isomorphism from $\mathrm{Ext}(A, B)$ to $KK(A, \mathcal{Q}(B))$ which assigns $[e]$ to $KK(\tau_e)$, where τ_e is the Busby invariant of e. Hence

$$KK(\alpha) \cdot [e] = KK(\alpha) \cdot [\tau_e] = [\tau_e \circ \alpha].$$

Note that there is a commutative diagram for $e\alpha$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B & \longrightarrow & E' & \longrightarrow & D & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow{\scriptstyle \alpha} & & \\
0 & \longrightarrow & B & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow{\scriptstyle \tau_e} & & \\
0 & \longrightarrow & B & \longrightarrow & M(B) & \longrightarrow & \mathcal{Q}(B) & \longrightarrow & 0.
\end{array}$$

Then $\tau_e \circ \alpha$ is the Busby invariant of $e\alpha$. Therefore, we have $[e\alpha] = KK(\alpha) \cdot [e]$. □

COROLLARY 3.6

Let A be a separable nuclear C^*-algebra and B a σ-unital C^*-algebra. Let $e \in \mathcal{E}\mathrm{xt}(A, B)$, $\alpha \in \mathrm{Hom}(D, A)$ and $\beta\colon B \to C$ a surjective homomorphism. If e is an invertible extension and τ_e lifts to a completely positive map $\varphi\colon A \to M(B)$, i.e., $\tau_e = \pi_B \circ \varphi$, then $\varphi \circ \alpha$ and $\bar{\beta} \circ \varphi$ are completely positive liftings of $\tau_{e\alpha}$ and $\tau_{\beta e}$, respectively.

Proof. By Theorem 3.5, we have $\tau_{e\alpha} = \tau \circ \alpha$ and $\tau_{\beta e} = \tilde{\beta} \circ \tau$. Note that $\tau = \pi \circ \varphi$, so we have $\tau_{e\alpha} = \pi_B \circ \varphi \circ \alpha$. Since $\varphi \circ \alpha\colon D \to M(B)$ is a completely positive map, $\tau_{e\alpha}$ lifts to the completely positive map $\varphi \circ \alpha$ from D to $M(B)$.

Note that $\tilde{\beta} \circ \pi_B = \pi_C \circ \bar{\beta}$. It follows that $\tau_{\beta e} = \tilde{\beta} \circ \pi_B \circ \varphi = \pi_C \circ (\bar{\beta} \circ \varphi)$. Hence $\tau_{\beta e}$ lifts to the completely positive map $\bar{\beta} \circ \varphi$ from A to $M(C)$. □

References

[1] Arveson W, Notes on extensions of C^*-algebras, *Duke Math. J.* **44(2)** (1977) 329–355

[2] Blackadar B, K-theory for operator algebras, Second edition, Mathematical Sciences Research Institute Publications, vol. 5 (Cambridge: Cambridge University Press) (1998)

[3] Brown L G, Douglas R G and Fillmore P A, Unitary equivalence modulo the compact operators and extensions of C^*-algebras, Proceedings of a Conference on Operator Theory (Dalhousie Univ., Halifax, N.S., 1973) Lecture Notes in Math. (Berlin: Springer) (1973) vol. 345, pp. 58–128

[4] Brown L G, Douglas R G and Fillmore P A, Extensions of C^*-algebras and K-homology, *Ann. Math. (2)* **105(2)** (1977) 265–324

[5] Dadarlat M, Approximate unitary equivalence and the topology of $\mathrm{Ext}(A, B)$, C^*-algebras (Munster, 1999) (Berlin: Springer) (2000) pp. 42–60

[6] Elliott G A, On the classification of C^*-algebras of real rank zero, *J. Reine Angew. Math.* **443** (1993) 179–219

[7] Elliott G A and Gong Guihua, On the classification of C^*-algebras of real rank zero. II, *Ann. Math. (2)* **144(3)** (1996) 497–610
[8] Jensen K K, Thomsen K, Elements of KK-theory (Boston, Basel, Berlin: Birkhauser) (1991)
[9] Lin Huaxin, On the classification of C^*-algebras of real rank zero with zero, K_1, *J. Operator Theory* **35(1)** (1996) 147–178
[10] Lin Huaxin, A classification theorem for infinite Toeplitz algebras, Operator algebras and operator theory (Shanghai, 1997) pp. 219–275, *Contemp. Math.* 228 (Providence, RI: Amer. Math. Soc.) (1998)
[11] Lin Huaxin, Classification of simple C^*-algebras of tracial topological rank zero, *Duke Math. J.* **125(1)** (2004) 91–119
[12] Lin Huaxin and Su, Hongbing, Classification of direct limits of generalized Toeplitz algebras, *Pacific J. Math.* **181(1)** (1997) 89–140
[13] MacLane S, Homology (Berlin, FRG: Springer-Verlag) (1963)
[14] Phillips N C, A classification theorem for nuclear purely infinite simple C^*-algebras, *Doc. Math.* **5** (2000) 49–114
[15] Rørdam M, Classification of extensions of certain C^*-algebras by their six term exact sequences in K-theory, *Math. Ann.* **308(1)** (1997) 93–117